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Concentration-Cancellation for the Velocity Fields in Two Dimensional Incompressible Fluid Flows

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Abstract. We show that the weak-L^2 limit of a sequence of solutions of the two dimensional incompressible Euler equation is still a solution, provided that a (strong) concentration set for the reduced defect measure has locally finite one dimensional Hausdorff measure in space and time.

1. Introduction

In studying the two dimensional incompressible Euler equation with vortex sheet initial data, DiPerna and Majda [4] proved that concentration for the so-called *reduced defect measure* θ could occur on a set of (cylindrical) Hausdorff dimension at most 1 in space and time. Recently, Alinhac [1] proved that if a concentration set for the *weak star defect measure* σ is of "finite type," one still obtains a weak solution in the limit. A set is of "finite type" if, speaking informally, it consists of a finite number of C^1-curves plus a "small" part. In this paper we generalize Alinhac's result in two respects. First, we enlarge the allowed set of concentration by an arbitrary set of finite one dimensional Hausdorff measure. Second, we employ the more precise notion of a *strong concentration set* for θ (see below or Definition 1) instead of the notion of a concentration set for σ. In particular, our result allows for an everywhere dense strong concentration set which could be purely $(H^1, 1)$-unrectifiable, so long as it has finite one dimensional Hausdorff measure. The proof combines the shielding technique employed in the aforementioned papers [4] and [1] and a structure theorem of Federer [7].

We define a set E to be a strong concentration set for θ if $\theta\ (V^c)=0$ for all open sets $V \supset E$. A concentration set for θ (in the sense of DiPerna–Majda [4]) is defined to be the intersection of a (decreasing) sequence of open sets whose complements are null sets of θ. It can be seen that each such open set is a strong concentration set. This connection actually also gives us an estimate on strong

★ Permanent Address: Tancheng, Shandong, People's Republic of China concentration sets, since DiPerna–Majda's estimate on concentration sets is expressed through an estimate on strong concentration sets. Detailed properties of strong concentration sets and relations between various concentration sets can be found in the next section. We point out that the advantages of strong concentration sets are first that one can use its structure as well as its size to enhance the weak convergence, and second strong concentration sets are generally much smaller than concentration sets for σ. The disadvantage of strong concentration sets is that they are generally larger than concentration sets for θ in the sense of DiPerna and Majda.

There is therefore still quite a distance between the present result and the final expectation, which is to establish the existence of weak solutions to the two dimensional Euler equation with vortex sheet initial data. See DiPerna–Majda [4–5], Greengard–Thomann [8], and Evans [6] (and references therein) for more background on this problem.

2. Concentration Sets

We shall consider a sequence of measurable functions $\{v^k\}_{k=1}^{\infty}$ mapping $\mathbb{R}^2 \times [0, T]$ into $\mathbb{R}^2$ such that, for each $R > 0$ there exists a constant C_R with

$$\sup_{0 \leqq t \leqq T} \int_{|x| \leqq R} |v^k(x,t)|^2 \, dx \leqq C_R, \quad k = 1, 2, \dots . \tag{1}$$

By Passing to a subsequence, we can assume that this sequence converges weakly in $L^2_{\text{loc}}(\mathbb{R}^2 \times [0, T])$ to a function $v \in L^\infty([0, T], L^2_{\text{loc}}(\mathbb{R}^2_x))$:

$$v^k \to v \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}^2 \times [0, T]). \tag{2}$$

We may also assume, by still passing to a subsequence, that

$$v^k \otimes v^k \xrightarrow{\omega^*} (\mu_{ij})_{2 \times 2} \equiv \mu, \tag{3}$$

where μ denotes a 2×2 matrix-valued Radon measure on $\mathbb{R}^2 \times [0, T]$ and $v^k \otimes v^k \equiv (v_i^k v_j^k)_{2 \times 2}$. This means

$$\int \phi : v^k \otimes v^k \, dx\, dt \to \int \phi : d\mu$$

for all matrix-valued test functions $\phi \in C_c(\mathbb{R}^2 \times (0, T))$. Here ":" denotes inner product of matrices, $\mathrm{A{:}B} = \operatorname{trace}(A^t B)$. The **weak star defect measure** σ in the present context is defined by requiring

$$\int \phi |v^k - v|^2 \, dx\, dt \to \int \phi \, d\sigma$$

for all $\phi \in C_c(\mathbb{R}^2 \times (0, T))$. The **reduced defect "measure"** θ is defined on each Borel set $E \subset \mathbb{R}^2 \times [0, T]$ by setting

$$\theta(E) \equiv \limsup_{k \to \infty} \int_E |v^k - v|^2 \, dx\, dt.$$

Note that θ is only finitely subadditive. **A concentration set for** σ is any Borel set $E \subset \mathbb{R}^2 \times [0, T]$ such that

$$\sigma(A) = \sigma(A \cap E)$$

for all Borel sets $A \subset \mathbb{R}^2 \times [0, T]$. **A concentration set for** θ in the sense of DiPerna and Majda [4] (see also Greengard and Thomann [8]) is a set $E \subset \mathbb{R}^2 \times [0, T]$ for which there exists a decreasing sequence of open sets

$$V_n \supset V_{n+1}, \quad n = 1, 2, 3, \dots$$

such that

$$\theta(V_n^c) = 0, \quad n = 1, 2, 3, \dots, \quad \text{and} \quad E = \bigcap_{n=1}^{\infty} V_n.$$

Note that an empty concentration set in the sense of DiPerna and Majda does not necessarily imply strong convergence (see also Ball–Murat [2]). We introduce the following

Definition 1. *A set* $E \subset \mathbb{R}^2 \times [0, T]$ *is called* **a strong concentration set** *for* θ *if* $\theta(V^c) = 0$ *for all open sets* $V \supset E$.

An empty strong concentration set for θ now implies strong convergence. For a fixed sequence $\{v^k\}_{k=1}^{\infty}$, any G_δ strong concentration set for θ is a concentration set for θ in the sense of DiPerna–Majda, and any concentration set for σ is a strong concentration set for θ. The latter is true because

$$0 \leqq \theta(V^c) \leqq \sigma(V^c) = \sigma(V^c \cap E) = 0$$

for any open set $V \supset E$, if E is a concentration set for σ. Also, any one of the open sets $\{V^n\}_{n=1}^{\infty}$ defining a concentration set in the sense of DiPerna–Majda is itself a strong concentration set. Therefore from DiPerna–Majda [4], we obtain an estimate of a sequence of strong concentration sets $\{V_n\}$ for a sequence $\{v^k\}_{k=1}^{\infty}$ of approximate weak solutions to two dimensional Euler equation with uniformly bounded energy and total vorticity:

$$H_{r_n}^{\delta, 1+\delta}(V_n) \leqq C_\delta, \quad n = 1, 2, 3, \dots, \quad r_n \to 0^+,$$

where δ is any positive number and $H_r^{\delta, 1+\delta}$ denotes the $(\delta, 1 + \delta)$-order cylindrical Hausdorff premeasure at level r. This premeasure is determined by the most efficient countable cover with cylinders of height h_j and sectional radius r_j:

$$H_r^{\delta, 1+\delta}(E) = \inf\{\Sigma r_j^\delta h_j^{1+\delta} : E \subset \cup C(r_j, h_j)\},$$

where $r_j \leqq r$, $h_j \leqq r$ and $C(r_j, h_j)$ denotes a cylinder. On the other hand, Greengard–Thomann [8] have an example where σ is concentrated on the entire box $[0, 1]^2 \times [0, T]$. In summary, the heuristic relation between these three concentration sets is as follows:

> a strong concentration set is "slightly bigger" than a concentration set in the sense of DiPerna–Majda, and "much smaller" than a concentration set for σ.

3. Main Theorem

Theorem 1. *Suppose* $\{v^k\}_{k=1}^{\infty}$ *satisfying* (1), (2) *and* (3) *are weak solutions to the two dimensional Euler equation in* $\mathbb{R}^2 \times (0, T)$,

$$\begin{cases} v_t^k + \operatorname{div}(v^k \otimes v^k) = -\nabla p^k + f^k \\ \operatorname{div} v^k = 0 \end{cases} \tag{4}$$

with $f^k \in L^1_{loc}$, $f^k \rightharpoonup f$ *weakly in* L^1_{loc}. *Assume there exists a strong concentration set* $E \subset \mathbb{R}^2 \times [0, T]$ *for the reduced defect measure* θ, *which has locally finite one dimensional Hausdorff measure*:

$$H^1(E \cap (B_R(0) \times [0, T])) < \infty \quad (0 < R < \infty).$$

Then v *is a weak solution to the two dimensional Euler Equation*:

$$\begin{cases} v_t + \operatorname{div}(v \otimes v) = -\nabla p + f \\ \operatorname{div} v = 0 \end{cases}. \tag{5}$$

The proof of this theorem will be given in the last section. The precise meaning of (5) is

$$\int_0^T \int_{\mathbb{R}^2} (v \cdot \phi_t + v \otimes v : \nabla \phi) dx dt = -\int_0^T \int_{\mathbb{R}^2} f \cdot \phi dx dt,$$

$$\int_0^T \int_{\mathbb{R}^2} v \cdot \nabla \psi dx dt = 0$$

for all divergence-free vector fields $\phi = (\phi_1, \phi_2) \in C^1_c(\mathbb{R}^2 \times (0, T))$, with div $\phi = 0$ and all $\psi \in C^1_c(\mathbb{R}^2 \times (0, T))$.

Remarks.

1. The theorem is still true if there exists a strong concentration set E which consists of two parts $E_1 \cup E_2$, where E_1 has locally finite H^1 measure and E_2 is of "finite type" in the sense of Alinhac [1]. The proof is similar.
2. The theorem is still true if there exists a strong concentration set E which takes the form

$$E = \{(x, t) | x - \phi(t) \in S, 0 \leqq t \leqq T\},$$

where $S \subset \mathbb{R}^2$ has zero one dimensional Hausdorff measure, and $\phi = (\phi_1, \phi_2) \in C^1[0, T]$. This is not contained in the theorem, nor is it contained in Remark 1. The stationary solution provides such an example ($\phi =$ constant). Its proof is contained in the proof of the theorem.

The proof of the theorem depends largely on a structure theorem of Federer [7], which we recall next.

4. Federer's Structure Theorem

Let k be an integer, $1 \leqq k \leqq n$. According to Federer [7], we say a set $A \subset \mathbb{R}^n$ is **countably** (H^k, k)**-rectifiable** if

$$A = A^* \cup \bigcup_{j=1}^{\infty} f_j(A_j),$$

where $A_j \subset \mathbb{R}^k$, $f_j : A_j \to \mathbb{R}^n$ is Lipschitz ($j = 1, 2, \ldots$) and

$$H^k(A^*) = 0.$$

We say a set $A \subset \mathbb{R}^n$ is **purely** (H^k, k)**-unrectifiable** if

$$H^k(A \cap B) = 0$$

for every countably (H^k, k)-rectifiable set B. Suppose now that $E \subset \mathbb{R}^2 \times [0, T]$ is an arbitrary set with finite one dimensional Hausdorff measure: $H^1(E) < \infty$. Federer's structure theorem (Federer [7, 3.3.13 & 3.3.18]; see also Ross [11]) asserts that (i) there exists a countably $(H^1, 1)$-rectifiable set $R \subset E$ for which $U \equiv E \backslash R$ is a purely $(H^1, 1)$-unrectifiable set. (ii) For a countably $(H^1, 1)$-rectifiable set R with $H^1(R) < \infty$ and any $\varepsilon > 0$, there exists a one dimensional C^1-imbedded submanifold $M \subset \mathbb{R}^2 \times [0, T]$ with

$$H^1((R - M) \cup (M - R)) < \varepsilon.$$

(iii) For a purely $(H^1, 1)$-unrectifiable set U with $H^1(U) < \infty$, the following projection property holds:

$$H^1(P_a(U)) = 0 \quad \text{for} \quad H^2 - \text{a.e. } a \in S^2.$$

Here P_a denotes the projection operator on to the line which passes through the origin with direction a, and $S^2 = \{(x, t) \mid |x|^2 + t^2 = 1\}$. Furthermore, a one dimensional C^1 imbedded submanifold $M \subset \mathbb{R}^2 \times [0, T]$ with $H^1(M) < \infty$ can be covered by

$$M \subset \bigcup_{j=1}^{N} g_j[0, 1] \cup M',$$

where $g_j(\cdot) = (x_{1j}(\cdot), x_{2j}(\cdot), t_j(\cdot)) \in C^1[0, 1]$, $j = 1, 2, \ldots, N < \infty$, and $H^1(M') < \varepsilon$.

We now use the above knowledge concerning the structure of the set E to build up an appropriate sequence of open covers for E.

Let us suppose we can find a direction $a \in S^2$ in the (x_1, t) plane:

$$a = (1, 0, \gamma)/\sqrt{1 + \gamma^2} \in S^2$$

such that

$$H^1(P_a(U)) = 0.$$

Each curve g_j has a C^1 projection on to (x_1, t) plane:

$$\{(x_{1j}(s), t_j(s)) \mid 0 \leqq s \leqq 1\}.$$

By Lemma 4.1 of Alinhac [1], for any given $\varepsilon > 0$ there exist a finite number K_j of functions $\phi_{jk} \in C^1[0, T]$, $k = 1, 2, \ldots, K_j$ and a set G_j such that

$$P_{x_1 t}(g_j[0, 1]) \subset \bigcup_{k=1}^{K_j} \{(x_1, t) \mid x_1 - \phi_{jk}(t) = 0\} \cup G_j$$

with

$$H^1(P_t G_j) < \varepsilon/N,$$

where $P_{x_1 t}$ and P_t denote respectively the projection operators onto the plane (x_1, t) and t-axis.

Therefore

$$P_{x_1 t}(M) \subset \bigcup_{j=1}^{N} \left(\bigcup_{k=1}^{K_j} \{x_1 - \phi_{jk}(t) = 0\} \cup G_j \right) \cup P_{x_1 t}(M')$$

with

$$H^1\left(P_t\left(\bigcup_{j=1}^{N} G_j \cup P_{x_1 t}(M') \right) \right) < 2\varepsilon.$$

Let $\mathbb{R}_{x_2} \equiv \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\}, \mathbb{R}_x^2 \equiv \{(x_1, x_2, 0) \mid (x_1, x_2) \in \mathbb{R}^2\}$, $\mathbb{R}_t^+ \equiv \{(0, 0, t) \mid 0 \leqq t \in \mathbb{R}\}$ and $\mathbb{R}_{a^\perp}^2 \equiv \{(x_1, x_2, t) \mid x_1 + \gamma t = 0\}$. We then have

$$\begin{aligned} E &= R \cup U \\ &\subset M \cup (R \backslash M) \cup U \\ &\subset (P_{x_1 t} M \times \mathbb{R}_{x_2}) \cup (P_t(R \backslash M) \times \mathbb{R}_x^2) \cup (P_a U \times \mathbb{R}_{a^\perp}^2) \\ &\subset \left\{ \bigcup_{j=1}^{N} \bigcup_{k=1}^{K_j} \{x_1 - \phi_{jk}(t) = 0\} \times \mathbb{R}_{x_2} \right\} \cup (P_a U \times \mathbb{R}_{a^\perp}^2) \\ &\quad \cup \left\{ P_t\left(\bigcup_{j=1}^{N} G_j \cup P_{x_1 t}(M' \cup (R \backslash M)) \right) \times \mathbb{R}_x^2 \right\} \end{aligned} \tag{6}$$

and

$$H^1 \left\{ P_t\left(\bigcup_{j=1}^{N} G_j \cup P_{x_1 t}(M' \cup (R \backslash M)) \right) \right\} < 3\varepsilon.$$

Take an open set $W \subset \mathbb{R}_t^+$ such that

$$W \supset P_t\left(\bigcup_{j=1}^{N} G_j \cup P_{x_1 t}(M' \cup (R \backslash M)) \right),$$

$$H^1(W) < 4\varepsilon.$$

Finally take open sets $\{V_m\}_{m=1}^{\infty}$ on the line passing through a such that

$$V_m \supset P_a(U) \quad \text{and}$$

$$H^1(V_m) < \frac{1}{m}, \quad m = 1, 2, \ldots.$$

Then we have

$$E \subset \left\{ \bigcup_{j=1}^{N} \bigcup_{k=1}^{K_j} \left\{ |x_1 - \phi_{jk}(t)| < \frac{1}{n_{jk}} \right\} \times \mathbb{R}_{x_2} \right\} \cup \{W \times \mathbb{R}_x^2\} \cup \{V_m \times \mathbb{R}_{a^\perp}^2\}. \tag{7}$$

The sets on the right-hand side of (7) are open for all choices of integers $\{n_{jk}\}_{j=1,k=1}^{N,\ K_j}$ and m as above.

5. Cut-Off Functions

Let us take

$$\chi_m(s) = \begin{cases} 0, & \text{if} \quad sa/\sqrt{1+\gamma^2} \in V_m \\ 1, & \text{if} \quad sa/\sqrt{1+\gamma^2} \notin V_m, \end{cases}$$

and

$$\chi_n(s) = \begin{cases} 1, & \text{if} \quad |s| > 2/n \\ 0, & \text{if} \quad |s| < 1/n \end{cases}$$

with $\chi_n \in C^\infty$, $0 \leqq \chi_n \leqq 1$. The index n will later be replaced by other integral indices $\{n_{jk}\}$.

Take finally

$$\chi(x_1,t) \equiv \chi_m(x_1+\gamma t)\prod_{j=1}^{N}\prod_{k=1}^{K_j}\chi_{n_{jk}}(x_1-\phi_{jk}(t)). \tag{8}$$

We observe that χ_m is not necessarily continuous. However the smoothness of $\chi_{n_{jk}}$ is needed later in the proof. We also point out that the most important property of this function is that it vanishes on an open set containing $E\backslash(W\times\mathbb{R}_x^2)$.

6. A Lemma

Lemma 1. *Let $\phi\in C^1(\mathbb{R})$ and let $\{\chi_i\}_{i=1}^{\infty}$ be a sequence of measurable functions such that $0\leqq\chi_i\leqq 1$, $\chi_i(s)\to 1$ for* a.e. *$s\in\mathbb{R}$ as $i\to\infty$. Let $K\in C^1(\mathbb{R}\times[0,\infty))$. Assume for any $T>0$, there exists an $R_T>0$ such that $K(x,t)=0$ on $[-R_T,R_T]^c\times[0,T]$. Then for all $\psi\in C_c^\infty(\mathbb{R}^2\times\mathbb{R}^+)$ and $v=(v_1,v_2)\in L^\infty_{\mathrm{loc}}(\mathbb{R}^+,L^1_{\mathrm{loc}}(\mathbb{R}_x^2))$ we have*

$$I_i\equiv\int_{\mathbb{R}^3}\nabla^\perp\partial_t(\eta_i\psi)\cdot v\,dx\,dt\to 0$$

as $i\to\infty$, where

$$\eta_i(x_1,t)\equiv\int_{-\infty}^{x_1}\int_{-\infty}^{z-\phi(t)}(1-\chi_i(s))K(s+\phi(t),t)\,ds\,dz.$$

(Here $\nabla^\perp\eta\equiv(-\partial_{x_2}\eta,\partial_{x_1}\eta)$).

Proof. We have

$$I_i=\int_{\mathbb{R}^3}(\nabla^\perp\partial_t\psi)\cdot v\eta_i+(\nabla^\perp\psi)\cdot v\partial_t\eta_i+(\partial_t\psi)(\partial_{x_1}\eta_i)v_2+\psi(\partial_{x_1}\partial_t\eta_i)v_2\,dx\,dt.$$

Notice

$$\eta_i=\int_{-\infty}^{x_1-\phi(t)}\left(\int_{-\infty}^{z}(1-\chi_i(s))K(s+\phi(t),t)\,ds\right)dz$$

Thus

$$\begin{aligned}\partial_{x_1}\eta_i&=\int_{-\infty}^{x_1-\phi(t)}(1-\chi_i(s))K(s+\phi(t),t)\,ds\\&\to 0,\\ \partial_t\eta_i&=-\phi'(t)\int_{-\infty}^{x_1-\phi(t)}(1-\chi_i(s))K(s+\phi(t),t)\,ds\\&\quad+\int_{-\infty}^{x_1-\phi(t)}\int_{-\infty}^{z}(1-\chi_i(s))\frac{dK}{dt}(s+\phi(t),t)\,ds\,dz\\&\to 0\end{aligned}$$

for each (x_1,t). Since also

$$\eta_i\to 0,$$

we thus conclude that all the terms in I_i go to zero except perhaps the last one. For the last term we have

$$\int_{\mathbb{R}^3} v_2\psi(\partial_{x_1}\partial_t\eta_i)dx\,dt = \int_{\mathbb{R}^1}\int_{\mathbb{R}^2} -\phi'(t)(1-\chi_i(x_1-\phi(t))K(x_1,t)v_2\psi\,dx\,dt$$
$$+\int_{\mathbb{R}^3}\left(\int_{-\infty}^{x_1-\phi(t)}(1-\chi_i(s))\frac{dK}{dt}(s+\phi(t),t)ds\right)v_2\psi\,dx\,dt.$$

The second integral goes to zero. For the first integral, we notice

$$\left|\int_{\mathbb{R}^2}(1-\chi_i(x_1-\phi(t)))K(x_1,t)v_2\psi\,dx\right| \leqq M$$

uniformly in t and

$$\int_{\mathbb{R}^2}(1-\chi_i(x_1-\phi(t)))K(x_1,t)v_2\psi\,dx \to 0$$

for each t. Thus the expression goes to zero by Lebesgue's Dominated Convergence Theorem.

We remark that Alinhac [1] has a similar lemma. The difference is that we do not require χ_i to be smooth.

The proof in the next section follows very much the same way as Alinhac [1].

7. Proof of Theorem 1

a. For any $\phi \in C_0^\infty(\mathbb{R}^2\times(0,T);\mathbb{R}^2)$, with $\operatorname{div}\phi = 0$, we write $\phi = \nabla^\perp\eta$ for $\eta\in C_0^\infty$. To show (5) is valid in the weak sense, it is sufficient to prove

$$\int \nabla^\perp\partial_t\eta\cdot v\,dx\,dt + \int \nabla\nabla^\perp\eta : v\otimes v\,dx\,dt = -\int \nabla^\perp\eta\cdot f\,dx\,dt. \tag{9}$$

However, from (4) we have for all $k=1,2,\dots$, and $\eta\in C_0^\infty$

$$\int \nabla^\perp\partial_t\eta\cdot v^k dx\,dt + \int \nabla\nabla^\perp\eta : v^k\otimes v^k dx\,dt = -\int \nabla^\perp\eta\cdot f^k dx\,dt. \tag{10}$$

By approximation this identity is valid if η belongs to $W^{2,\infty}(\mathbb{R}^2\times(0,T))$ and has compact support.

b. Let $\psi\in C_c^\infty(\mathbb{R}_x^2\times(0,T))$ be fixed, with $\operatorname{spt}\psi\subset\{x\in\mathbb{R}^2 \,|\, |x|\leqq R_0\}\times(0,T)$. Then the strong concentration set E restricted to $\operatorname{spt}\psi$ has one dimensional measure finite. We will still use E to denote this restriction. According to the decomposition of E in Sect. 4, we have

$$E = R\cup U,$$

where R is countably $(H^1,1)$-rectifiable and U is purely $(H^1,1)$-unrectifiable. There exists a dense set D of points on S^2 so that we always have zero one dimensional Hausdorff measure of the projection of U onto the straight line passing both through the origin and any point of the set D. Since Euler's equation is rotational covariant, we suppose without loss of generality that the (x_1,t) plane is one of the planes which contain at least one such point, and we further assume $(1,\gamma)$ is such a direction in the (x_1,t) plane. Taking $\xi\in C_c^\infty(\mathbb{R})$, we want to establish a modified version of (9) for $\eta(x_1,x_2,t)=\psi(x_1,x_2,t)\xi(x_1+\gamma t)$.

Given $\varepsilon_0 > 0$, we can choose ε so small that

$$4\varepsilon C_{R_0} \sup|\psi\xi''| < \varepsilon_0/2. \tag{11}$$

For the above $\varepsilon > 0$, E has an open cover as in (7). Using $\chi(x_1, t)$ as in (8), we then set

$$\begin{aligned}\zeta(x_1,t) &\equiv \int_{-\infty}^{x_1}\int_{-\infty}^{x} \chi(s,t)\xi''(s+\gamma t)\,ds\,dz \\ &= \int_{-\infty}^{x_1}\int_{-\infty}^{z+\gamma t} \chi(s-\gamma t,t)\xi''(s)\,ds\,dz \\ &= \int_{-\infty}^{x_1}\int_{-\infty}^{z+\gamma t} \chi_m(s)\prod_{j=1}^{N}\prod_{k=1}^{K_j}\chi_{n_{jk}}(s-\gamma t-\phi_{jk}(t))\xi''(s)\,ds\,dz.\end{aligned}$$

Finally take $\eta = \psi\zeta$ in (10).

c. We find

$$\begin{aligned}\int \nabla\nabla^{\perp}\eta : v^k \otimes v^k dx\,dt &= \int \psi(\partial_{x_1}^2\zeta)v_1^k v_2^k dx\,dt + \int \nabla\psi\otimes\nabla^{\perp}\zeta : v^k\otimes v^k dx\,dt \\ &\quad + \int \nabla\zeta\otimes\nabla^{\perp}\psi : v^k\otimes v^k dx\,dt + \int \nabla\nabla^{\perp}\psi : (v^k\otimes v^k)\zeta\,dx\,dt \\ &\equiv A + B + C + D\end{aligned}$$

and

$$\partial_{x_1}^2\zeta(x_1,t) = \chi(x_1,t)\xi''(x_1+\gamma t) \quad \text{for} \quad \text{a.e. } (x_1,t).$$

Split A into the two integrals

$$\begin{aligned}A &= \int_W\int_{\mathbb{R}^2} \psi\chi\xi'' v_1^k v_2^k dx\,dt + \int_{W^c}\int_{\mathbb{R}^2} \psi\chi\xi'' v_1^k v_2^k dx\,dt \\ &\equiv A_1 + A_2.\end{aligned}$$

We obtain $|A_1| \leqq \varepsilon_0/2$ by the choice of W and (11).

We then pass the limit as $k\to\infty$ in terms A_2, B, C and D. By construction, the integral in A_2 lies on a compact set disjoint from E; on such a compact set, the convergence of $\{v^k\}_{k=1}^{\infty}$ to v is strong in L^2. Therefore

$$A_2 \to \int_{W^c}\int_{\mathbb{R}^2} \psi\chi\xi'' v_1 v_2\,dx\,dt.$$

The terms B, C and D go to terms B', C' and D' of the same form, where $v^k\otimes v^k dx\,dt$ is replaced by $d\mu$, according to (3). Finally

$$\int \nabla^{\perp}\partial_t\eta\cdot v^k dx\,dt \to \int \nabla^{\perp}\partial_t\eta\cdot v\,dx\,dt$$

and

$$\int \nabla^{\perp}\eta\cdot f^k \to \int \nabla^{\perp}\eta\cdot f\,dx\,dt.$$

d. From c, we have obtained

$$\left|\int \nabla^{\perp}\partial_t\eta\cdot v\,dx\,dt + \int_{W^c}\int_{\mathbb{R}^2} \psi\chi\xi'' v_1 v_2\,dx\,dt + B' + C' + D' + \int \nabla^{\perp}\eta\cdot f\,dx\,dt\right| \leqq \frac{\varepsilon_0}{2}.$$

Therefore

$$\left|\int \nabla^{\perp}\partial_t\eta\cdot v\,dx\,dt + \int \psi\chi\xi'' v_1 v_2\,dx\,dt + B' + C' + D' + \int \nabla^{\perp}\eta\cdot f\,dx\,dt\right| \leqq \varepsilon_0.$$

Now we let the indices m and $\{n_{jk}\}$ go to infinity one by one; i.e., let $m \to \infty$, then $n_{11} \to \infty$, and so on. To deal with the first term $\int \nabla^{\perp}\partial_t \eta \cdot v dx dt$, we need Lemma 1 repeatedly. In the limit $m \to \infty$, we apply Lemma 1 with $\phi = -\gamma t$ and

$$K(s,t) = \prod_{j=1}^{N} \prod_{k=1}^{K_j} \chi_{n_{jk}}(s - \phi_{jk}(t))\xi''(s+\gamma t) \equiv K_0(s,t)$$

and obtain

$$\int \nabla^{\perp}\partial_t(\psi\zeta)\cdot v dx dt \to \int \nabla^{\perp}\partial_t\left(\psi \int_{-\infty}^{x_1}\int_{-\infty}^{z} K_0(s,t) ds dz\right)\cdot v dx dt$$

as $m \to \infty$. We have thus eliminated χ_m in the cut-off function χ. In the limit $n_{11} \to \infty$, we will eliminate $\chi_{n_{11}}$ in χ by applying Lemma 1 with $\phi(t) = \phi_{11}(t)$, $\chi_i(s) = \chi_{n_{11}}(s)$ and $K(s,t) = K_0(s,t)$ without the factor $\chi_{n_{11}}(s - \phi_{11}(t))$. Let $n_{12}, n_{21}, \ldots, n_{NK_N} \to \infty$ in a similar way, we obtain in the end

$$\int \nabla^{\perp}\partial_t(\psi\zeta)\cdot v dx dt \to \int \nabla^{\perp}\partial_t(\psi\xi)\cdot v dx dt.$$

Using Lebesgue's Dominated Convergence Theorem, we pass the limits m, $\{n_{jk}\} \to \infty$ in the second term to obtain

$$\int \psi\chi\xi'' v_1 v_2 dx dt \to \int \psi\xi'' v_1 v_2 dx dt.$$

The terms B', C', D' and $\int \nabla^{\perp}\eta \cdot f dx dt$ converge also to terms of their own forms, only with $\eta = \psi\zeta$ replaced by $\eta = \psi\xi$. This is because ζ and $\partial_{x_1}\zeta$ converge uniformly to ξ and $\partial_{x_1}\xi$ respectively on compact sets. To see this, let $(x,t) \in B_{R_0}(0) \times [0,T]$, we have

$$\begin{aligned}\left|\partial_{x_1}\zeta(x_1,t) - \int_{-\infty}^{x_1} K_0(s,t) ds\right| &\leqq \left|\int_{-\infty}^{x_1} (1 - \chi_m(s+\gamma t))K_0(s,t) ds\right| \\ &\leqq \|K_0\|_{L^\infty}\left|\int_{-\infty}^{\infty} (1 - \chi_m(s)) ds\right| \leqq cH^1(V_m)\end{aligned}$$

and

$$\left|\zeta(x_1,t) - \int_{-\infty}^{x_1}\int_{-\infty}^{z} K_0(s,t) ds dz\right| \leqq cH^1(V_m).$$

So

$$\partial_{x_1}\zeta(x_1,t) \to \int_{-\infty}^{x_1} K_0(s,t) ds,$$

$$\zeta \to \int_{-\infty}^{x_1}\int_{-\infty}^{z} K_0(s,t) ds dz$$

uniformly on $B_{R_0}(0) \times [0,T]$ as $m \to \infty$. Similarly, we can let $\{n_{jk}\} \to \infty$ to conclude that ζ and $\partial_{x_1}\zeta$ converge uniformly to ξ and $\partial_{x_1}\xi$ respectively on compact sets.

We finally deduce $|I| \leqq \varepsilon_0$, where

$$\begin{aligned} I = &\int \nabla^{\perp}\partial_t(\psi\xi)\cdot v dx dt + \int \psi\nabla\nabla^{\perp}\xi : (v\otimes v) dx dt \\ &+ \int \nabla\psi \otimes \nabla^{\perp}\xi : d\mu + \int \nabla\xi \otimes \nabla^{\perp}\psi : d\mu + \int \xi\nabla\nabla^{\perp}\psi : d\mu + \int \nabla^{\perp}(\psi\xi)\cdot f dx dt\end{aligned}$$

with $\xi = \xi(x_1 + \gamma t)$.

Since ε_0 is arbitrary, we have

$$I = 0. \tag{12}$$

e. We actually have proved that (12) holds for all functions of the form $\xi(d\cdot x + \gamma t)$, $\xi \in C_c^\infty(\mathbb{R})$, where $\{d\cdot x + \gamma t = 0\}$ forms a dense set of straight lines through the origin. By Radon transform [9], which asserts that finite linear combinations of functions of the form $\xi(d\cdot x + \gamma t)$ can approximate a given test function $\eta \in C_2^2$ in C^2-norm, we pass the limit in (12) to find

$$\begin{aligned}&\int \nabla^\perp \partial_t(\psi\eta)\cdot v\,dx\,dt + \int \psi\nabla\nabla^\perp\eta : v\otimes v\,dx\,dt\\ &\quad + \int \nabla\psi\otimes\nabla^\perp\eta : d\mu + \int \nabla\eta\otimes\nabla^\perp\psi : d\mu + \int \eta\nabla\nabla^\perp\psi : d\mu + \int \nabla^\perp(\psi\eta)\cdot f\,dx\,dt = 0.\end{aligned}$$

Letting $\psi \equiv 1$ on $\operatorname{spt}\eta$, we obtain (9). □

8. A Special Extension

Comparing Theorem 1 with DiPerna–Majda's result [4] that there exists a concentration set of "cylindrical" Hausdorff dimension at most 1, we see it would be interesting to shield a strong concentration set comprising a one dimensional, time-like curve defined on $[0, T]$ with infinite one dimensional Hausdorff measure.

A typical set of this kind is a nowhere differentiable curve $\Phi \in C^{0,\alpha}([0, T], \mathbb{R})$ (the space of all Hölder continuous functions with Hölder exponent α) for all $0 < \alpha < 1$, but not $\alpha = 1$. See Federer [7] or Ross [11] for explicit examples. More generally, it is shown in Besicovitch and Ursell [3] that a set of the following form

$$E = \{(x_1, t) \in \mathbb{R} \times [0, T] \mid x_1 = \Phi_1(t) \in C^{0,\alpha}([0, T], \mathbb{R})\}$$

can have Hausdorff dimension at most $2 - \alpha$. And there exist examples for each $0 < \alpha < 1$ such that the dimension $2 - \alpha$ is achieved. In the following theorem, we actually shield a strong concentration set of the more general form

$$E = \{(x_1, x_2, t) \in \mathbb{R}^2 \times [0, T] \mid (x_1, x_2) = (\Phi_1(t), \Phi_2(t)) \in C^{0,\alpha}([0, T], \mathbb{R}^2)\}, \tag{13}$$

where $\alpha \geqq 2/3$. This set E can have Hausdorff dimension at most $3 - 2\alpha$: see Mandelbrot [10] and references therein.

Theorem 2. *Suppose $\{v^k\}_{k=1}^\infty$ satisfying (1), (2) and (3) are weak solutions to the two dimensional Euler equation in $\mathbb{R}^2 \times (0, T)$*

$$\begin{cases} v_t^k + \operatorname{div}(v^k \otimes v^k) = -\nabla p^k + f^k \\ \operatorname{div} v^k = 0 \end{cases}$$

with $f^k \in L^1_{\mathrm{loc}}$, $f^k \to f$ weakly in L^1_{loc}. Assume there exists a strong concentration set $E \subset \mathbb{R}^2 \times [0, T]$ for the reduced defect measure θ which has the form of (13) where $\alpha \geqq 2/3$. Then v is a weak solution to the two dimensional Euler equation:

$$\begin{cases} v_t + \operatorname{div}(v \otimes v) = -\nabla p + f \\ \operatorname{div} v = 0. \end{cases}$$

Proof. We shall mollify $\Phi(t) = (\Phi_1(t), \Phi_2(t))$ by a standard mollifier:

$$m_\varepsilon(s) = \frac{1}{\varepsilon} m\left(\frac{s}{\varepsilon}\right), \quad m \in C_c^\infty(-1, 1), \quad 0 \leqq m \leqq 1, \quad \int_{-\infty}^{+\infty} m(s)\,ds = 1.$$

Extend Φ by constant outside $[0, T]$ so that it is continuously defined on $\mathbb{R}$. For $\varepsilon > 0$, let

$$\Phi^\varepsilon(t) = m_\varepsilon * \Phi.$$

It is easy to check that

(i) $\Phi^\varepsilon(t) \in C^\infty[0, T]$,
(ii) $|\Phi^\varepsilon(t) - \Phi(t)| \leqq C\varepsilon^\alpha$, $t \in [0, T]$,
(iii) $|d\Phi^\varepsilon(t)/dt| \leqq C\varepsilon^{\alpha-1}$, $t \in [0, T]$.

a) Similar to the proof of Theorem 1, we shall use Radon transform. To show (9), we need only to show that (12) holds for all

$$\eta = \xi(x_1 + \gamma t)\psi(x_1, x_2, t), \tag{14}$$

where $\xi(s) \in C_c^\infty(\mathbb{R})$, $\gamma \in \mathbb{R}$ and $\psi \in C_c^\infty(\mathbb{R}^2 \times (0, T))$.

b) Take

$$V_\varepsilon = \{(x_1, t) \in \mathbb{R}^2 \mid |x_1 - \Phi_1^\varepsilon(t)| < 2C\varepsilon^\alpha\} \times \mathbb{R}^1_{x_2}$$

where C is the constant appearing in (ii). Then V_ε is open and

$$V_\varepsilon \supset E.$$

Choose a $\chi_\varepsilon(s) \in C^\infty(\mathbb{R})$ such that $0 \leqq \chi_\varepsilon(s) \leqq 1$ and

$$\chi_\varepsilon(s) = \begin{cases} 0 & \text{if} \quad |s| < 2C\varepsilon^\alpha \\ 1 & \text{if} \quad |s| > 3C\varepsilon^\alpha. \end{cases}$$

Set

$$\zeta_\varepsilon(x_1, t) = \int_{-\infty}^{x_1} \int_{-\infty}^{z} \xi''(s + \gamma t)\chi_\varepsilon(s - \Phi_1^\varepsilon(t))\,ds\,dz,$$

then $\zeta_\varepsilon(x_1, t) \in C^\infty$ and

$$\partial_{x_1}^2 \zeta_\varepsilon(x_1, t) = 0 \quad \text{on} \quad V_\varepsilon.$$

Finally take

$$\eta_\varepsilon(x_1, x_2, t) = \zeta_\varepsilon(x_1, t)\psi(x_1, x_2, t).$$

The idea is to put η_ε into (10), let $k \to +\infty$, then $\varepsilon \to 0+$, we will recover (12) for η of the form (14).

c) Now put $\eta_\varepsilon = \zeta_\varepsilon \psi \in C_c^\infty$ into (10) and let $k \to +\infty$. We find

$$\int \nabla^\perp \partial_t \eta^\varepsilon \cdot v\,dx\,dt + \int \psi(\partial_{x_1}^2 \zeta_\varepsilon) v_1 v_2\,dx\,dt + \tag{15}$$

$$+ \int (\nabla\psi \otimes \nabla^\perp \zeta_\varepsilon + \nabla\zeta_\varepsilon \otimes \nabla^\perp \psi + \zeta_\varepsilon \nabla\nabla^\perp \psi) : d\mu \tag{16}$$

$$= -\int \nabla^\perp \eta^\varepsilon \cdot f\,dx\,dt. \tag{17}$$

d) Let $\varepsilon \to 0+$. Notice

$$\begin{aligned} &\zeta_\varepsilon(x_1, t) \to \xi(x_1 + \gamma t) \text{ uniformly on spt}\,\psi, \\ &\partial_{x_1}\zeta_\varepsilon(x_1, t) \to \partial_{x_1}\xi(x_1 + \gamma t) \text{ uniformly on spt}\,\psi, \\ &\partial_{x_1}^2 \zeta_\varepsilon(x_1, t) \to \partial_{x_1}^2 \xi(x_1 + \gamma t),\ \forall t \text{ a.e. in } x, \\ &|\partial_{x_1}^2 \zeta_\varepsilon(x_1, t)| \leqq M,\ \forall \varepsilon > 0, x_1, t. \end{aligned}$$

We find that the second term in (15) and all terms in (16–17) go to the right limits. for the first term in (15), we have

$$\nabla^{\perp}\partial_t\eta^{\varepsilon}\cdot v = (\partial_{x_1}\partial_t\zeta_\varepsilon)v_2\psi + (\partial_t\zeta_\varepsilon)v_2\partial_{x_1}\psi - (\partial_t\zeta_\varepsilon)v_1\partial_{x_2}\psi \tag{18}$$

$$+ (\partial_{x_1}\zeta_\varepsilon)v_2\partial_t\psi + \zeta_\varepsilon v_2\partial_{x_1}\partial_t\psi - \zeta_\varepsilon v_1\partial_{x_2}\partial_t\psi. \tag{19}$$

The three terms in (19) go to the right limits. To find the limits for the second and third terms in (18), we carry out the differentiation in $\partial_t\zeta_\varepsilon$, then use $\partial_s\chi_\varepsilon(s) = -\partial_s(1-\chi_\varepsilon(s))$ and integration by parts to turn the differentiation on $\partial_s(1-\chi_\varepsilon(s))$ onto $\partial^2_{x_1}\xi$, to find

$$\partial_t\zeta_\varepsilon = \gamma \int_{-\infty}^{x_1}\int_{-\infty}^{z} \xi'''(s+\gamma t)\chi_\varepsilon(s-\Phi_1^\varepsilon(t))ds\,dz \tag{20}$$

$$+ \Phi_1^{\varepsilon\prime}(t)\left\{\int_{-\infty}^{x_1} \xi''(z+\gamma t)[1-\chi_\varepsilon(z-\Phi_1^\varepsilon(t)]dz\right. \tag{21}$$

$$\left.-\int_{-\infty}^{x_1}\int_{-\infty}^{z} \xi'''(s+\gamma t)[1-\chi_\varepsilon(s-\Phi_1^\varepsilon(t)]ds\,dz\right\}. \tag{22}$$

Using (ii) and (iii), we find that both (21) and (22) are of order $\varepsilon^{2\alpha-1}$, and the left-hand side of (20) equals $\partial_t\xi(x_1+\gamma t) + O(\varepsilon^\alpha)$. Therefore,

$$\partial_t\zeta_\varepsilon = \partial_t\xi(x_1+\gamma t) + O(\varepsilon^\alpha + \varepsilon^{2\alpha-1}).$$

So the second and the third terms in (18) go to the right limits also.

For the first term in (15), we notice that

$$\begin{aligned}\partial_t\partial_{x_1}\zeta_\varepsilon &= \gamma\int_{-\infty}^{x_1} \xi'''(z+\gamma t)\chi_\varepsilon(z-\Phi_1^\varepsilon(t))dz + \Phi_1^{\varepsilon\prime}(t)\left\{\xi''(x_1+\gamma t)[1-\chi_\varepsilon(x_1-\Phi_1^\varepsilon(t)]\right.\\ &\quad\left.-\int_{-\infty}^{x_1}\xi'''(z+\gamma t)[1-\chi_\varepsilon(z-\Phi_1^\varepsilon(t)]dz\right\}\\ &= \partial_t\partial_{x_1}\xi + O(\varepsilon^\alpha) + \Phi_1^{\varepsilon\prime}(t)\xi''(x_1+\gamma t)[1-\chi_\varepsilon(x_1-\Phi_1^\varepsilon(t)] + O(\varepsilon^{2\alpha-1}).\end{aligned}$$

Thus, the first term in (15) goes to the right limit if we can show that

$$I_\varepsilon \equiv \int_0^T\int_{\mathbb{R}^2} \Phi_1^{\varepsilon\prime}(t)\xi''(x_1+\gamma t)[1-\chi_\varepsilon(x_1-\Phi_1^\varepsilon(t)]v_2\psi\,dx\,dt \to 0.$$

In fact, if we denote

$$U_\varepsilon \equiv \{(x_1,t)\,|\,|x_1-\Phi_1^\varepsilon(t)| < 3C\varepsilon^\alpha\}\times\mathbb{R}^1_{x_2},$$

then

$$\begin{aligned}|I_\varepsilon| &\leqq C\varepsilon^{\alpha-1}\int_0^T\int_{\mathbb{R}^2}[1-\chi_\varepsilon(x_1-\Phi_1^\varepsilon(t)]|v_2|\,dx\,dt\\ &\leqq C\varepsilon^{\alpha-1}\int_{U_\varepsilon^c}|v_2|\,dx\,dt\\ &\leqq C\varepsilon^{\alpha-1}\varepsilon^{\alpha/2}\left(\int_{U_\varepsilon^c}|v_2|^2\,dx\,dt\right)^{1/2}\end{aligned}$$

$$\leq C \| v_2 \|_{L^2(U_\varepsilon^c)}$$
$$\to 0$$

as $\varepsilon \to 0$. Therefore (12) holds for the choice of η in (14). $\square$

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