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Quantization of the Poisson $SU(2)$ and Its Poisson Homogeneous Space – The 2-Sphere*

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Abstract. We show that deformation quantizations of the Poisson structures on the Poisson Lie group $SU(2)$ and its homogeneous space, the 2-sphere, are compatible with Woronowicz's deformation quantization of $SU(2)$'s group structure and Podles' deformation quantization of 2-sphere's homogeneous structure, respectively. So in a certain sense the multiplicativity of the Lie Poisson structure on $SU(2)$ at the classical level is preserved under quantization.

Introduction

In the area of quantization of (symplectic manifolds), there have been two major approaches, namely geometric quantization and deformation quantization. In this paper, we shall work with the second approach, which seems to be more realistic physically, although the first approach is mathematically beautiful and intriguing. In the seventies, Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer first formalized the concept of deformation quantization of the symplectic (or Poisson) structure of a manifold in terms of formal power series [Ba–Fl–Fr–Li–St]. Since then there has been a lot of research in this direction. Recently, Marc A. Rieffel formulated such a theory in the context of C^*-algebras and obtained interesting results [Ri1, 2, 3]. From a certain point of view, this formulation has the advantage of being closer to the traditional way of quantization using operators on Hilbert spaces.

Parallel to the above quantization of geometric structures, there is a theory of deformation quantization of group structures, namely the theory of quantum groups [Dr]. Also recently, S. L. Woronowicz developed such a theory in the

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context of Hopf C^*-algebras [Wo1, 2]. In particular, he studied $S_\mu U(2)$, "the twisted $SU(2)$" or the deformed Lie group $SU(2)$, in great detail [Wo1]. Meanwhile, P. Podles constructed and studied the "quantum sphere" $S^2_{\mu c}$ as the twisted homogeneous 2-sphere of $S_\mu U(2)$ [Po].

What makes the whole situation more interesting is the discovery of "multiplicative" Poisson structures on $SU(n)$ (and more general Lie groups) and corresponding Poisson structures on homogeneous spheres by J. H. Lu and A. Weinstein [Lu–We1, 2].

A natural question now arises [Lu–We1, Ri2], namely, whether there are deformation quantizations (in the sense of Rieffel) of the Poisson structures on $SU(2)$ and S^2 which are "compatible" with Woronowicz's deformation quantization of $SU(2)$'s group structure and Podles' deformation quantization of 2-sphere's homogeneous structure. In this paper we shall give a positive answer to the above question. This suggests in a sense that the multiplicativity of the Poisson structure on $SU(2)$ on the classical level is preserved on the quantum level under deformation quantization.

Section 1. The Quantum $SU(2)$ and Quantum 2-Sphere

Let us first recall the idea of quantum $SU(2)$ of Woronowicz. In [Wo1], Woronowicz showed that the universal C^*-algebras $C(S_\mu U(2))$ generated by α and γ satisfying the following equations:

$$\begin{aligned} \alpha\alpha^* + \mu^2\gamma\gamma^* = 1, \quad & \mu\gamma\alpha = \alpha\gamma, \quad \gamma\gamma^* = \gamma^*\gamma, \\ \alpha^*\alpha + \gamma\gamma^* = 1, \quad & \mu\gamma^*\alpha = \alpha\gamma^*, \end{aligned} \tag{1.1}$$

with $-1 \leqq \mu \leqq 1$, are examples of certain Hopf C^*-algebras, called compact matrix pseudogroups [Wo2], and in a sense form a deformation of the classical Hopf algebra of continuous functions on the Lie group $SU(2)$ since $C(S_1 U(2)) \cong C(SU(2))$. In other words, the corresponding underlying "pseudogroups" form a "noncommutative" deformation of the topological group structure of $SU(2)$. In the appendix of [Wo1], it was shown that the $C(S_\mu U(2))$'s are isomorphic as C^*-algebras for all $|\mu| < 1$ and that the structure of the C^*-algebra $C(S_\mu U(2))$ is described as an extension [Do] of $C(\mathbb{T})$ by $C(\mathbb{T}) \otimes \mathscr{K}$, i.e. there is a short exact sequence of C^*-algebras,

$$0 \to C(\mathbb{T}) \otimes \mathscr{K} \to C(S_\mu U(2)) \to C(\mathbb{T}) \to 0.$$

It was M. A. Rieffel who first observed that this structure of $C(S_\mu U(2))$ coincides with that of the unitized group C^*-algebra [Pe] $C^*(SB(2, \mathbb{C}))^+$ of the solvable Lie group $SB(2, \mathbb{C})$, which is the dual group of $SU(2)$ in the sense of [Dr, Lu–We1]. This discovery has encouraged the investigation which led to the results of this paper, since Rieffel [Ri1] has also related the group C^*-algebra of such (and more general) groups with the deformation quantization of linear Poisson structures on their Lie algebra duals.

Before we turn to the quantum 2-sphere, we will describe the structure of $C(S_\mu U(2))$ in a more constructive way to be used later. Recall that the C^*-algebra $C^*(\mathscr{S})$ generated by the unilateral shift $\mathscr{S}$, a bounded linear operator defined by $\mathscr{S}(e_n) = e_{n+1}$ for an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of a Hilbert space $\mathscr{H}$, is an extension of $C(\mathbb{T})$ by $\mathcal{K}$ [Col], i.e. there is a short exact sequence

$$0 \to \mathcal{K} \to C^*(\mathcal{S}) \xrightarrow{\sigma} C(\mathbb{T}) \to 0,$$

where σ is the symbol map if we identify $C^*(\mathcal{S})$ with the Toeplitz C^*-algebra on the circle $\mathbb{T}$ [Do]. We shall recall some equivalent realizations of $C^*(\mathcal{S})$ as Toeplitz algebras at the end of this section. Note that $\sigma(\mathcal{S})$ is the identity function $\mathbf{z}$ on $\mathbb{T}$.

For various reasons (as we shall see later in this paper), it is suggestive to regard $C^*(\mathcal{S})$ as a "non-commutative" (cf. [Con1, 2]) unit disc $\bar{D}$ with boundary, or more precisely, a foliation C^*-algebra of the singular foliation (cf. [Da, Sh1]) of $\bar{D}$ with the interior D as one leaf and each point on $\mathbb{T} = \partial D$ as a leaf. The interior corresponds to the ideal $\mathcal{K}$ of $C^*(\mathcal{S})$, while the boundary corresponds to the quotient $C(\mathbb{T})$. With this concept in mind, we introduce another "non-commutative" singular foliation, namely

$$(\mathrm{id} \otimes \sigma)^{-1}(1 \otimes C(\mathbb{T})) = \{f \in C(\mathbb{T}, C^*(\mathcal{S})) \mid \sigma(f(z)) = \sigma(f(1)) \text{ for all } z \text{ in } \mathbb{T}\},$$

where $\mathrm{id} \otimes \sigma : C(\mathbb{T}, C^*(\mathcal{S})) = C(\mathbb{T}) \otimes C^*(\mathcal{S}) \to C(\mathbb{T}) \otimes C(\mathbb{T})$ with id the identify map of $C(\mathbb{T})$. This algebra is in some sense the fibered product of a circle family of $C^*(\mathcal{S})$ over $C(\mathbb{T})$, or the pull-back of a circle family of the same symbol map $\sigma : C^*(\mathcal{S}) \to C(\mathbb{T})$. Conceptually, the corresponding singular foliation is the one constructed by glueing a circle family of disc leaves along their boundary circles to one circle consisting of point leaves.

During the preparation of this paper, we learned that the following result may also be obtained by combining the results of [Va–So] and [Ro]. But we shall present a proof based on the universality of $C(S_\mu U(2))$ and $C^*(\mathcal{S})$ [Col].

Proposition 1.1. *For $|\mu| < 1$, we have, as C^*-algebras,*

$$C(S_\mu U(2)) \cong (\mathrm{id} \otimes \sigma)^{-1}(1 \otimes C(\mathbb{T})).$$

Proof. As we mentioned above, all $C(S_\mu U(2))$'s with $|\mu| < 1$ are isomorphic as C^*-algebras. So we shall assume that $\mu = 0$ in the following discussion.

Let us denote the algebra on the right-hand side of the above identity by $\mathcal{A}$. Then $\mathcal{A}$ contains two distinguished elements $\tilde{\alpha} = 1 \otimes \mathcal{S}^*$ and $\tilde{\gamma} = \mathbf{z} \otimes p$, where $p = I - \mathcal{S}\mathcal{S}^*$ is a rank 1 projection with $\sigma(p) = 0$ and $\mathbf{z}$ is the identity map on $\mathbb{T}$. It is easy to see that $\mathcal{A}$ is an extension of $C(\mathbb{T})$ by $C(\mathbb{T}) \otimes \mathcal{K}$, i.e. there is a short exact sequence

$$0 \to C(\mathbb{T}) \otimes \mathcal{K} \to \mathcal{A} \xrightarrow{\tilde{\sigma}} C(\mathbb{T}) \to 0,$$

where $\tilde{\sigma}(f) := \sigma(f(1))$ for $f \in \mathcal{A}$. Moreover $\mathcal{A}$ is generated by $\tilde{\alpha}$ and $\tilde{\gamma}$ since $\tilde{\sigma}(\tilde{\alpha}) = \bar{\mathbf{z}}$ generates $C(\mathbb{T})$, while $\tilde{\gamma}$ generates $C(\mathbb{T}) \otimes p$, and hence $C(\mathbb{T}) \otimes \mathcal{K}$ is contained in the C^*-algebra $C^*(\tilde{\alpha}, \tilde{\gamma})$ generated by $\tilde{\alpha}$ and $\tilde{\gamma}$.

It is easy to see that $\tilde{\alpha}$ and $\tilde{\gamma}$ satisfy the above relations (with $\mu = 0$) defining $C(S_0 U(2))$. We claim that they are actually universal with respect to those relations. More precisely, if α and γ satisfy (1.1), we prove that there is a homomorphism ϕ from $\mathcal{A}$ to $C^*(\alpha, \gamma)$ sending $\tilde{\alpha}$ to α and $\tilde{\gamma}$ to γ. Note that the identity $\alpha\alpha^* = 1$ of (1.1) implies that α^* is an isometry and $1 - \alpha^*\alpha$ is a projection (i.e. a self-adjoint idempotent).

In fact, if α is unitary then $\gamma = 0$ and it is well known that by functional calculus there is a homomorphism h from $C(\mathbb{T})$ to $C^*(\alpha)=C^*(\alpha,\gamma)$ sending $\mathbf{z}$ to α. So clearly we can take $\phi:=h\circ\tilde{\sigma}$.

If α is not unitary, then regarding $C^*(\alpha,\gamma)$ as a self-adjoint subalgebra of $B(\mathscr{H})$, the algebra of bounded linear operators on a Hilbert space $\mathscr{H}$, we have $q\gamma q=\gamma$, where $q=I-\alpha^*\alpha$ is the orthogonal projection onto the kernel of α. So by the well known result of [Ha] about decomposing an isometry into a direct sum of a block unilateral shift and a unitary (sometimes called the Wold decomposition) and by the universality of $C^*(\mathscr{S})$ [Co1], we get

$$C^*(\alpha,\gamma)\cong C^*(\{1\otimes\mathscr{S},q\gamma q\otimes p\})\subseteqq C^*(q\gamma q)\otimes C^*(\mathscr{S})\cong C^*(\gamma)\otimes C^*(\mathscr{S})$$

identifying α^* with $1\otimes\mathscr{S}$ and γ with $q\gamma q\otimes p=\gamma\otimes p$. Note that $1=q$ in $C^*(q\gamma q)\cong C^*(\gamma)$, since γ is a unitary on the range of $q=I-\alpha^*\alpha$. Now clearly there is a homomorphism from $C(\mathbb{T})$ to $C^*(\gamma)$ sending z to γ and hence a homomorphism from $\mathscr{A}$ to $C^*(\alpha,\gamma)$ sending $\tilde{\alpha}=1\otimes\mathscr{S}^*$ and $\tilde{\gamma}=\mathbf{z}\otimes p$ to α and γ respectively. Thus $\mathscr{A}$ with $\tilde{\alpha}$ and $\tilde{\gamma}$ is universal with respect to the relations (1.1) (with $\mu=0$) and hence $\mathscr{A}\cong C(S_0U(2))$. Q.E.D

Now we recall Podles' results on the quantum spheres. In [Po], a family of C^*-algebras $C(S^2_{\mu c})$ are constructed and shown to be co-modules of the Hopf C^*-algebras $C(S_\mu U(2))$, and hence the corresponding imaginary "spaces" are regarded as quantum (or pseudo-) homogeneous spaces of the quantum (or pseudo-) groups $S_\mu U(2)$. Here we only consider $S^2_{\mu c}$ with $|\mu|\leqq 1$ and $c\geqq 0$. Note that $C(S^2_{1c})\cong C(S^2)$. Recall that $C(S^2_{\mu c})$ is a unital C^*-algebra generated by two elements A and B (and 1) [Po].

Using the classification of all irreducible representations of $C(S^2_{\mu c})$ (as C^*-algebras) in [Po], we can get an explicit description of the C^*-algebra structure of $C(S^2_{\mu c})$ as follows.

Proposition 1.2. *Let $|\mu|<1$. (1) If $c>0$, then $C(S^2_{\mu c})$ is the pull-back of two copies of the symbol map of $C^*(\mathscr{S})$, namely,*

$$C(S^2_{\mu c})\cong C^*(\mathscr{S})\oplus_\sigma C^*(\mathscr{S}):=\{(x,y)\mid x,y\in C^*(\mathscr{S})\text{ and }\sigma(x)=\sigma(y)\},$$

and so there is a short exact sequence

$$0\to\mathscr{K}\oplus\mathscr{K}\to C(S^2_{\mu c})\to C(\mathbb{T})\to 0.$$

(2) If $c=0$, then $C(S^2_{\mu c})\cong\tilde{\mathscr{K}}$ is the unitization of $\mathscr{K}$ and so we have a short exact sequence

$$0\to\mathscr{K}\to C(S^2_{\mu 0})\to\mathbb{C}\to 0.$$

Proof. (1) It is well known that the direct sum of all irreducible representations of a C^*-algebra is faithful [Pe]. So by Proposition 4(1) of [Po], we get a faithful representation

$$\pi:=\pi_+\oplus\pi_-\oplus\sum_{z\in\mathbb{T}}^{\oplus}\pi_z$$

of $C(S^2_{\mu c})$, where π_z's are one-dimensional representations sending A to 0 and B to $c^{1/2}z$, while $\pi_\pm$ are represented on separable (i.e. countably infinite dimensional) Hilbert spaces so that $\pi_\pm(A)$ are "weight operators" and $\pi_\pm(B)$ are weighted shifts as described in the following. With respect to a suitably chosen orthonormal basis

$\{e_n\}$, $n=0,1,2,\ldots$,

$$\pi_\pm(A)(e_n)=\lambda_\pm\mu^{2n}e_n \quad\text{and}\quad \pi_\pm(B)(e_n)=c_\pm(n)^{1/2}e_{n-1},$$

where

$$c_\pm(n)=\lambda_\pm\mu^{2n}-(\lambda_\pm\mu^{2n})^2+c \quad\text{and}\quad \lambda_\pm=(1/2)\pm(c+(1/4))^{1/2}$$

with $e_{-1}:=0$. One can easily check that $\lim_n(c_\pm(n))=c$, $\lambda_+\geqq 1$ and $\lambda_-<0$. So we get

$$(\pi_\pm(B)-c^{1/2}\mathscr{S}^*)(e_n)=(c_\pm(n)^{1/2}-c^{1/2})e_{n-1}$$

with $\lim_n[(c_\pm(n)^{1/2}-c^{1/2})]=0$ and hence $\pi_\pm(B)-c^{1/2}\mathscr{S}^*$ is a compact operator in $\mathscr{K}$. On the other hand, since $|\mu|<1$ and hence $\lim_n(\lambda_\pm\mu^{2n})=0$, we get $\pi_\pm(A)$ also a compact operator. Thus we have $\pi_\pm(C(S^2_{\mu c}))\subseteqq C^*(\mathscr{S})$ and $\sigma(\pi_\pm(A))=0$ while

$$\sigma(\pi_\pm(B))=c^{1/2}\sigma(\mathscr{S}^*)=c^{1/2}\bar{\mathbf{z}}.$$

Now it is easy to see that π_z factors through $\pi_\pm$, namely, $\pi_z=\phi_z\circ\sigma\circ\pi_\pm$, where $\phi_z:C(\mathbb{T})\to\mathbb{C}$ is the evaluation at $\bar{z}\in\mathbb{T}$. From this, we get that $\pi_+\oplus\pi_-$ is also a faithful representation of $C(S^2_{\mu c})$.

It is clear that $(\pi_+\oplus\pi_-)(C(S^2_{\mu c}))\subseteqq C^*(\mathscr{S})\oplus_\sigma C^*(\mathscr{S})$. So it remains to prove the inverse inclusion. Note that $(\pi_+\oplus\pi_-)(A)$ is a self adjoint operator with $e_n\oplus 0$ and $0\oplus e_m$ as eigenvectors of distinct eigenvalues $\lambda_+\mu^{2n}$ and $\lambda_-\mu^{2m}$ respectively, and hence the (orthogonal) projection to each one-dimensional space spanned by $e_n\oplus 0$ or $0\oplus e_m$ is in the C^*-algebra $(\pi_+\oplus\pi_-)(C(S^2_{\mu c}))$ by functional calculus. Now from this and the fact that $\pi_\pm(B)$ are weighted shifts with nonvanishing weights, it is easy to see that the matrix elements $\varepsilon_{ij}\oplus 0$ and $0\oplus\varepsilon_{ij}$ for all $i,j\geqq 0$, which generate $\mathscr{K}\oplus\mathscr{K}$, are in the C^*-algebra $(\pi_+\oplus\pi_-)(C(S^2_{\mu c}))$, where $\varepsilon_{ij}(e_k)=\delta_{jk}e_i$ for all k. So we get $\mathscr{K}\oplus\mathscr{K}\subseteqq(\pi_+\oplus\pi_-)(C(S^2_{\mu c}))\subseteqq C^*(\mathscr{S})\oplus_\sigma C^*(\mathscr{S})$ such that $\tilde{\sigma}[(\pi_+\oplus\pi_-)(C(S^2_{\mu c}))]=C(\mathbb{T})$, where $\tilde{\sigma}(x,y):=\sigma(x)=\sigma(y)$ for (x,y) in $C^*(\mathscr{S})\oplus_\sigma C^*(\mathscr{S})$. This shows that $(\pi_+\oplus\pi_-)(C(S^2_{\mu c}))=C^*(\mathscr{S})\oplus_\sigma C^*(\mathscr{S})$.

(2) When $c=0$, the above description of irreducible representations is still valid except that we do not have the representation π_-. Now

$$\pi_+(A)(e_n)=\mu^{2n}e_n \quad\text{and}\quad \pi_+(B)(e_n)=c_+(n)^{1/2}e_{n-1},$$

where $c_+(n)=|\mu|^{2n}(1-\mu^{2n})$ converges to 0, while $\pi_z(A)=\pi_z(B)=0$. So $\pi_+(A)$ and $\pi_+(B)$ are compact operators and hence $\pi_+(C(S^2_{\mu c}))\subseteqq\tilde{\mathscr{K}}$ (remember that $C(S^2_{\mu c})$ contains the identity). Clearly π_z still factors through π_+ and so π_+ is faithful. Again, as in (1), the matrix units ε_{ij} are in $\pi_+(C(S^2_{\mu c}))$, so we get the conclusion.

Q.E.D

Remark. We learned that results similar to part (2) of the above proposition have also been obtained by Soibelman.

As in the case of $SU(2)$, we may regard $C(S^2_{\mu c})$ with $|\mu|<1$ as the foliation C^*-algebra of a singular foliation on S^2 constructed by glueing two disc leaves along their boundaries to a circle of point leaves if $c>0$, and by glueing a disc leaf along the boundary to a single point leaf if $c=0$.

In the rest of this section, we shall recall a few well known realizations of $C^*(\mathscr{S})$ as Toeplitz algebras.

Let $H^2(\partial D)$ be the Hardy space over the unit disc D in $\mathbb{C}$, i.e. the closure of the space of continuous functions on ∂D, which can be extended continuously to holomorphic functions on D, in the Hilbert space $L^2(\partial D)$ (with respect to the normalized arc length measure on $\mathbb{T} = \partial D$). Then it is easy to see that $\{\mathbf{z}^n\}_{n=0}^{\infty}$ form an orthonormal basis. Let P be the orthogonal projection from $L^2(\partial D)$ onto $H^2(\partial D)$. The Toeplitz C^*-algebra $\mathscr{T}(\partial D) = \mathscr{T}(\mathbb{T})$ is defined to be the C^*-algebra generated by the operators $T_\phi := PM_\phi$ restricted to $H^2(\partial D)$, where ϕ is a continuous function on $\partial D = \mathbb{T}$ and M_ϕ is the multiplication operator by ϕ on $L^2(\partial D)$. It is easy to see that the Toeplitz operator $T_{\mathbf{z}}$ on $H^2(\partial D)$ is simply a unilateral shift with respect to the orthonormal basis $\{\mathbf{z}^n\}$ and hence $\mathscr{T}(\partial D) \cong C^*(\mathscr{S})$. It is well known that the map sending T_ϕ to ϕ can be extended to a homomorphism σ, called the symbol map, from $\mathscr{T}(\partial D)$ onto $C(\partial D) = C(\mathbb{T})$.

Replacing $H^2(\partial D)$ by $H^2(D)$, the Bergman space over D, i.e. the Hilbert subspace of $L^2(D)$ consisting of holomorphic L^2-functions over D with the Lebesgue measure, and replacing $\phi \in C(\partial D)$ by $\phi \in C(\bar{D})$, we can repeat the above process to construct the Toeplitz algebra $\mathscr{T}(D)$ on $H^2(D)$. It is well known [Co2] that $\mathscr{T}(D) \cong \mathscr{T}(\partial D)$ (see [Sh2] for a more general result) and the corresponding symbol map on $\mathscr{T}(D)$ extends the map sending T_ϕ to $\phi|_{\partial D}$.

What we need in this paper is the third realization, which is less well known and can be derived easily from the work of Berger and Coburn [Be–Co1, 2] as follows. Let μ be the Gaussian measure on $\mathbb{C}$, i.e.

$$d\mu(z) = \exp(-|z|^2)\,dx\,dy \quad \text{for} \quad z = x + iy \text{ in } \mathbb{C}.$$

We consider the Segal–Bargmann space $H^2(\mathbb{C})$, i.e. the Hilbert subspace of $L^2(\mathbb{C}, \mu)$ consisting of holomorphic (μ-L^2-)functions on $\mathbb{C}$. As before, P denotes the orthogonal projection from $L^2(\mathbb{C}, \mu)$ onto $H^2(\mathbb{C})$. Then a Toeplitz operator T_ϕ with symbol $\phi \in L^\infty(\mathbb{C}, \mu)$ is defined as the restriction of PM_ϕ to $H^2(\mathbb{C})$. In [Be–Co1, 2], a through study has been done on the maximal possible symbol space $Q \subseteq L^\infty(\mathbb{C}, \mu)$ such that $\mathscr{T}(Q)/\mathscr{K}$ is commutative, where $\mathscr{T}(Q)$ is the C^*-algebra generated by T_ϕ with $\phi \in Q$. What we need in this paper is a much smaller symbol space $C(\bar{\mathbb{C}})$ consisting of continuous functions on $\mathbb{C}$ uniformly converging radially to a continuous function on $\mathbb{T}$, i.e. $\phi_r(z) := \phi(rz)$, $z \in \mathbb{T}$, converges uniformly to $\phi_\infty(z)$ as r goes to ∞ and ϕ_∞ is a continuous function on $\mathbb{T}$. By the results of [Be–Co2] that $\mathscr{K} = \mathscr{T}(C_0(\mathbb{C}))$ and that there is an isomorphism from $\mathscr{T}(Q)/\mathscr{K}$ to $C_b(\mathbb{C}) \cap ESV/C_0(\mathbb{C})$ identifying $[T_\phi]$ with $[\phi]$ for ϕ in $C_b(\mathbb{C}) \cap ESV$, where ESV is the space of functions which "oscillate" slowly at ∞, it is easy to deduce that

$$\mathscr{T}(\bar{\mathbb{C}})/\mathscr{K} \cong [C(\bar{\mathbb{C}}) \cap C_b(\mathbb{C}) \cap ESV]/[C(\bar{\mathbb{C}}) \cap C_0(\mathbb{C})] = C(\bar{\mathbb{C}})/C_0(\mathbb{C}) \cong C(\mathbb{T})$$

which identifies $[T_\phi]$ with ϕ_∞ for ϕ in $C(\bar{\mathbb{C}})$, where $\mathscr{T}(\bar{\mathbb{C}}) := \mathscr{T}(C(\bar{\mathbb{C}}))$. Thus we get a short exact sequence

$$0 \to \mathscr{K} \to \mathscr{T}(\bar{\mathbb{C}}) \xrightarrow{\sigma} C(\mathbb{T}) \to 0,$$

where σ sends T_ϕ to ϕ_∞. In order to see that $\mathscr{T}(\bar{\mathbb{C}}) \cong C^*(\mathscr{S})$, we only need to note that $T_{\mathbf{z}/|\mathbf{z}|} \in \mathscr{T}(\bar{\mathbb{C}})$ is a weighted shift (with respect to the orthonormal basis $\{\mathbf{z}^n/\|\mathbf{z}^n\|_2\}$) with positive weights α_n converging to 1 [Be–Co1] and hence $T_{\mathbf{z}/|\mathbf{z}|} - \mathscr{S}$ is compact. (Note that although the bounded function $\mathbf{z}/|\mathbf{z}|$ is not continuous at the origin, it can be checked that $T_{\mathbf{z}/|\mathbf{z}|}$ can be approximated by T_ϕ with $\phi \in C(\bar{\mathbb{C}})$ uniformly bounded and equal to $\mathbf{z}/|\mathbf{z}|$ except in shrinking neighborhoods of the origin.)

Section 2. Deformation Quantization of Poisson Structure

In this section, we shall recall Rieffel's strict deformation quantization of the Poisson structure on a manifold and introduce a variation of this notion.

Definition 2.1. [Rieffel] Given a Poisson manifold $(M, \{,\})$ and a *-subalgebra A of $C_b^\infty(M)$ containing $C_c^\infty(M)$ and closed under the Poisson bracket $\{,\}$, a family of normed involutive algebra structures $(A, \#_h, *h, \| \quad \|_h)$ on A, with $\#_h, *h$ and $\| \quad \|_h$ as the product, involution and norm respectively for all $0 \leqq h < \varepsilon$ and some $\varepsilon > 0$, is called a strict deformation quantization of the Poisson algebra A if $\| \quad \|_h$ is a C^*-norm for all h and $(A, \#_0, *_0, \| \quad \|_0)$ is the normed involutive algebra structure of A inherited from $C_b(M)$ (with multiplication, conjugation and sup-norm), such that

(1) $\|f\|_h$ is continuous in $h \in [0, \varepsilon)$ for any fixed $f \in A$,
(2) $\|(ih)^{-1}(f \#_h g - g \#_h f) - \{f, g\}\|_h$ converges to 0 as h goes to 0 for any fixed f and g in A.

We shall denote by A_h the C^*-algebra completion of the pre-C^*-algebra $(A, \#_h, *h, \| \quad \|_h)$.

Note that the above definition can be applied to any *-subalgebra A of $C_b(M)$ (containing $C_c^\infty(M)$), on which a Poisson bracket consistent with the original one on $C_c^\infty(M)$ is well defined.

Since every C^*-algebra can be realized as a closed *-subalgebra of bounded linear operators on some Hilbert space. We may regard $(A, \#_h, *h, \| \quad \|_h)$ as a *-subalgebra of operators and state the above definition in a way similar to the next definition (but slightly stronger). Note that conditions (1) and (2) in the above definition are the core idea of quantizing the Poisson bracket $\{,\}$ by deformation and we feel that the other conditions may be loosened slightly without losing the spirit of deformation quantization. In particular, we shall not require A to be closed under the product $\#_h$ and just consider it as a *-subspace of operators. So we introduce the following slightly weaker notion needed in stating the result of this paper.

Definition 2.2. Given a Poisson manifold $(M, \{,\})$ and a *-subalgebra A of $C_b^\infty(M)$ containing $C_c^\infty(M)$ and closed under the Poisson bracket $\{,\}$, a family of pairs (A, ρ_h) with $0 \leqq h < \varepsilon$ for some $\varepsilon > 0$ is called an operator deformation quantization of A if ρ_0 is the embedding of A into the commutative C^*-algebra $C_b(M)$ (realized as an algebra of operators through some faithful representation) and ρ_h is a *-preserving linear isomorphism from A onto a dense linear subspace of a *-subalgebra (or equivalently, of a C^*-subalgebra) of bounded linear operators on some Hilbert space for $0 < h < \varepsilon$, such that

(1) $\|\rho_h(f)\|$ is continuous in $h \in [0, \varepsilon)$ for any fixed $f \in A$,
(2) $\|(ih)^{-1}[\rho_h(f), \rho_h(g)] - \rho_h(\{f, g\})\|$ converges to 0 as h goes to 0 for any fixed f and g in A.

We shall use A_h to denote the C^*-algebra obtained by completing $\rho_h(A)$.

In [Ri1, 2], many interesting examples of strict deformation quantization have been found. What we need in this paper is a more classical result from the Weyl calculus. In the following, we summarize the relevant facts (extracted from [Vo]) and refer the interested reader to [Ho, Hö, Vo] for a detailed account and historical references.

We define the symbol space $\mathbf{S}^m$ to be the space of $a \in C^\infty(\mathbb{R}^2)$ with the property that there exists an asymptotic expansion

$$a(\xi, x) \sim \sum_{j=0}^{\infty} a_{m-j}(\xi, x)$$

in the sense that for all $n, \alpha, \beta \in \mathbb{N}$ there is a $C > 0$ such that

$$\left| D_\xi^\alpha D_x^\beta \left(a - \sum_{j=0}^{n-1} a_{m-j} \right)(\xi, x) \right| \leqq C(1 + \xi^2 + x^2)^{(m-n-\alpha-\beta)/2}$$

for all (ξ, x) outside a compact subset of $\mathbb{R}^2$, where a_k are homogeneous functions in (ξ, x) of degree k. It is easy to check that $\mathbf{S}^m$ is a Lie subalgebra of $C_b^\infty(\mathbb{R}^2)$, i.e. closed under the standard Poisson bracket $\{,\}$. On $\mathbf{S}^m$, we have a Fréchet space structure defined by the seminorms

$$\| a \|_k = \sup \{ | D_\xi^\alpha D_x^\beta a(\xi, x) | (1 + \xi^2 + x^2)^{(\alpha+\beta-m)/2} : (\xi, x) \in \mathbb{R}^2 \text{ and } \alpha + \beta \leqq k \}.$$

Given $a \in \mathbf{S}^m$, we define a (possibly unbounded) linear operator $W_h(a)$ on $L^2(\mathbb{R})$, for $h > 0$,

$$W_h(a)u(x) = (2\pi h)^{-1} \int a(\xi, (x + x')/2) \exp(i\xi(x - x')/h) u(x') d\xi dx'$$

for $u \in C_c^\infty(\mathbb{R})$. It is easy to check that $W_h(\bar{a}) = W_h(a)^*$. When $m \leqq 0$, $W_h(a)$ is bounded. We define the Moyal product (or twisted product) $a \#_h b$ by

$$W_h(a \#_h b) = W_h(a) W_h(b).$$

Then $\#_h$ defines a map from $\mathbf{S}^k \times \mathbf{S}^l$ to $\mathbf{S}^{k+l}$, and we have an asymptotic expansion for $\#_h$, namely

$$(a \#_h b)(\xi, x) \sim \sum_{k=0}^{\infty} (ih/2)^k (k!)^{-1} (D_\xi D_y - D_\eta D_x)^k |_{\eta = \xi, y = x} [a(\xi, x) b(\eta, y)],$$

and one for its commutator, namely

$$(a \#_h b - b \#_h a)(\xi, x) \sim \sum_{k=0}^{\infty} ih(-h^2/4)^k ((2k+1)!)^{-1} (D_\xi D_y - D_\eta D_x)^{2k+1} |_{\eta = \xi, y = x} [a(\xi, x) b(\eta, y)].$$

Note that for $h > 0$, the operators $W_h(a)$ and $W(a_h)$ have the same norm since they are unitarily equivalent, where $a_h(\xi, x) = a(\sqrt{h}\xi, \sqrt{h}x)$ and $W := W_1$. We set $W_0(a) = a \in C_b(\mathbb{R}^2)$ which can be represented faithfully as multiplication operators on $L^2(\mathbb{R}^2)$ and hence it is natural to define $a \#_0 b = ab$.

Although the enormous literature on the Weyl calculus seems to guarantee that W_h does form an operator deformation quantization of $\mathbf{S}^0$, we are unable to find theorems written in exactly the form we need and so we shall briefly describe how to verify the requirements using known results from the literature. (In fact, it is known that the Weyl correspondence does give rise to a strict deformation quantization of the Schwarz functions on $\mathbb{R}^2$ [Ri1], but for this bigger class $\mathbf{S}^0$ of symbols, the proof is trickier.)

First we point out that the proof of Theorem 3.1.3 in [Ho] shows that if $A \subseteqq \mathbf{S}^0$ is bounded with respect to the seminorm $\| \quad \|_4$ then there is a constant C such that for all $a \in A$ and $0 < h < 1$, $\| W_h(a) \| \leqq C \| a_h \|_4$. Clearly $A_\delta = \{a_h - a_t | 1 > t, h > \delta\}$ with $\delta > 0$ is such an example of A. For $h > 0$, we have a_t approximating a_h in $\| \quad \|_4$ when t converges to h, and so by the above remark, $W_t(a)$ converges to $W_h(a)$ in norm as t goes to h. Thus $\| W_h(a) \|$ is continuous in $h > 0$.

Before we prove the continuity of $\| W_h(a) \|$ at $h = 0$, we recall that [Gr–Lo–St] if we represent $a \in \mathbf{S}^0$ as an operator acting on $L^2(\mathbb{R}^2)$ (instead of on $L^2(\mathbb{R})$) by the twisted product $a \#_h \cdot$, we get a representation (of the algebra $(\mathbf{S}^0, \#_h)$) equivalent to W_h (or more precisely a countable direct sum of the representation W_h) and hence $\| W_h(a) \| = \| a \#_h \cdot \|$.

We shall first prove that $\| W_h(a) \|$ is upper semi-continuous at $h = 0$, i.e. $\operatorname*{lim\text{-}sup}_{h \to 0} \| W_h(a) \| \leqq \| W_0(a) \| = \| a \|_\infty$. By Proposition 4 of [Gr–Lo–St], the period 2 Fourier transform $\tilde{a}$ (and similarly $\tilde{a_h}$) of a (and a_h) used in [Ho] is a smooth distribution on $\mathbb{R}^2 \backslash \{0\}$ vanishing at infinity rapidly. Since $(\partial_x^4 + \partial_\xi^4) a \in \mathbf{S}^{-4} \subseteqq L^1(\mathbb{R}^2)$, we have

$$|\tilde{a_h}(\xi, x)| \leqq M h \pi^{-4} (x^4 + \xi^4)^{-1}$$

for $(\xi, x) \neq (0,0)$, where M is the L^1-norm of $(\partial_x^4 + \partial_\xi^4) a$. Let $1 \geqq \phi_r \geqq 0$ be a C^∞-function on $\mathbb{R}^2$ supported in $K_r := \{(\xi, x) | x^4 + \xi^4 < r\}$ and equal to 1 on $K_{r/2}$. Then

$$\| ((1 - \phi_r) \tilde{a_h})\check{\ } \|_\infty \leqq \| (1 - \phi_r) \tilde{a_h} \|_1 \leqq M_r h,$$

where M_r is the integral of $M \pi^{-4} (x^4 + \xi^4)^{-1}$ over $\mathbb{R}^2 \backslash K_{r/2}$. Thus from

$$a_h = (\phi_r \tilde{a_h})\check{\ } + ((1 - \phi_r) \tilde{a_h})\check{\ },$$

we easily get $\lim_{h \to 0} \| (\phi_r \tilde{a_h})\check{\ } \| = \| a \|_\infty$ because $\| a_h \|_\infty = \| a \|_\infty$. Since $\phi_r \tilde{a_h}$ is a compactly supported distribution with smooth Fourier transform, we get by the remark following Theorem 3.1.1 of [Ho] that

$$\| W((\phi_r \tilde{a_h})\check{\ }) \| \leqq (1 + c_0 \sqrt[4]{2r})^2 \| (\phi_r \tilde{a_h})\check{\ } \|_\infty$$

for some constant c_0 independent of r and h, since $\operatorname{supp}(\phi_r \tilde{a_h}) \subseteqq K_r \subseteqq B_{\sqrt[4]{2r}}$ the ball of radius $\sqrt[4]{2r}$. On the other hand, by identifying $\#_h$ with the twisted convolution [K, Ri1], one can derive

$$\| W(((1 - \phi_r) \tilde{a_h})\check{\ }) \| \leqq \| (1 - \phi_r) \tilde{a_h} \|_1 \leqq h M_r,$$

and so

$$\| W(a_h) \| \leqq (1 + c_0 \sqrt[4]{2r})^2 \| (\phi_r \tilde{a_h})\check{\ } \|_\infty + h M_r,$$

which shows that for any fixed $r > 0$, $\operatorname*{lim\text{-}sup}_{h \to 0} \| W(a_h) \| \leqq (1 + c_0 \sqrt[4]{2r})^2 \| a \|_\infty$. So we get

$$\operatorname*{lim\text{-}sup}_{h \to 0} \| W(a_h) \| \leqq \| a \|_\infty .$$

Next we shall prove the lower semi-continuity of $\| W_h(a) \|$ at $h = 0$. By setting $\mathbf{a}(h, \xi, x) = a(\xi, x)$ for $a \in \mathbf{S}^0$, we get an admissible symbol of [Vo]. By Theorem 3.4.1

of [Vo], for any $u, v \in C_c^\infty(\mathbb{R}^2)$, we have that

$$\{h^{-1}[(\mathbf{a}\#\mathbf{u})(h,\cdot,\cdot) - \mathbf{a}(h,\cdot,\cdot)\mathbf{u}(h,\cdot,\cdot)] \,|\, \varepsilon > h > 0\}$$

or equivalently

$$\{h^{-1}(a\#_h u - au) \,|\, \varepsilon > h > 0\}$$

is a bounded subset of $\mathbf{S}^{-2}$ if ε is sufficiently small. So

$$\langle a\#_h u - au, v\rangle_{L^2(\mathbb{R}^2)} = \int\int [(a\#_h u) - au](\xi, x)\overline{v(\xi, x)}\, d\xi\, dx$$

converges to 0 as h goes to 0 since v has compact support. From the earlier remarks, we know that $\| W_h(a)\|$ and hence $\| a\#_h \cdot \|$ is uniformly bounded for fixed $a \in \mathbf{S}^0$. By this fact and that

$$\langle a\#_h u, v\rangle \text{ converges to } \langle a\#_0 u, v\rangle = \langle au, v\rangle$$

for all $u, v \in C_c^\infty(\mathbb{R}^2) \subseteqq L^2(\mathbb{R}^2)$, it is an easy exercise to show that $a\#_h \cdot$ converges to the multiplication operator $a\cdot$ weakly and hence $\| W_h(a)\| = \| a\#_h \cdot \|$ is lower semi-continuous at $h = 0$. Now we have proved the continuity of $\| W_h(a)\|$ for $h \geqq 0$.

It remains to show that $(ih)^{-1}[W_h(a), W_h(b)] - W_h(\{a, b\})$ converges to 0 in norm as h goes to 0. By Theorem 3.4.1 (ii) of [Vo], for any $a, b \in \mathbf{S}^0$, the set

$$\{h^{-2}[(\mathbf{a}\#\mathbf{b} - \mathbf{b}\#\mathbf{a})(h,\cdot,\cdot) - ih\{\mathbf{a}(h,\cdot,\cdot), \mathbf{b}(h,\cdot,\cdot)\}] \,|\, \varepsilon > h > 0\}$$

or equivalently

$$\{h^{-2}(a\#_h b - b\#_h a - ih\{a, b\}) \,|\, \varepsilon > h > 0\}$$

is bounded in $\mathbf{S}^{-4}$ for ε sufficiently small. So there is $C > 0$ such that

$$\begin{aligned} &h^{-2} \| W_h(a) W_h(b) - W_h(b) W_h(a) - ih W_h(\{a, b\})\| \\ &\quad = h^{-2} \| W_h(a\#_h b - b\#_h a - ih\{a, b\})\| \leqq C, \end{aligned}$$

which is uniformly bounded for all $\varepsilon > h > 0$, and hence we get what we want. Let us remark that the same kind of argument used in this paragraph can be used to prove that $\| W_h(f) W_h(g) - W_h(fg)\| = \| W_h(f\#_h g - fg)\|$ converges to 0 as h goes to 0.

Thus we have proved the following theorem.

Theorem 2.1. $(\mathbf{S}^0, W_h)$ *is an operator deformation quantization of* $\mathbf{S}^0$ *and* $(\mathbf{S}^0, \#_h, {}^*h, \| \;\; \|_h)$ *is a strict deformation quantization of* $\mathbf{S}^0$ *over the symplectic manifold* $\mathbb{R}^2$, *where* ${}^*h$ *is just the ordinary complex conjugation of functions and* $\| a\|_h := \| W_h(a)\|$.

In the rest of this section, we shall relate the C^*-algebra $\mathscr{W}_h(\mathbb{C})$ generated by the Weyl operators $W_h(a)$ with $a \in \mathbf{S}^0$ to the Toeplitz algebra $\mathscr{T}(\bar{\mathbb{C}})$. We first state some known facts about Weyl operators and Toeplitz operators and refer the reader to the paper [Gu] of Guillemin and references there for details. Since $W_h(a(\cdot,\cdot)) = W_1(a(h\cdot,\cdot))$, we have $\mathscr{W}_h(\mathbb{C}) = \mathscr{W}_1(\mathbb{C})$ and so we shall work with $W := W_1$ only.

The fact we need from [Gu, Gu–St] is that

$$W(a) \equiv T_\phi \text{ mod smoothing operators } (\subseteqq \mathscr{K})$$

for $a \in \mathbf{S}^0$ with an asymptotic expansion

$$a(\xi, x) \sim \sum_{k=0}^{\infty} a_{-2k}(\xi, x)$$

and $\phi \in \mathbf{S}^0$ with an asymptotic expansion

$$\phi(\xi, x) \sim \sum_{n=0}^{\infty} \phi_{-2n}(\xi, x)$$

such that $\phi_{-2n} = \left[\sum_{k=0}^{n} (1/k!)(\Delta/2)^{n-k} a_{-2k} \right]$, where Δ is the Laplacian $\partial^2/\partial z \partial \bar{z}$. It is well known that Weyl operators (respectively Toeplitz operators) with rapidly decreasing (or even compactly supported) smooth symbols a (respectively ϕ) generate $\mathscr{K}$ as a C^*-algebra. On the other hand, Weyl operators and Toeplitz operators with symbols a and ϕ of order no greater than -1 are compact and hence can be approximated by the same kinds of operators with rapidly decreasing smooth symbols. From these facts, we get $\mathscr{W}(\mathbb{C}) = \mathscr{T}(\bar{\mathbb{C}})$ and that for $a \in \mathbf{S}^0$ with asymptotic expansion

$$a(\xi, x) \sim \sum_{k=0}^{\infty} a_{-k}(\xi, x),$$

with a_{-k} smooth and homogeneous of degree $-k$,

$$\sigma(W(a)) = a_0 \circ \mathbf{z}|_{\mathbb{T}} = \lim_{r \to \infty} a(r\mathbf{z}|_{\mathbb{T}}) = a_{\infty}.$$

We summarize in the following proposition.

Proposition 2.1. *$\mathscr{W}_h(\mathbb{C})$ and $\mathscr{T}(\bar{\mathbb{C}})$ are isomorphic as C^*-algebras for $h > 0$, and $\sigma(W_h(a)) = a(h\cdot, \cdot)_{\infty}$ for all $a \in \mathbf{S}^0$.*

Section 3. Deformation Quantization of Poisson $SU(2)$ and S^2

In this section, we shall construct operator deformation quantizations of smooth functions over the Poisson $SU(2)$ and Poisson spheres. We refer readers to [Lu–We1, 2] by Lu and Weinstein for the geometric properties of Poisson $SU(2)$ and Poisson spheres used in this section.

We recall that the singular foliation of Poisson $SU(2)$ by symplectic leaves consists of a circle family of open disks, each one symplectomorphic to the canonical symplectic $\mathbb{R}^2$, and a circle (namely, $U(1)$) of 0-dimensional symplectic leaves, to which the open disks are glued together along their boundaries. We can find a diffeomorphism $f: \mathbb{T} \times \mathbb{R}^2 \to SU(2) \backslash U(1)$ such that on each $\{z\} \times \mathbb{R}^2, f$ is a symplectomorphism onto a symplectic leaf of $SU(2)$, and $\lim_{r \to \infty} f(z, r\mathbf{e}(\theta)) = \mathbf{e}(\theta) \in U(1)$, where $\mathbf{e}(\theta) := \exp(2\pi i \theta)$ for $0 \leqq \theta \leqq 1$. (We shall freely consider $\mathbf{e}(\theta)$ to be in $\mathbb{T} \subseteqq \mathbb{C}$, in $\mathbb{R}$ or in $U(1) \subseteqq SU(2)$ according to the context.) In fact, we can fix a non-degenerate symplectic leaf L of $SU(2)$ and a symplectomorphism $g: \mathbb{R}^2 \to L$ (e.g. $g(r\mathbf{e}(\theta)) = (1 - R^2)^{1/2} \mathbf{e}(\theta) + R \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, where $R = \exp(-r^2/4)$) such that $\lim_{r \to \infty} g(r\mathbf{e}(\theta)) = \mathbf{e}(\theta) \in U(1)$, and define $f(z, \zeta) := z \cdot g(z^{-1}\zeta)$ for $z \in \mathbb{T} = U(1)$

and $\zeta \in \mathbb{C} = \mathbb{R}^2$. Then we have

$$f^*(C^\infty(SU(2))) \subseteqq f^*(C^\infty(SU(2)^\sim))$$
$$= \left\{ s \in C_b^\infty(\mathbb{T} \times \mathbb{R}^2) \mid s(z,\cdot) \sim \sum_{j=0}^{\infty} s_{-j}(z,\cdot) \text{ with } s_0 \text{ independent of } z \right\},$$

where the asymptotic expansion is in the sense that for all $n, \alpha, \beta \in \mathbb{N}$ there is a $C > 0$ such that

$$\left| D_\xi^\alpha D_x^\beta \left(s - \sum_{j=0}^{n-1} s_{-j} \right)(z,\xi,x) \right| \leqq C(1+\xi^2+x^2)^{(-n-\alpha-\beta)/2}$$

for all $z \in \mathbb{T}$ and $x, \xi \in \mathbb{R}$, with $s_k(z,\cdot)$ homogeneous functions in (ξ, x) of degree k (and smooth in z), and the subalgebra $C^\infty(SU(2)^\sim)$ of $C(SU(2))$ is defined by the last equality so that for any $a \in C^\infty(SU(2)^\sim)$, we have $(a \circ f(z,\cdot))_\infty = a|_{U(1)}$, i.e.

$$\lim_{r \to \infty} a \circ f(z, r\mathbf{e}(\theta)) = a(\mathbf{e}(\theta)).$$

It is easy to see that we can extend the Poisson bracket on $C^\infty(SU(2))$ to $C^\infty(SU(2)^\sim)$ by defining for any $a, b \in C^\infty(SU(2)^\sim)$,

$$(\{a,b\} \circ f)(z,\cdot) = \{a \circ f(z,\cdot), b \circ f(z,\cdot)\}_{\mathbb{R}^2}$$

and $\{a,b\}(p) = 0$ for $p \in U(1)$, since the original Poisson bracket on $C^\infty(SU(2))$ coincides with the bracket defined by the symplectic structure on each symplectic leaf (or more precisely, the Poisson 2-tensor of $SU(2)$ is "tangent" to the symplectic leaves) [We]. So we get a Poisson algebra $C^\infty(SU(2)^\sim)$ containing $C^\infty(SU(2))$ as a sub-Poisson algebra.

It is easy to see that for $a \in C^\infty(SU(2)^\sim)$, the set $\{a \circ f(z,\cdot) \mid z \in \mathbb{T}\}$ is bounded in $\mathbf{S}^0$, and hence by Theorem 3.1.3 of [Ho] (as used in Sect. 2), for any fixed $h > 0$, the map $\rho_h(a)$ sending $z \in \mathbb{T}$ to $(W_h(a \circ f(z,\cdot))) \in \mathscr{F}(\bar{\mathbb{C}})$ is (norm) continuous. Actually $\rho_h(a)$ belongs to $(\mathrm{id} \otimes \sigma)^{-1}(1 \otimes C(\mathbb{T})) \subseteqq C(\mathbb{T}) \otimes C^*(\mathscr{S})$, since

$$\sigma(W_h(a \circ f(z,\cdot))) = a|_{U(1)} \circ \tau_h$$

is independent of $z \in \mathbb{T}$, where τ_h is the map on $\mathbb{T}$ (the space of rays in $\mathbb{R}^2$) induced by the linear map on $\mathbb{R}^2$ sending (ξ, x) to $(h\xi, x)$.

We claim that ρ_h form an operator deformation quantization of $C^\infty(SU(2))$ (remember that any C^*-algebra can be realized as a *-algebra of operators) and give rise to a strict deformation quantization of $C^\infty(SU(2)^\sim)$. (Note that the essential difference here is that $\rho_h(C^\infty(SU(2)))$ is not a subalgebra of $(\mathrm{id} \otimes \sigma)^{-1}(1 \otimes C(\mathbb{T}))$, while obviously $\rho_h(C^\infty(SU(2)^\sim))$ is because $W_h(\mathbf{S}^0)$ is an algebra as discussed in Sect. 2.) The proof is similar to what we did for $\mathbf{S}^0$ in Sect. 2 and we only need to note that the argument used there is still valid when a smooth parameter $z \in \mathbb{T}$ of the symbol a is introduced. We shall omit this routine verification.

It is easy (using the partitions of unity on $\mathbb{T}$) to check that $C(\mathbb{T}) \otimes \mathscr{K} \subseteqq C^*(\{\rho_h(a) \mid a \in C_c^\infty(SU(2) \backslash U(1))\})$ for $h > 0$ since $W_h(C_c(\mathbb{R}^2))$ is dense in $\mathscr{K}$, and to construct an $a \in C^\infty(SU(2))$ with $a|_{U(1)}$ equal to any given $\phi \in C(\mathbb{T})$. So we get

$$C^*(\{\rho_h(a) \mid a \in C^\infty(SU(2))\}) = (\mathrm{id} \otimes \sigma)^{-1}(1 \otimes C(\mathbb{T})).$$

Thus for $1 \geqq h \geqq 0$ and $\mu = 1 - h$, by Proposition 1.1, we have

$$C^\infty(SU(2))_h = C^\infty(SU(2)^\sim)_h \cong C(S_\mu U(2)).$$

This shows, in a sense, that the deformation quantizations of the group structure and of the (multiplicative) Poisson structure on $SU(2)$ are consistent, and hence the group structure and the Poisson structure on $SU(2)$ are compatible on the quantum level as they are on the classical level.

We summarize in the following theorem.

Theorem 3.1. $(C^\infty(SU(2))), \rho_h)$ *form an operator deformation quantization of* $C^\infty(SU(2))$, *and* $(C^\infty(SU(2)^\sim), \#_h, *h, \|\ \|_h)$ *is a strict deformation quantization of* $C^\infty(SU(2)^\sim)$, *where* $\#_h$ *and* $\|\ \|_h$ *are the product and the norm pulled back from* $C(\mathbb{T})\otimes C^*(\mathscr{S})$ *through* ρ_h *respectively* (*for* $h>0$), *and* $*h$ *is the conjugation of functions. Moreover*

$$C^\infty(SU(2))_h = C^\infty(SU(2)^\sim)_h \cong C(S_\mu U(2))$$

for $1 \geqq h \geqq 0$ and $\mu = 1-h$.

From the results of Lu and Weinstein [Lu–We1,2], there is a family of "homogeneous" Poisson structures on the 2-sphere (with a suitable parameter c), and for $c>0$, the Poisson sphere $(S^2, \{,\}_c)$ consists of two disks, each one symplectically isomorphic to the canonical $\mathbb{R}^2$, glued along their boundaries to one circle of 0-dimensional leaves, while for $c=0$, the Poisson sphere degenerates to one disk leaf (symplectically isomorphic to $\mathbb{R}^2$) attached to one singleton leaf $\{n\}$. So using arguments similar to the above, by "glueing" together two copies of Weyl quantization of S^0, we can get, for $c>0$, operator deformation quantizations $\rho_{h,c}$ of $C^\infty(S^2)$ with respect to $\{,\}_c$ and the associated C^*-algebras $C^\infty(S^2)_{h,c}$ are isomorphic to $C(S^2_{\mu c})$, where $\mu = 1-h$. Similarly, when $c=0$, let $f:\mathbb{R}^2 \to S^2\backslash\{n\}$ be a symplectomorphism, then $\rho_{h,0}(a) := W_h(a\circ f)$ for $a \in C^\infty(S^2)$ defines an operator deformation quantization of $C^\infty(S^2)$ with respect to $\{,\}_0$ and $C^\infty(S^2)_{h,0} \cong C(S^2_{\mu 0})$ with $\mu = 1-h$. Since the detail is similar to the case of $SU(2)$ (but simpler), we shall omit it.

Appendix. Classification of $SU(2)$-Covariant Poisson Structures on S^2

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A Poisson Lie group is a Lie group G with a Poisson structure such that the multiplication map $G\times G \to G$ is a Poisson map. Let $\sigma: G\times P \to P$ be an action of a Poisson Lie group G on a Poisson manifold P with Poisson structure π_P. We say that π_P is G-invariant if for each $g\in G$, the map $\sigma_g \in \mathrm{Diff}(P)$ preserves π_P. We say that π_P is G-covariant if σ is a Poisson map, where $G\times P$ has the product Poisson structure. In this case, σ is called a Poisson action of G on P. The difference of two G-covariant Poisson structures on P is G-invariant, but not necessarily Poisson (see Theorem 2.6 in [Lu–We]).

Consider the Lie group

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

We choose the following basis

$$e_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad e_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

for the Lie algebra $su(2)$ of $SU(2)$ to identify $su(2) \cong \mathbb{R}^3$. Then the adjoint action of $SU(2)$ on $su(2)$ becomes the action by rotations. The isotropy subgroup of the point $e_1 = (1,0,0)$ is $S^1 = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}$, and its orbit is the unit sphere S^2 in $\mathbb{R}^3$. This way $S^2 \cong SU(2)/S^1$ becomes a homogeneous $SU(2)$-space, and the natural projection $p: SU(2) \to S^2$ is given by

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \mapsto (|\alpha|^2 - |\beta|^2, -i(\alpha\beta - \bar{\alpha}\bar{\beta}), -(\alpha\beta + \bar{\alpha}\bar{\beta})).$$

A Poisson structure on $SU(2)$ can be defined by [Lu–We]

$$\pi(g) = r_g \Lambda - l_g \Lambda, \quad g \in G,$$

where r_g and l_g respectively denote the right and left translations on G by g, as well as their differential maps extended to multivector fields, and $\Lambda = e_2 \wedge e_3 \in su(2) \wedge su(2)$. The Lie brackets of the functions $\alpha, \bar{\alpha}, \beta$ and $\bar{\beta}$ are given as follows:

$$\{\alpha, \bar{\alpha}\} = -i\beta\bar{\beta}, \quad \{\alpha, \beta\} = \frac{i}{2}\alpha\beta, \quad \{\alpha, \bar{\beta}\} = \frac{i}{2}\alpha\bar{\beta}, \quad \{\beta, \bar{\beta}\} = 0.$$

$SU(2)$ together with π becomes a Poisson Lie group. We will describe all $(SU(2), \pi)$-covariant Poisson structures on S^2.

One can check that π vanishes on the subgroup S^1; hence there is an induced Poisson structure π_1 on $SU(2)/S^1 \cong S^2$ such that the projection $p: SU(2) \to S^2$ is a Poisson map [Lu–We]. Moreover, the left action of $SU(2)$ on S^2 is a Poisson action. Therefore π_1 is one $SU(2)$-covariant Poisson structure on S^2. By using the explicit formula for p, we get the Lie brackets of the coordinate functions x_1, x_2 and x_3 on S^2 as follows:

$$\{x_1, x_2\} = (1 - x_1)x_3, \quad \{x_2, x_3\} = (1 - x_1)x_1, \quad \{x_3, x_1\} = (1 - x_1)x_2.$$

Notice that $\pi_1 = (1 - x_1)\pi_0$, where π_0 is the standard $SU(2)$-invariant Poisson (symplectic) structure on S^2, i.e.,

$$\{x_1, x_2\}_{\pi_0} = x_3, \quad \{x_2, x_3\}_{\pi_0} = x_1, \quad \{x_3, x_1\}_{\pi_0} = x_2.$$

In fact, π_1 can also be considered as an $SU(2)$-covariant Poisson structure on $\mathbf{R}^3 \cong su(2)^*$ with the unit 2-sphere as a Poisson submanifold.

Theorem. *Any $SU(2)$-covariant Poisson structure on S^2 is of the form $\pi_1 + c\pi_0$, where c is a constant.*

Proof. Let $\tilde{\pi}$ be any $SU(2)$-covariant Poisson structure on S^2. Then $\tilde{\pi} - \pi_1$ is $SU(2)$-invariant. Since π_0 is nondegenerate, there exists $f \in C^\infty(S^2)$ such that $\tilde{\pi} - \pi_1 = f\pi_0$. But f has to be $SU(2)$-invariant, whence f must be a constant c, and $\tilde{\pi} = \pi_1 + c\pi_0$.

Since $\pi_1 = (1 - x_1)\pi_0$, every $SU(2)$-covariant Poisson structure π on S^2 is of the form $(c - x_1)\pi_0$ for some constant c. When $c < -1$ or $c > 1$, π is nondegenerate and therefore symplectic. When $c = \pm 1$, π has one point and the remaining 2-cell as its symplectic leaves. When $-1 < c < 1$, the symplectic leaves of π are two open discs and the points of the circle separating them. The symplectic leaf space in each case coincides with the primitive ideal space of the sphere algebra of Podles [Po]. See also [Va–So].

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