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q-Weyl Group and a Multiplicative Formula for Universal R-Matrices*

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Abstract. We define the q-version of the Weyl group for quantized universal enveloping algebras of simple Lie group and we find explicit multiplicative formulas for the universal R-matrix.

1. For any semisimple complex Lie algebra $\mathscr{G}$ there is a natural deformation of its universal enveloping algebra $U\mathscr{G}$ as a Hopf algebra over the formal power series over **C** [D1, J]. This deformation $U_h\mathscr{G}$ is called a quantum universal enveloping algebra or quantum group [D1]. These algebras are important in the theory of quantum integrable systems [F] because with each $U_h\mathscr{G}$ one can associate a certain canonical element R in $(U_h\mathscr{G})^{\otimes 2}$ which satisfies the Yang-Baxter equation

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}.$$

Here $R_{ij}\in U_h\mathscr{G}^{\otimes 3}$, $R_{12}=R\otimes 1$, $R_{23}=1\otimes R$ and $R_{13}=\sum_i \alpha_i\otimes 1\otimes\beta_i$ if we rewrite R as $R=\sum_i \alpha_i\otimes\beta_i$, α_i, $\beta_i\in U_h\mathscr{G}$.

But up to now there was no explicit formula for R, except for the cases $g=sl_2$ [D1], $\mathscr{G}=sl_n$ [Ro2]. Drinfeld (private communication) conjectured that there is a relation between the Weyl group and the universal R-matrix for general simple Lie algebras. In this paper we define a completion $U_h\mathscr{G}$ by the Weyl elements of sl_2 triples corresponding to simple roots. This completion gives us a description of the quantum Weyl group as well as explicit formulas for the element R.

2. Let $\mathscr{G}$ be a semisimple Lie algebra of rank n, a_{ij} its Cartan matrix, and d_i the length of the i-th root (then $d_ia_{ij}=a_{ji}d_j$).

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Let h be a formal variable. For integers n and m we use the notations:

$$[n]_h = \frac{sh\left(\frac{nh}{2}\right)}{sh\left(\frac{h}{2}\right)}, \quad [n]_h! = [n]_h[n-1]_h \dots [1]_h,$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_h = \frac{[h]_h!}{[m]_h!\,[n-m]_h!}.$$

Following [D1, J] we consider an algebra $U_h\mathcal{G}$ over $\mathbf{C}[\![h]\!]$ with generators H_i, X_i, Y_i and relations:

$$[H_i, H_j] = 0, \quad [H_i, H_j] = a_{ij}X_j, \tag{1}$$

$$[H_i, Y_j] = -a_{ij}Y_j, \quad [X_i Y_j] = \delta_{ij}\frac{sh\left(\frac{d_jH_j}{2}\right)}{sh\left(\frac{hd_i}{2}\right)}\delta_{ij},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{d_ih} X_i^k X_j X_i^{1-a_{ij}-k} = 0, \quad i \neq j,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{d_ih} Y_i^k Y_j Y_i^{1-a_{ij}-k} = 0, \quad i \neq j.$$

This is a Hopf algebra with comultiplication $\Delta : U_n\mathcal{G} \to (U_h\mathcal{G})^{\otimes 2}$:

$$\Delta H_i = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta X_i = X_i \otimes e^{\frac{hH_id_i}{4}} + e^{-\frac{hH_id_i}{4}} \otimes X_i,$$

$$\Delta Y_i = Y_i \otimes e^{\frac{hH_id_i}{4}} + e^{-\frac{hH_id_i}{4}} \otimes Y_i.$$

An antipode S and counit ε is defined by the Hopf algebra axioms:

$$S(H_i) = -H_i, \quad S(X_i) = -e^{\frac{hd_i}{2}} X_i,$$

$$S(Y_i) = -e^{-\frac{hd_i}{2}} Y_i,$$

$$\varepsilon(H_i) = \varepsilon(Y_i) = \varepsilon(X_i) = 0.$$

In $U_h\mathcal{G}$ there are important Hopf subalgebras U_hb_+ generated by 1, H_i, X_i and U_hb_- generated by 1, H_i, Y_i. They are dual to each other over $\mathbf{C}[\![h^{-1}, h]\!]$ with respect to the pairing

$$\langle H_i, H_j \rangle = \frac{2}{h} d_i a_{ij}, \quad \langle X_i, Y_j \rangle = \delta_{ij}(1 - e^{-hd_i})^{-1}, \tag{2}$$

defined on the generators. The pairing between other elements can be found from the Hopf algebra structure on $U_hb_\pm$,

$$\langle a \otimes b, \Delta(c) \rangle = \langle ba, c \rangle, a, b \in U_hb_+, c \in U_hb_-,$$

$$\langle \Delta a, c \otimes b \rangle = \langle a, cb \rangle, a \in U_hb_+, b, c \in U_hb_-.$$

The algebras $U_h\mathscr{G}$ are quasitriangular Hopf algebras, i.e. for each $\mathscr{G}$ there exists an element R belonging to an appropriate completion of $(U_h\mathscr{G})^{\otimes 2}$ in h-adic topology satisfying the relations:

$$\Delta'(a)=R\Delta(a)R^{-1}, \quad (\Delta\otimes \mathrm{id})R=R_{13}R_{23}, \quad (\mathrm{id}\otimes\Delta)R=R_{13}R_{12}.$$

From the description of $U_h\mathscr{G}$ as a double of U_hb_+ it follows that this element is unique and it is the canonical element under the pairing (2) between U_hb_+ and U_hb_-. The first coefficient in the expansion of R in powers of X_i, $Y_i(B_{ij}=d_ia_{ij})$ has the form

$$\begin{aligned} R&=\exp\left(\frac{h}{2}(B^{-1})_{ij}H_i\otimes H\right)\\ &\quad\times\left(1+\sum_{i=1}^{n}2sh\left(\frac{hd_i}{2}\right)e^{-\frac{hd_i}{2}}e^{\frac{hH_i}{4}}X_i\otimes e^{-\frac{hH_i}{4}}Y_i+\dots\right)\\ &=\left(1+\sum_{i=1}^{n}2sh\left(\frac{hd_i}{2}\right)e^{-\frac{hd_i}{2}}e^{\frac{-hd_iH_i}{4}}X_i\otimes e^{\frac{hH_id_i}{4}}Y_i+\dots\right)\\ &\quad\times\exp\left(\frac{h}{2}(B^{-1})_{ij}H_i\otimes H_j\right). \end{aligned}$$

For any Hopf algebra A one can define the adjoint action of A on itself by

$$a\circ b=\sum_i a^ibS(a_i), \tag{3}$$

where a^i and a_i are the components of $\Delta(a)$: $\Delta(a)=\sum_i a^i\otimes a_i$. The action

$$a\bullet b=S^{-1}(a\circ S(b))=\sum_i a_ibS^{-1}(a^i)$$

defines another adjoint action on A is not equivalent to (3) for noncommutative Hopf algebras. For $A=U_h\mathscr{G}$ we have

$$H_i\circ a=[H_i,a], \tag{4}$$

$$X_i\circ a=X_ia\exp\left(\frac{hH_id_i}{4}\right)-e^{\frac{hd_i}{2}}\exp\left(-\frac{hH_id_i}{4}\right)aX_i, \tag{5}$$

$$Y_i\circ a=Y_ia\exp\left(\frac{hH_id_i}{4}\right)-e^{-\frac{hd_i}{2}}\exp\left(-\frac{hH_id_i}{4}\right)aY_i. \tag{6}$$

Remark. Let $\mathscr{G}_h$ be the minimal nontrivial orbit in $U_h\mathscr{G}$ under the adjoint action (4–6). Because $\mathscr{G}_h$ is an irreducible representation of $U_h\mathscr{G}$ and at $h=0$ this is the adjoint representation of $\mathscr{G}$, we have $\dim\mathscr{G}_h=\dim\mathscr{G}$ [L, Ro1]. Fix e_i a linear basis in $\mathscr{G}_h$, then the action of these elements on itself defines the quantum version of Lie brackets on $\mathscr{G}$.

In quasitriangular Hopf algebras an important role is played by the element

$$u=\sum_i S(\beta_i)\alpha_i,$$

where α_i and β_i are coordinates of the element R: $R=\sum_i \alpha_i\otimes\beta_i$. One can show [D2] that

$$S^2(a)=uau^{-1}$$

and for $U_h\mathscr{G}$ we have

$$v=u\exp\left(-\frac{hH_\varrho}{2}\right)\in\text{center of } U_h\mathscr{G}.$$

Here H_ϱ is an element corresponding to the half of the sum of positive roots in Cartan subalgebra $U(H) \subset U_h\mathcal{G}$ generated by elements H_i, $i=1,\dots n$.

3. According to the decomposition (3) let us introduce regular generators on $U_h\mathcal{G}$:

$$E_i = e^{\frac{hd_iH_i}{4}} X_i, \quad F_i = e^{-\frac{hd_iH_i}{4}} Y_i,$$
$$\bar{E}_i = e^{-\frac{hd_iH_i}{4}} X_i, \quad \bar{F}_i = e^{\frac{hd_iH_i}{4}} Y_i.$$

Proposition 1. 1. *The maps* φ *and* $\bar{\varphi}$

$$\varphi(H_i) = H_i, \quad \varphi(X_i) = E_i, \quad \varphi(Y_i) = F_i,$$
$$\Psi(H_i) = H_i, \quad \Psi(X_i) = \bar{E}_i, \quad \Psi(Y_i) = \bar{F}_i$$

preserve the relations (2).

2. $$E_i \bar{F}_j = q_i^{\frac{a_{ij}}{2}} \bar{F}_j E_i, \quad \bar{E}_i F_j = q_i^{\frac{a_{ij}}{2}} F_j \bar{E}_i,$$

where $q_i = \exp(hd_i)$.

Let us define now the q-commutator as

$$[A, B]_q = ABq - BAq^{-1}.$$

Proposition 2.

$$(F_i)^n \circ F_j = q_i^{1/4(na_{ij}+n(n-1))} \left[F_i, \dots \left[F_i, [F_i, F_j]_{q_i^{\frac{a_{ij}}{4}}}\right]_{q_i^{\frac{a_{ij}+2}{4}}}\right]_{q_i^{\frac{a_{ij}+2(n-1)}{4}}}$$
$$(\bar{E}_i)^n \circ \bar{E}_j = q_i^{1/4(na_{ij}+n(n-1))} \left[\bar{E}_i, \dots \left[\bar{E}_i, [\bar{E}_i, \bar{E}_j]_{q_i^{\frac{a_{ij}}{4}}}\right]_{q_i^{\frac{a_{ij}+2}{4}}}\right]_{q_i^{\frac{a_{ij}+2n-2}{4}}}$$

The proof follows from (7) by induction in n

Proposition 3. *The q-Serre relations* (2) *are equivalent to the following ones:*

$$(F_i)^{-a_{ij}+1} \circ F_j = 0, \quad (\bar{E}_i)^{-a_{ij}+1} \circ \bar{E}_j = 0.$$

The adjoint action of regular generators has the following form:

$$\begin{aligned} \bar{E}_i \circ b &= \bar{E}_i b - K_i^{-2} b K_i^2 \bar{E}_i, \\ \bar{F}_i \circ b &= (\bar{F}_i b - b\bar{F}_i) K_i^{-2}, \\ E_i \circ b &= (E_i b - bE_i) K_i^{-2}, \\ F_i \circ b &= F_i b - K_i^{-2} b K_i^2 F_i. \end{aligned} \tag{7}$$

Representations of $U_h\mathcal{G}$ are isomorphic as a linear spaces to corresponding representations $U\mathcal{G}$. If V^λ is a representation of $U_h\mathcal{G}$ with highest weight λ, then

$$vV^\lambda = \exp(-h(\lambda \mid \lambda + 2\varrho)) V^\lambda.$$

4. Let $\mathcal{G} = sl_2$. An irreducible finite dimensional representation V^j of $U_h sl_2$ is parametrised by half integers $j = 0, \frac{1}{2}, 1, \dots$. The action of generators H, X, Y, in the weight basis e_m^j, $m = -j, -j+1, \dots j$ of the space V^j has the following form:

$$He_m^j = me_m^j, \quad Xe_m^j = \sqrt{[j-m][j+m+1]}\, e_{m+1}^j,$$
$$Ye_m^j = \sqrt{[j+m][j-m+1]}\, e_{m-1}^j.$$

The universal R-matrix for $U_h sl_2$ has the following form

$$R=R(H,X,Y\,|\,h)=\exp\left(\frac{h}{2}H\otimes H\right)\sum_{n\geqq 0}\frac{(1-q^{-1})^n}{[n]_h!}q^{\frac{n(n-1)}{4}}(e^{\frac{hH}{4}}X)^n\otimes(e^{-\frac{hH}{4}}Y)^n$$
$$=\left(\sum_{n\geqq 0}\frac{(1-q^{-1})^n}{[n]_h!}q^{\frac{n(n-1)}{4}}(e^{-\frac{hH}{4}}X)^n\otimes(e^{\frac{hH}{4}}Y)^n\right)\exp\left(\frac{h}{4}H\otimes H\right). \tag{8}$$

It is easy to check that this is the canonical element in $U_hb_+\otimes U_hb_-$ with pairing (2). The algebra U_hsl_2 can be completed by the element w, defined in each irreducible representation as

$$we^i_m=(-1)^{j-m}e^{-h\frac{j(j+1)}{2}+\frac{mh}{2}}e^j_{-m}. \tag{9}$$

Let us denote this completion by $\overline{U_hsl_2}$.

Theorem [KR].

1. *The element w satisfies the relation*

$$wXw^{-1}=-q^{1/2}Y,\quad wYw^{-1}=-q^{-\frac{1}{2}}X,\quad wHw^{-1}=-H. \tag{10}$$

2. *$\overline{U_hsl_2}$ is a Hopf algebra with*

$$\Delta w=R^{-1}w\otimes w,\quad s(w)=we^{\frac{hH}{2}},\quad \varepsilon(w)=1,$$

where R is the universal R-matrix for U_hsl_2.

3. *Let $u=\sum_i S(\beta_i)\alpha_i$ be the element describing the square of the antipode, then*

$$w^2=v\varepsilon=u\varepsilon^{\frac{hH}{2}}\varepsilon,$$

where ε is the unipotent central element $\varepsilon^2=1$, $\varepsilon V^j=(-1)^{2j}V^j$.

The element w has another interesting interpretation [VS] in representation theory of dual Hopf algebra to U_hsl_2.

5. In each $U_h\mathscr{G}$ module we can define the action of the Weyl elements of sl_2 – triples corresponding to simple roots of $\mathscr{G}$. Because $U_h\mathscr{G}$ is a semisimple algebra it is enough to define the action of $\check{w}_i$ in irreducible representations. Let $V^\lambda=\overset{\oplus}{j}(W^\lambda_j\otimes V^j)$ be the decomposition of V^λ into irreducible $(U_hsl_2)_i$ submodules. Define the action of w_i in V^λ as $w_i=\overset{\oplus}{j}(I^\lambda_{w_j}\otimes(w_i)_j)$, where $(w_i)_j$ is the action of $\check{w}$ in V^j, (see (9)).

Let us denote the algebra $U_h\mathscr{G}$ extended by w_i, $i=1,\dots,$ rank $\mathscr{G}$ as $\overline{U_h\mathscr{G}}$. The definition of w_i implies the following relations in $U_h\mathscr{G}$:

$$w_iH_jw_i^{-1}=H_j-a_{ij}H_i,\quad w_iX_iw_i^{-1}=-Y_iq_i^{1/2},\quad w_iY_iw_i^{-1}=-X_iq_i^{-1/2}. \tag{11}$$

also,

$$\Delta w_i=R(i)^{-1}w_i\otimes w_i,$$

where $R(i)\equiv R(H_i,X_i,Y_i|hd_i)$ and $R(H,X,Y|h)$ is defined by (8).

Theorem 1. *The following relations hold in the algebra* $\bar{U}_h\mathscr{G}$:

$$w_i\bar{E}_jK_i^{a_{ij}}w_i^{-1}=(-1)^{a_{ij}}q^{\frac{a_{ij}}{4}-\frac{a_{ij}(a_{ij}-2)}{8}}\frac{1}{[-a_{ij}]_{hd_i}!}(\bar{E}_i)^{-a_{ij}}\circ\bar{E}_j, \tag{Ad1}$$

$$w_iS(F_j)K_i^{-a_{ij}}w_i^{-1}=q_i^{-\frac{a_{ij}}{4}-\frac{a_{ij}(a_{ij}-2)}{8}}\frac{1}{[-a_{ij}]_{hd_i}!}S((F_i)^{-a_{ij}}\circ F_j). \tag{Ad2}$$

Proof. Let us first prove two auxiliary lemmas.

Lemma 1.

$$w_i\circ F_j=S(w_i)^{-1}K_i^{a_{ij}}F_jS(w_i),$$
$$w_i\circ\bar{E}_j=w_i\bar{E}_jK_i^{a_{ij}}w_i^{-1}.$$

Proof. Let α_k and β_k be coordinates of $R(H_i,X_i,Y_i\,|\,hd_i)=\sum_k\alpha_k\otimes\beta_k$,

$$\begin{aligned}
w_i\circ\bar{E}_j&=\sum_k S(\alpha_k)w_i\bar{E}_jS(w_i)S(\beta_k)=\sum_k\alpha_kw_i\bar{E}_jS(w_i)\beta_k\\
&=\sum_{n,m\geqq 0}a_m\frac{\left(\frac{hd_i}{4}\right)^n}{n!}w_i(\bar{F}_i)^mH_i^n\bar{E}_j(\bar{E}_i)^mH_i^nq_i^mS(w_i)\\
&=\sum_{n,m\geqq 0}a_m\frac{\left(\frac{hd_i}{4}\right)^n}{n!}w_i\bar{E}_j\bar{F}_i^m(H_i+a_{ij})^n\bar{E}_i^mH_i^nq_i^mS(w_i)\\
&=w_i\bar{E}_j\sum_k\beta_kS^2(\alpha_k)K_i^{a_{ij}}S(w_i)=w_i\bar{E}_jK_i^{a_{ij}}u_i^{-1}S(w_i)\\
&=w_i\bar{E}_jK_i^{a_{ij}}w_i^{-1}.
\end{aligned}$$

The similar calculations give us the action of w_i on F_j:

$$\begin{aligned}
w_i\circ F_j&=\sum_k S(\alpha_k)w_iF_jS(w_i)S(\beta_k)\\
&=\sum_{n,m\geqq 0}a_m\frac{\left(\frac{h_id_i}{4}\right)^n}{n!}H_i^nE_i^nw_iF_jS(w_i)H_i^nF_i^n\\
&=\sum_{n,m}a_m\frac{(hd_i)}{n!}w_iH_i^nF_i^mq_i^mF_jH_i^nE_i^mS(w_i)\\
&=\sum_{n,m\geqq 0}a_m\frac{\left(\frac{hd_i}{4}\right)}{n!}w_iF_i^m(H_i-2m)^n(H_i+a_{ij})^nE_i^mq_i^mF_jS(w_i)\\
&=w_i\sum_{m\geqq 0}a_m\exp\left(\frac{hd_i}{4}(H_i^2+2mH_i+H_ia_{ij}0)\right)F_i^mE_i^mq_i^mF_jS(w_i)\\
&=w_iK_i^{a_{ij}}\sum_k\beta_kS^2(\alpha_k)F_jS(w_i)=w_iw^{-1}K_i^{a_{ij}}F_jS(w_i)\\
&=S(w_i)^{-1}K_i^{a_{ij}}F_jS(w_i).
\end{aligned}$$

Lemma 2. *The linear spaces* $V_{ij}=\{(F_i)^n\circ F_j\}_{n=0}^{-a_{ij}}$, $\bar V_{ij}=\{(\bar E_i)^n\circ \bar E_j\}_{n=0}^{-a_{ij}}$ *are irreducible* $(U_h sl_2)_i$ *modules with highest weight* $-a_{ij}$.

Proof. From relations (1) and from Proposition 1 we obtain the following structure of the adjoint action of $(U_h sl_2)_i$ in these spaces:

$$\begin{aligned}
F_i\circ(F_i^n\circ F_j)&=F_i^{n+1}\circ F_j,\\
E_i\circ((F_i)^n\circ F_j)&=[-a_{ij}+1-n]_{hd_i}[n]_{hd_i}F_i^{n-1}\circ F_j,\\
H_i\circ(F_i^nF_j)&=(-a_{ij}-2n)F_i^n\circ F_j,\\
\bar E_i\circ(\bar E_i^n\circ \bar E_j)&=E_i^{n+1}\circ \bar E_j,\\
\bar F_i\circ(\bar E_i^n\circ \bar E_j)&=[-a_{ij}+1-n]_{hd_i}[n]_{hd_i}\bar E_i^{n-1}\circ \bar E_j,\\
H_i\circ(\bar E_i^n\circ \bar F_j)&=(a_{ij}+2n)\bar E_i^n\circ \bar F_j.
\end{aligned}$$

The maps

$$\sigma(F_i^n\circ F_j)=\sqrt{\frac{[n]_{hd_i}!}{[-a_{ij}-n]_{hd_i}!}}\; e^{-\frac{a_{ij}}{s}}_{\frac{a_{ij}}{2}-n},$$

$$\tau(\bar E_i^n\circ \bar E_j)=\sqrt{\frac{[-a_{ij}-n]_{hd_i}!}{[n]_{hd_i}!}}\; e^{-\frac{a_{ij}}{2}}_{-\frac{a_{ij}}{2}+n}$$

obviously define an isomorphism between V_{ij}, $\bar V_{ij}$, and $V^{-a_{ij}}$.

Now, to prove Theorem 1 let us combine these two lemmas with the explicit action of the Weyl element for $U_h sl_2$ and we immediately obtain relations (Ad1, Ad2).

Theorem 2. *The elements* w_i *satisfy the following relations:*

$$\begin{aligned}
w_iw_jw_i&=w_jw_iw_j, & a_{ij}&=-1,\\
w_iw_jw_iw_j&=w_jw_iw_jw_i, & a_{ij}&=-2,\\
w_iw_jw_iw_jw_i&=w_jw_iw_jw_iw_j, & a_{ij}&=-3.
\end{aligned}\tag{12}$$

To prove this theorem it is sufficient to consider only rank $\mathscr{G}=2$ cases. From the relations (Ad1, Ad2) it follows that the left-hand side and right-hand side parts of (12) can differ only by a central element (in the appropriate rank 2 algebra, A_2 for $a_{ij}=-1$, B_2 for $a_{ij}=-2$, G_2 for $a_{ij}=-3$). Acting by left-hand side and right-hand side parts on the h.w. vector we immediately obtain that this central element is unit.

The following two lemmas are useful for simplification of formulas (Ad1, Ad2).

Lemma 3.

$$\bar E_i^n\circ \bar E_j=K_i^{-n}K_j^{-1}\Big[X_i,\dots[X_i,X_j]_{q_i^{\frac{a_{ij}}{4}}}\dots\Big]_{q_i^{\frac{a_{ij}+2n-2}{4}}},$$

$$F_i^n\circ F_j=K_i^{-n}K_j^{-1}\Big[Y_i,\dots[Y_i,Y_j]_{q_i^{-\frac{a_{ij}}{4}}}\dots\Big]_{q_i^{-\frac{a_{ij}+2n-2}{4}}}.$$

Lemma 4.

$$S([Y_i,\dots,[Y_i,Y_j]_{q^{-n}}]_{q^{-n+2}}\dots]_{q^{n-2}})=-q_i^{-n/2}q_j^{-1/2}[Y_i,\dots[Y_iY_j]_{q^n}]_{q^{n-2}}\dots]_{q^{-n+2}}.$$

Now, we can rewrite relations (Ad1, Ad2) in the following more explicit form:

$$w_iX_jw_i^{-1}=(-1)^{a_{ij}}q^{\frac{a_{ij}}{8}+\frac{a_{ij}}{2}}\frac{1}{[a_{-ij}]_{hd_i}!}\left[\left[X_i,\dots,[X_i,X_j]_{q_i^{\frac{a_{ij}}{4}}}\right]_{q_i^{\frac{a_{ij}+2}{4}}}\right]_{q_i^{\frac{-a_{ij}-2}{4}}}K_i^{a_{ij}},$$

$$w_iY_jw_i^{-1}=q_i^{-\frac{a_{ij}}{8}-\frac{a_{ij}}{2}}\frac{1}{[-a_{ij}]_{hd_i}!}\left[\left[Y_i,\dots,[Y_i,Y_j]_{q_i^{\frac{a_{ij}}{4}}}\right]_{q_i^{\frac{a_{ij}+2}{4}}}\dots\right]_{q_i^{\frac{-a_{ij}-2}{4}}}K_i^{-a_{ij}},$$

6. Consider elements

$$w_i=\check{w}_iq_i^{\frac{H_i^2}{8}}$$

and define automorphisms

$$T_i(a)=\check{w}_i^{-1}a\check{w}_i.$$

From the relations between w_i and generators of $U_h\mathscr{G}$ we obtain

$$\begin{aligned}
&T_i(K_j)=K_jK_i^{-a_{ij}},\quad T_i(X_i)=Y_iK_i^{-2},\quad T_i(Y_i)=-K_i^2X_i,\\
&T_i(X_j)=(-1)^{a_{ij}}\frac{1}{[-a_{ij}]!}\left[\left[X_i,\dots[X_i,X_j]_{q_i^{\frac{a_{ij}}{4}}}\right]_{q_i^{\frac{a_{ij}+2}{4}}}\dots\right]_{q_i^{\frac{-a_{ij}-2}{4}}},\\
&T_i(Y_j)=\frac{1}{[-a_{ij}]!}\left[\left[Y_i,\dots[Y_i,Y_j]_{q_i^{\frac{a_{ij}}{4}}}\right]_{q_i^{\frac{a_{ij}+2}{4}}}\dots\right]_{q_i^{\frac{-a_{ij}-2}{4}}},
\end{aligned}\tag{13}$$

which coincides with Lusztig's automorphisms [L].

Lemma 5. *The elements $\check{w}_i$ satisfy the Weyl group relations:*

$$\underbrace{\check{w}_i\check{w}_j\check{w}_i\dots}_{-a_{ij}+2}=\underbrace{\check{w}_j\check{w}_i\check{w}_j\dots}_{-a_{ij}+2}.$$

It follows from Theorem 2 and relations (11).

7. From the definition of $\check{w}_i$ we obtain the action of the comultiplication on the elements $\check{w}_i$:

$$\Delta\check{w}_i=\tilde{R}^{-1}(i)\check{w}_i\otimes\check{w}_i,$$

where

$$\tilde{R}(i)=\sum_{n\geq 0}\frac{(1-q_i^{-1})^n}{[n]_{hd_i}!}q_i^{\frac{n(n-1)}{4}}E_i^n\otimes F_i^n.$$

Let $s_0=s_{i_1}\dots s_{i_k}$ be a decomposition of the element of Weyl group with maximal length in the minimal product of elementary reflections.

From relation Lemma 5 follows that the element

$$\check{w}_0=\check{w}_{i_1}\dots\check{w}_{i_k}$$

is well defined and does not depend on the choice of decomposition of s_0.

Theorem 3. *The universal R-matrix for $U_h\mathcal{G}$ has the following form:*

$$R=\exp\left(\frac{h}{2}\sum_{ij=1}^{n}(B^{-1})_{ij}H_i\otimes H_j\right)(\check{w}_0\otimes\check{w}_0)\Delta(\check{w}_0)^{-1}$$

or

$$R=\exp\left(\frac{h}{2}\sum_{ij=1}^{n}(B^{-1})_{ij}H_i\otimes H_j\right), \quad \tilde{R}(i_k\,|\,s_{i_1}\dots s_{i_{k-1}})\dots\tilde{R}(i_2\,|\,s_{i_1})\tilde{R}(i_1), \tag{14}$$

where

$$\tilde{R}(i_l\,|\,s_{i_1}\dots s_{i_{l-1}})=(T_{i_1}^{-1}\otimes T_{i_1}^{-1})\dots(T_{i_{l-1}}^{-1}\otimes T_{i_{l-1}}^{-1})\tilde{R}(i_l)$$

and T_i are the authomorphisms in (14).

To prove this theorem it is convenient to introduce the following enumeration of positive roots. Let $s_0=s_{i_1}\dots s_{i_k}$ be the decomposition of the maximal element of the Weyl group. The set of positive roots Δ_+ can be considered as a set of roots α_{i_1}, $s_{i_1}\alpha_{i_2},\dots, s_{i_1}\dots s_{i_{k-1}}\alpha_{i_k}$ [B, L]. According to this enumeration introduce elements

$$E(p)=T_{i_1}^{-1}\dots T_{i_{p-1}}^{-1}E_{ip}, \quad F(p)=T_{i_1}^{-1}\dots T_{i_{p-1}}^{-1}F_{i_p}.$$

From relations in $U_h\mathcal{G}$ it follows (see [L] for details) that the elements

$$H_1^{m_1}\dots H_n^{m_n} \quad E(1)^{n_1}\dots E(k)^{b_k}, \tag{15}$$

$$(H_1^v)^{m_1}\dots(H_n^v)^{m_n} \quad F(1)^{n_1}\dots F(k)^{n_k}, \tag{16}$$

where

$$H_i^v=\frac{h}{2}\sum_j(B^{-1})_{ij}H_j$$

form the bases in U_hb_+ and U_ab_- respectively.

Lemma 6. *With respect to the pairing* (2) *we have:*

$$\langle E(s),F(t)\rangle=\delta_{st}(1-e^{-hd_{i_s}})^{-1}. \tag{17}$$

It can be derived from the pairing (2) and from the definition of $E(p)$, $F(p)$. From the formula for the action of comultiplication on $\check{w}_i$ and from the definition of T_i it follows

$$\Delta(T_i^{-1}(a))=\tilde{R}(i)^{-1}((T_i^{-1}\otimes T_i^{-1})\Delta(s))\tilde{R}(i).$$

This formula gives us the action of comultiplication on elements $E(i)$.

Lemma 7. *Bases* (16) *and* (17) *are dual with respect to the pairing* (2) *between U_hb_+ and U_hb_-:*

$$\langle H_1^{m_1}\dots H_n^{m_n}E(1)^{n_1}\dots E(k)^{n_k},(H_1^v)^{m'_1}\dots(H_n^v)^{m'_n}F(1)^{n'_1}\dots F(k)^{n'_k}\rangle$$
$$=\prod_{j=1}^{n}\delta_{m_jm'_j}m_j!\prod_{p=1}^{k}\delta_{n_pn'_p}\frac{[n_p]_{hd_{ip}}!}{(1-e^{-hd_{i_p}})^{n_p}}e^{-\frac{hn_p(n_p-1)}{4}d_{i_p}}.$$

The proof follows from the lemma and formula (18).

So for the canonical element R we have the representation (15).
8. Let us describe more precisely authomorphisms T_i as an authomorphism of Hopf algebras.

Theorem 4. *Let z be an invertible element of the quasitriangular Hopf algebra A. Then the triple $(A, \Delta^{(z)}, R^{(z)})$, where*

$$\Delta^{(z)}(a) = (z \otimes z)\Delta(z^{-1}az)z^{-1} \otimes z^{-1},$$
$$R^{(z)} = z^{-1} \otimes z^{-1} R z \otimes z$$

also forms a quasitriangular Hopf algebra.

Proof. Associativity of $\Delta^{(z)}$ is a consequence of the following equalities:

$$(\Delta^{(z)} \otimes \mathrm{id})\Delta^{(z)}(a) = (z \otimes z \otimes z)(\Delta \otimes \mathrm{id})\Delta(a) z^{-1} \otimes z^{-1} \otimes z^{-1},$$
$$(\mathrm{id} \otimes \Delta^{(z)})\Delta^{(z)}(a) = (z \otimes z \otimes z)(\mathrm{id} \otimes \Delta)\Delta(a)(z^{-1} \otimes z^{-1} \otimes z^{-1}).$$

From the definition of $R^{(z)}$ we have the relation

$$\Delta^{(z)}(a)' = R^{(z)} \Delta^{(z)}(a) R^{(z)-1}.$$

The quasitriangular relations also follow from the structure of $R^{(z)}$ and from quasitriangularity of A.

Consider $z = \check{w}_{i_1}^{-1} \ldots \check{w}_{i_{k-1}}^{-1} \equiv \check{w}$ and denote the corresponding Hopf algebra structure on $U_y\mathscr{G}$ by $(U_h\mathscr{G})_w$. As an algebra this is $U_h\mathscr{G}$ but the comultiplication now differs from the previous one for $U_h\mathscr{G}$ and has the form:

$$\Delta^{(w)}(a) = (T_w \otimes T_w)(\Delta(T_w^{-1}(a))),$$

where $T_w(a) = \check{w} a \check{w}^{-1}$.

So, we see that automorphisms T_i are not automorphisms of $U_h\mathscr{G}$ as a Hopf algebra, $T_i^{-1} : (U_h\mathscr{G})_{\check{w}} \to (U_h\mathscr{G})_{\check{w}_i \check{w}}$. But they are automorphisms of the Hopf algebra $U_h\mathscr{G}$ in the sense of the Theorem 4.

9. *Remark 1.* The same construction gives us the quantum version of a Weyl group for Kac-Moody algebras. The relations (14) are still true.

Remark 2. Elements $\check{w}_{i_1} \ldots \check{w}$ describes irreducible representations of the quantized algebra of algebraic functions over G [S]. The multiplicative formula for the R-matrix together with the construction of the dual double given in [RST] make explicit the way for a description of cell decomposition of $\mathbf{C}_q(G)$.

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