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Unitary Dressing Transformations and Exponential Decay Below Threshold for Quantum Spin Systems. Parts III and IV

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Abstract. This is the continuation of a series of articles concerning a class of quantum spin systems with Hamiltonian operators of the form

$$H_\lambda = \sum_{x\in\Lambda} s_x + \sum_{\gamma_0\subset\Lambda} \lambda^{|\gamma_0|_c - 1} t_{\gamma_0},$$

where Λ is a graph, λ is a small parameter and s_x has a gap $\geqq 1$ for all $x\in\Lambda\backslash\mathcal{S}$. In the singular set $\mathcal{S}\subset\Lambda$, the gap of s_x can be arbitrarily small. Part III is devoted to the proof of a preliminary result, while in Part IV we consider the case in which $\mathcal{S}$ is a subset of finite density of Λ. This completes the first iteration step of the deterministic part of the proof of localization in the ground state of the random field quantum XY model.

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Part III. A New Representation with the Ground State of Compact Support

7. Introduction, Notations and Results

In this part, I consider the same model studied in Part II of [1], but we make two additional hypotheses. The Hamiltonian operator is

$$H_\lambda = \sum_{x \in \Lambda} s_x + \sum_{\gamma_0 \subset \Lambda} \lambda^{|\gamma_0|_c - 1} t_{\gamma_0}, \tag{7.1}$$

where the operators s_x have a gap $\geqq 1$ between the ground state and the first excited state for all $x \in \Lambda \backslash \mathscr{S}$. However, as in Part II, such gap could be smaller for $x \in \mathscr{S}$. Here, we introduce the following new assumptions:

Additional Hypotheses:

(i) There is a finite gap $2g > 0$ among the ground state and the first excited state of (7.1).

(ii) The size $|\partial\bar{\mathscr{S}}_n|$ of the set

$$\partial\bar{\mathscr{S}}_n = \{x \in \Lambda \text{ such that } (n-1) \leqq d(x, \mathscr{S}) \leqq n\} \tag{7.2}$$

grows at most exponentially fast in n, i.e. there is a constant c_0 such that

$$|\partial\bar{\mathscr{S}}_n| \leqq c_0^n |\mathscr{S}|. \tag{7.3}$$

for all $n = 1, 2, \ldots$.

In Part II, I consider the unitary dressing transformation $U_0(\lambda)$ computed for the regularized Hamiltonian

$$H_\lambda^{\text{reg}} \equiv \sum_{x \in \Lambda \backslash \mathscr{S}} s_x + \sum_{x \in \mathscr{S}} (1 - P_{|0\rangle_x}) + \sum_{\gamma_0 \subset \Lambda} \lambda^{|\gamma_0|_c - 1} t_{\gamma_0} \tag{7.4}$$

with the method indicated in Part I. By applying such transformation to the original Hamiltonian (7.1), one finds the self adjoint operator

$$S + V(\lambda) + W(\lambda) \equiv U(\lambda)^{-1} H_\gamma U(\lambda) - E_0^{\text{reg}}(\lambda), \tag{7.5}$$

where $E_0^{\text{reg}}(\lambda)$ is the ground state energy of H_λ^{reg}. Let E_0 be the ground state energy of the dressed Hamiltonian (7.3). If

$$u = \sum_\gamma u_\gamma \tau_\gamma |0\rangle \tag{7.6}$$

is an eigenstate of $S + V(\lambda) + W(\lambda)$ with energy $< E_0 + \frac{1}{2}$, then thanks to Corollary

4.2, we have

$$\sum_{\gamma: d_{\bar{\mathscr{S}}_{k_0}}(\emptyset, s(\gamma)) \geqq k} u_\gamma^2 \leqq (c_0\lambda)^k \tag{7.7}$$

for all $\lambda < c$ and $k = 1, 2, \ldots$, where n_0 is an integer to be fixed in Sect. 8 and whose order of magnitude is

$$n_0 \cong \frac{\log c_0 |\mathscr{S}|}{|\log c_0 \lambda|}. \tag{7.8}$$

Let n be an integer $> n_0$ and let u_0 be the ground state of the operator

$$\tilde{S}(\lambda) \equiv S + V_{\bar{\mathscr{S}}_n}(\lambda) + W_{\bar{\mathscr{S}}_n}(\lambda) - E_{0n}(\lambda), \tag{7.9}$$

where $E_{0n}(\lambda)$ is the ground state energy of $S + V_{\bar{\mathscr{S}}_n}(\lambda) + W_{\bar{\mathscr{S}}_n}(\lambda)$. Clearly, u_0 has no excitations outside $\bar{\mathscr{S}}_n$. The aim of this Part of the paper is to construct a unitary dressing transformation that solves perturbatively the ground state problem for $S + V(\lambda) + W(\lambda)$, by starting from u_0. More precisely, if we set

$$\tilde{V}(\lambda) \equiv V_{\partial\bar{\mathscr{S}}_n}(\lambda) + V_{\sim\bar{\mathscr{S}}_n}(\lambda) + W_{\partial\bar{\mathscr{S}}_n}(\lambda), \tag{7.10}$$

our aim is to compute a skew symmetric operator $\bar{R}(\lambda)$, analytic for $\lambda \leqq c$, such that

$$e^{-\bar{R}(\lambda)}(\tilde{S}(\lambda) + \tilde{V}(\lambda))e^{\bar{R}(\lambda)}u_0 = E_0(\lambda)u_0. \tag{7.11}$$

For fixed $\lambda \leqq c$, $\bar{R}(\lambda)$ is constructed as the value at $\beta = \lambda^{1/4}$ of an operator-valued function expressed by a convergent power expansion

$$\bar{R}_\lambda(\beta) = \sum_{j=1}^{\infty} \beta^j \bar{R}_{\lambda,j}. \tag{7.12}$$

$\bar{R}_\lambda(\beta)$ is such that

$$e^{-\bar{R}_\lambda(\beta)}(\tilde{S}(\lambda) + \tilde{V}(\beta\lambda^{3/4}))e^{\bar{R}_\lambda(\beta)}u_0 = E(\beta)u_0 \tag{7.13}$$

for all $\beta \in [0, \lambda^{1/4}]$. In the following, the subscript λ of $\bar{R}_\lambda(\beta)$ is omitted.

The operators $\bar{R}_j$ have the form

$$\bar{R}_j = \sum_{s(\gamma) \subset \sim\bar{\mathscr{S}}_n} r_{j\gamma}\bar{\tau}_{j\gamma}, \tag{7.14}$$

where $\bar{\tau}_\gamma$ is the operator

$$\bar{\tau}_\gamma \equiv \prod_{x \in (s(\gamma) \setminus x_0)} T_{x,\gamma(x)} T'_{x_0,\gamma(x_0)} T_{j,\gamma}. \tag{7.15}$$

Here, x_0 is any point of $s(\gamma)$. $T_{x,\alpha}$ and $T'_{x,\alpha}$ are operators acting only on the spin in the site x and are defined as follows:

$$T_{x,\alpha} = |\alpha\rangle_{x\,x}\langle 0| + |0\rangle_{x\,x}\langle\alpha|, \tag{7.16}$$

$$T'_{x,\alpha} = |\alpha\rangle_{x\,x}\langle 0| - |0\rangle_{x\,x}\langle\alpha|, \tag{7.17}$$

where $|\alpha\rangle_x$ is an excited eigenstate of S_x. $T_{j,\gamma}$ acts on the spins inside $\bar{\mathscr{S}}_n$ and is an operator of the form

$$T_{j,\gamma} = |\alpha_{j,\gamma}\rangle\langle u_0| \pm |u_0\rangle\langle\alpha_{j,\gamma}|, \tag{7.18}$$

where the minus sign has to be taken in case $s(\gamma) = \emptyset$; otherwise, the plus sign is there. $|\alpha_{j,\gamma}\rangle$ can be either equal to $|u_0\rangle$, or it is a state of L^2-norm with excitations only inside $\bar{\mathscr{S}}_n$ and orthogonal to $|u_0\rangle$.

Let us remark that the operators presently defined are skewsymmetric, so that $e^{\bar{R}}$ is a unitary operator. Unlike the τ operators used in Part I, the operators (7.15) are strongly noncommutative. However, this is not a shortcoming here. In fact, since the problems is of local character, no cluster expansion is needed and the convergence of the perturbative expansion can be controlled by means of a global norm.

The following is the main result of this Part:

Theorem 7.1. *Under the hypothesis above, if $\lambda < c$ and*

$$n \geqq n_0 + \max\left(c\frac{\log c|\mathscr{S}|}{|\log c\lambda|}, c\frac{|\log cg|}{|\log c\lambda|}\right), \tag{7.19}$$

then there is one and only one operator $\bar{R}(\lambda)$ of the form (7.12) which solves the conjugacy problem (7.13). $\bar{R}(\lambda)$ admits a convergent expansion

$$\bar{R}(\lambda) = \sum_{k=1}^{\infty} \bar{R}_{\lambda,k}\lambda^{(1/4)k} \tag{7.20}$$

with

$$\sum_{k=1}^{\infty} \|\bar{R}_{\lambda,k}\|_{2,1} \leqq (c_0\lambda)^{(1/4)(n-n_0)}, \tag{7.21}$$

where $\|\cdot\|_{2,1}$ is the norm for which

$$\left\| \sum_{s(\gamma)\subset\sim\bar{\mathscr{S}}_n} \phi_\gamma \otimes (\tau_\gamma |0_{\sim\bar{\mathscr{S}}_n}\rangle)\right\|_{2,1} = \sum_{s(\gamma)\subset\sim\bar{\mathscr{S}}_n} \|\phi_\gamma\|_2, \tag{7.22}$$

where $\phi_\gamma \in \mathscr{H}(\bar{\mathscr{S}}_n)$.

In the rest of this section, the strategy for the proof of Theorem 7.1 is described, while the details are deferred to the next sections.

Let us introduce the operators $\tilde{V}_k(\lambda)$ such that

$$\tilde{V}(\beta\lambda^{3/4}) = \sum_{k=1}^{\infty} \tilde{V}_k(\lambda)\beta^k. \tag{7.23}$$

The operators $\bar{R}_j$ are uniquely determined by the requirements of having the form (7.12) and of solving the following recurrence relations that are obtained by expanding in powers of β both the members of (7.13):

$$\begin{aligned}\bar{R}_j|u_0\rangle = -\Pi_0\tilde{S}^{-1}\Bigg\{ &\sum_{\substack{i_1+\dots+i_k=j\\ k\geqq 2}} \frac{1}{k!}[\cdots[\tilde{S},\bar{R}_{i_1}]\cdots\bar{R}_{i_k}] \\ &+\sum_{l=1}\sum_{i_1+\dots+i_k=j-l}\frac{1}{k!}[\cdots[\tilde{V}_l,\bar{R}_{i_1}]\cdots\bar{R}_{i_k}]\Bigg\}|u_0\rangle,\end{aligned} \tag{7.24}$$

where

$$\Pi_0 = 1 - P_{|u_0\rangle} \tag{7.25}$$

and $P_{|u_0\rangle}$ is the orthogonal projection along $|u_0\rangle$.

In order to control the expansion generated by the identities (7.24), we use the numerical sequence

$$\bar{r}_n^* \equiv \left\| \mathring{S}\bar{R}_n | u_0 \rangle \right\|_{2,1}. \tag{7.26}$$

Here

$$\mathring{S} = \Pi^{\leqq 1/2} + \Pi^{>1/2}\tilde{S} \tag{7.27}$$

and $\Pi^{\leqq 1/2}$ (respectively $\Pi^{>1/2}$) is the orthogonal projection onto the eigenspace of $\tilde{S}$ with energy $\leqq \frac{1}{2}$ (respectively $\geqq \frac{1}{2}$).

In Sect. 10, we derive recurrence inequalities for the sequence (7.26) that permit, in Sect. 11, to conclude the proof of Theorem 7.1. To find such estimates, one has to treat separately low energy and high energy excitations, i.e. one has to control with different arguments the couplings with the states in $\Pi^{\leqq 1/2}\mathscr{H}(\Lambda)$ and those with the states in $\Pi^{>1/2}\mathscr{H}(\Lambda)$. In fact, the perturbation $\tilde{V}$ has a relative bound with respect to $\tilde{S}$ that is of order λ, while in typical applications we have $g \ll \lambda$. However, since $\tilde{V}$ is able to induce with large amplitude only excitations far away from $\mathscr{S}$, it can hardly induce transitions from the ground state to an eigenstate of low energy. In fact, such states are essentially different from the ground state only near $\mathscr{S}$, and they approach it exponentially fast away from $\mathscr{S}$. Hence, in the perturbation expansion, small divisors like g^{-1} appear multiplied by factors that are exponentially small in the distance $(n - n_0)$ between $\partial\bar{\mathscr{S}}_n$ and $\bar{\mathscr{S}}_{n_0}$. On the other hand, the transitions to states of high energy occur with amplitude of order λ, that is much smaller than the energy gap that for such states is $\gtrsim 1$.

To turn such intuitive arguments into a rigorous proof, one has to establish three sorts of bounds. First, one needs to prove that if n satisfies a bound of the form (7.19), then the operator $\tilde{S}$ has a gap g, $2g$ being the gap we assumed $S + V + W$ to have. Second, one has to bound in L^2 and in $L^{2,1}$ operator norm the operator $\tilde{V}\Pi^{<1/2}$ that contains the couplings with low energy excitations. Third, one has to find a relative boundedness estimate for $\tilde{V}$ with respect to $\tilde{S}$, that permits to control high energy excitations. The first two tasks require similar techniques and they are accomplished in Sect. 8, while Sect. 9 contains the relative boundedness result that is needed in Sect. 10.

8. Effective Coupling of Low Energy Excitations

This section has three goals. First, I fix the integer n_0 in (7.7). Second, I prove that if $S + V + W$ has gap $2g$ and n is large enough, then $\tilde{S}$ has a gap $\geqq g$. Finally, two relative boundedness estimates for $\tilde{V}_l$ with respect to $\tilde{S}$ are proven, one in L^2 norm and one in $L^{2,1}$ norm.

The methods in Part II apply also to the operator

$$\tilde{S}(\lambda) + \tilde{V}(\beta\lambda^{3/4}) \tag{8.1}$$

for $\beta \in [0, \lambda^{1/4}]$ and they permit one to conclude that any eigenfunction of (8.1) with energy less than the ground state energy $E(\beta)$ plus 1/2 fulfill the decay estimate

$$\left(\sum_{d_{\bar{\mathscr{S}}_{n_0}}(\emptyset,\gamma) \geqq k} |u_\gamma(\beta)|^2 \right)^{1/2} \leqq (c_0\lambda)^{(1/2)k} \tag{8.2}$$

for some

$$n_0 \leqq \frac{\log c|\mathscr{S}|}{|\log c\lambda|}, \tag{8.3}$$

where none of the constants here depend on β, as far as $\beta \in [0, \lambda^{1/4}]$. Let us fix n_0 as the minimum integer for which (8.2) holds. We have

Lemma 8.1. *Under the hypothesis of Theorem 7.1, the following is true:*

(i) *If $g(\beta)$ is the gap between the ground state and the first excited state of (8.1), then*

$$g(\beta) \geqq 2g - (c\lambda)^{1/3(n-n_0)} \geqq g \tag{8.4}$$

for all $\beta \in [0, \lambda^{1/4}]$

(ii) *We have*

$$\| \tilde{V}_l \Pi^{<1/2} \|_{2,1} \leqq (c_0\lambda)^{1/3(n-n_0)} \tag{8.5}$$

for all integers $l \geqq 1$.

(iii) *We have*

$$\| \tilde{V}_l \Pi^{<1/2} \|_2 \leqq (c_0\lambda)^{1/3(n-n_0)} \tag{8.6}$$

for all integers $l \geqq 1$.

Proof. (i) Let $E_0(\beta)$ and $E_1(\beta)$ be the ground state energy and the first excited level of the operator (8.1), respectively. Let $u_0(\beta)$ and $u_1(\beta)$ be the corresponding eigenstates. $E_0(\beta)$ and $E_1(\beta)$ are continuous functions and they are analytic in the interval $[0, \lambda^{1/4}]$, except for β belonging to a discrete set (see [4]), corresponding to those values for which there is an intersection of levels. In the points of analyticity, we have

$$\frac{d}{d\beta} E_0(\beta) = \langle u_0(\beta) | \tilde{V}'(\beta, \lambda) | u_0(\beta) \rangle, \tag{8.7}$$

where

$$\tilde{V}'(\beta, \lambda) = \frac{d}{d\beta} \tilde{V}(\beta\lambda^{3/4}) = \sum_{l=1}^{\infty} \tilde{V}_l(\lambda) l \beta^{l-1}. \tag{8.8}$$

The methods used to prove Theorem 1.2 (ii) and Lemma 5.2 lead to the bound

$$|\langle u | \tilde{V}'(\beta, \lambda) | u \rangle| \leqq \langle u | S^{\text{in}}_{\partial \mathscr{S}_n} + S_{\sim \bar{\mathscr{S}}_n} | u \rangle, \tag{8.9}$$

valid for all $u \in \mathscr{H}(\Lambda)$. We have

$$\begin{aligned} \langle u(\beta) | S_{\sim \bar{\mathscr{S}}_n} | u(\beta) \rangle &\leqq \sum_{k=1}^{\infty} \sum_{d_{\bar{\mathscr{S}}_n}(s(\gamma), \varnothing) = k} |u_\gamma(\beta)|^2 \langle \gamma | S_{\sim \bar{\mathscr{S}}_n} | \gamma \rangle \\ &\leqq \sum_{k=1}^{\infty} k|s| \sum_{d_{\bar{\mathscr{S}}_n}(s(\gamma), \varnothing) = k} |u_\gamma(\beta)|^2 \\ &\leqq |s| \sum_{k=1}^{\infty} k (c_0\lambda)^{(1/2)(k+n-n_0)} \leqq (c\lambda)^{(1/3)(n-n_0)}. \end{aligned} \tag{8.10}$$

Moreover, we have

$$\begin{aligned}\langle u(\beta)|S^{\text{in}}_{\partial\bar{\mathscr{S}}_n}|u(\beta)\rangle &\leqq \sum_{j=1}^{n-n_0}\sum_{d(x,\partial\bar{\mathscr{S}}_n)=j}(c_0\lambda)^{(1/2)(j+1)}\langle u(\beta)|s_x|u(\beta)\rangle\\ &\leqq \sum_{j=1}^{n-n_0}\sum_{d_{\bar{\mathscr{S}}_{n_0}}(s(\gamma),\emptyset)\geqq n-n_0-j}(c_0\lambda)^{(1/2)(j+1)}|u(\beta)|^2|s|\\ &\leqq \sum_{j=0}^{n-n_0}(c_0\lambda)^{(1/2)(n-n_0+1)}|s|\leqq(c\lambda)^{(1/3)(n-n_0)}.\end{aligned}\tag{8.11}$$

Hence, we have

$$\left|\frac{dE_0(\beta)}{d\beta}\right|\leqq(c\lambda)^{(1/3)(n-n_0)}.\tag{8.12}$$

Since the bound (8.2) holds also for $u_1(\beta)$, we have

$$\left|\frac{dE_1(\beta)}{d\beta}\right|\leqq(c\lambda)^{(1/3)(n-n_0)}\tag{8.13}$$

in all points of analyticity. Since the points of nonanalyticity form at most a discrete set, (8.4) follows by integration of the differential inequality

$$\left|\frac{dg(\beta)}{d\beta}\right|\leqq(c\lambda)^{(1/3)(n-n_0)}\tag{8.14}$$

that is true almost everywhere. Q.E.D.

(ii) The range of the projection $\Pi^{<1/2}$, is the space generated by the wavefunctions of the form

$$|u\rangle\otimes|0_{\sim\mathscr{S}_n}\rangle,\tag{8.15}$$

where $|u\rangle$ is an eigenstate of $S_{\bar{\mathscr{S}}_n}+V_{\bar{\mathscr{S}}_n}+W_{\bar{\mathscr{S}}_n}$ with energy $<E_{on}+\frac{1}{2}$. Due to Theorem 4.1 and our choice of n_0, such eigenstates satisfy the bound (8.2). Being a bound in L^2-norm, it must be valid for all wavefunctions in the range of $\Pi^{<1/2}$. Hence, it suffices to prove that

$$\|\tilde{V}_l(|u\rangle\otimes|0_{\sim\bar{\mathscr{S}}_n}\rangle)\|_{2,1}\leqq(c_0\lambda)^{(1/3)(n-n_0)}\tag{8.16}$$

for all states $|u\rangle\in\mathscr{H}(\bar{\mathscr{S}}_n)$ satisfying (8.2).

Following the same procedure used in the proof of Lemma 6.3, we can decompose $\tilde{V}_l$ as follows:

$$\tilde{V}_l=\sum_{\gamma_0}v_l(\gamma_0)=\sum_{\gamma_0}\operatorname{ad}v_l(\gamma_0),\tag{8.17}$$

where $v_l(\gamma_0)$ is an operator with support γ_0 and $\operatorname{ad}v_l(\gamma_0)$ is the operator such that

$$\operatorname{ad}v_l(\gamma_0)|\gamma\rangle=\operatorname{ad}v_l(\gamma_0)\tau_\gamma|0\rangle=[v_l(\gamma_0),\tau_\gamma]|0\rangle\tag{8.18}$$

for all excitations $|\gamma\rangle$. Let us also introduce the operators

$$\bar{v}_l(\gamma_0)=\sum_{\gamma_0'\cap\bar{\mathscr{S}}_n=\gamma_0}F_1\operatorname{ad}v_l(\gamma_0')F_2\tag{8.19}$$

for all $\gamma_0\subset\bar{\mathscr{S}}_n$, where F_1 and F_2 are defined as in (6.25) and (6.26). Due to the bounds in Sect. 2, for all $x \in \bar{\mathscr{S}}_n$ we have

$$\sum_{\substack{\gamma_0 : x \in \gamma_0 \\ \gamma_0 \subset \bar{\mathscr{S}}_n}} \| \bar{v}_l(\gamma_0) \|_1 \leqq (c_0 \lambda)^{1 + d(x, \partial \bar{\mathscr{S}}_n)} \tag{8.20}$$

Hence, we have

$$\| \tilde{V}_l(|u\rangle \otimes |0_{\sim \bar{\mathscr{S}}_n}\rangle \|_{2,1} \leqq \sum_{\gamma_0 \subset \bar{\mathscr{S}}_n} \| \bar{v}_l(\gamma_0) |u\rangle \|_2 \leqq \| S_{\partial \bar{\mathscr{S}}_n} |u\rangle \|_2, \tag{8.21}$$

where the last bound can be proven as Theorem 1.2 (iv). Due to (8.11), we have (8.5). Q.E.D.

(iii) This bound follows from the estimate

$$| \langle u | \tilde{V}_l | u \rangle | \leqq \langle u | S^{\text{in}}_{\partial \bar{\mathscr{S}}_n} + S_{\sim \bar{\mathscr{S}}_n} | u \rangle \tag{8.22}$$

holding for all $u \in \mathscr{H}(\Lambda)$ and from (8.10), (8.11). Q.E.D.

9. A Relative Boundedness Result

This section contains the proof of the following relative boundedness result that permits to control high energy excitations in the perturbative expansions considered in this part of the paper:

Lemma 9.1. *Under the hypothesis of Theorem 7.1, we have*

$$\| V_j \mathring{S}^{-1} \|_{2,1} \leqq 1. \tag{9.1}$$

Proof. It suffices to prove that for all $u \in \mathscr{H}(\Lambda)$ we have

$$\| V_j u \|_{2,1} \leqq \| \mathring{S} u \|_{2,1}. \tag{9.2}$$

Let us expand u as follows

$$u = \sum_{s(\gamma) \subset \sim \bar{\mathscr{S}}_n} \phi_\gamma \otimes \tau_\gamma | 0_{\sim \bar{\mathscr{S}}_n} \rangle \tag{9.3}$$

with $\phi_\gamma \in \mathscr{H}(\tilde{\mathscr{S}}_n)$. We have

$$\| \tilde{S} u \|_{2,1} \leqq \| \mathring{S} u \|_{2,1}. \tag{9.4}$$

In fact

$$\begin{aligned} &\left\| \mathring{S} \sum_{s(\gamma) \subset \sim \bar{\mathscr{S}}_n} \phi_\gamma \otimes \tau_\gamma | 0_{\sim \bar{\mathscr{S}}_n} \rangle \right\|_{2,1} \\ &= \left\| (\Pi^{<1/2} + \tilde{S} \Pi^{\geqq 1/2}) \phi_\varnothing \otimes | 0_{\sim \bar{\mathscr{S}}_n} \rangle + \tilde{S} \sum_{\substack{s(\gamma) \subset \sim \bar{\mathscr{S}}_n \\ s(\gamma) \neq \varnothing}} \phi_\gamma \otimes \tau_\gamma | 0_{\sim \bar{\mathscr{S}}_n} \rangle \right\|_{2,1} \\ &= \| (\Pi^{<1/2} + \tilde{S} \Pi^{\geqq 1/2}) \phi_\varnothing \|_2 + \left\| \tilde{S} \sum_{\substack{s(\gamma) \subset \bar{\mathscr{S}}_n \\ s(\gamma) \neq \varnothing}} \phi_\gamma \otimes \tau_\gamma | 0_{\sim \bar{\mathscr{S}}_n} \rangle \right\|_{2,1} \\ &= \| \tilde{S} \phi_\varnothing \|_2 + \left\| \tilde{S} \sum_{\substack{s(\gamma) \subset \sim \bar{\mathscr{S}}_n \\ s(\gamma) \neq \varnothing}} \phi_\gamma \otimes \tau_\gamma | 0_{\sim \bar{\mathscr{S}}_n} \rangle \right\|_{2,1} = \| \tilde{S} u \|_{2,1}. \end{aligned} \tag{9.5}$$

Moreover, we have

$$\left\| \mathring{S} \sum_{s(\gamma)\subset\sim\bar{\mathscr{S}}_n} \phi_\gamma \otimes \tau_\gamma |0_{\sim\bar{\mathscr{S}}_n}\rangle \right\|_{2,1} \geqq \frac{1}{2}\|\phi_\varnothing\|_2 + \left\| \sum_{\substack{s(\gamma)\subset\sim\bar{\mathscr{S}}_n \\ s(\gamma)\neq\varnothing}} \phi_\gamma \otimes \tau_\gamma |0_{\sim\bar{\mathscr{S}}_n}\rangle \right\|_{2,1} \geqq \frac{1}{2}\|u\|_{2,1}. \tag{9.6}$$

Hence

$$\tfrac{1}{2}\|\tilde{S}u\|_{2,1} + \tfrac{1}{4}\|u\|_{2,1} \leqq \|\mathring{S}u\|_{2,1} \tag{9.7}$$

and it suffices to prove that

$$\|V_j u\|_{2,1} \leqq \tfrac{1}{2}\|\tilde{S}u\|_{2,1} + \tfrac{1}{4}\|u\|_{2,1} \tag{9.8}$$

for all u.

We have

$$\|V_j u\|_{2,1} \leqq \sum_{s(\gamma)\subset\sim\bar{\mathscr{S}}_n} \|V_j\phi_\gamma\otimes\tau_\gamma|0_{\sim\bar{\mathscr{S}}_n}\rangle\|_{2,1} \tag{9.9}$$

and

$$\|\tilde{S}u\|_{2,1} = \sum_{s(\gamma)\subset\sim\bar{\mathscr{S}}_n} \|(\varepsilon(\gamma) + \tilde{S}_{\bar{\mathscr{S}}_{n_2}})\phi_\gamma\|_2. \tag{9.10}$$

Hence, it suffices to show that for all functions $\phi\in\mathscr{H}(\mathscr{S}_n)$ and all excitations γ with support in $\sim\bar{\mathscr{S}}_n$, we have

$$\|V_j\phi\otimes\tau_\gamma|0_{\sim\bar{\mathscr{S}}_n}\rangle\|_{2,1} \leqq \|(\tfrac{1}{4} + \tfrac{1}{2}\varepsilon(\gamma) + \tfrac{1}{2}\tilde{S}_{\bar{\mathscr{S}}_n})\phi\|_2. \tag{9.11}$$

Let A_j be the annular region

$$A_j = \{x : d(x, \partial\bar{\mathscr{S}}_n) \leqq j-1\}. \tag{9.12}$$

The operator V_j is given by a sum of clusters of operators $v(\gamma_0)$ with support $\gamma_0 \subset A_j$, i.e.

$$V_j = \sum_{\gamma_0\subset A_j} v_j(\gamma_0). \tag{9.13}$$

Since $V_j|0\rangle = 0$, we can express V_j in the following alternative way:

$$V_j = \sum_{\gamma_0\subset A_j} \operatorname{ad} v_j(\gamma_0), \tag{9.14}$$

where $\operatorname{ad} V_j(\gamma_0)$ is the operator acting as follows:

$$\operatorname{ad} v_j(\gamma_0)|\gamma'\rangle = \operatorname{ad} v_j(\gamma_0)\tau_{\gamma'}|0\rangle \equiv [v_j(\gamma_0), \tau_{\gamma'}]|0\rangle. \tag{9.15}$$

We have

$$\begin{aligned} \|V_j\phi\otimes\tau_\gamma|0_{\sim\bar{\mathscr{S}}_n}\rangle\|_{2,1} = & \sum_{\gamma_0\cap s(\gamma)\neq\varnothing} \|\operatorname{ad} v_j(\gamma_0)\phi\otimes\tau_\gamma|0_{\sim\bar{\mathscr{S}}_n}\rangle\|_{2,1} \\ & + \sum_{\gamma_0\cap s(\gamma)=\varnothing} \|\operatorname{ad} v_j(\gamma_0)\phi\otimes\tau_\gamma|0_{\sim\bar{\mathscr{S}}_n}\rangle\|_{2,1}. \end{aligned} \tag{9.16}$$

The first term in (9.16) is

$$\leqq (c_0\lambda)|s(\gamma)|\,\|\phi\|_2 \leqq (c_0\lambda)\varepsilon(\gamma)\|\phi\|_2. \tag{9.17}$$

The second term can be bounded by using the operators

$$\bar{v}_j(\gamma_0) = \sum_{\gamma_0' \cap \bar{\mathscr{S}}_n = \gamma_0} F_1 \operatorname{ad} v_j(\gamma_0) F_2 \tag{9.18}$$

with $\gamma_0 \subset \bar{\mathscr{S}}_n$ and F_1 and F_2 defined as in (6.25), (6.26). Due to the estimates in Sect. 2, we have

$$\sup_x \sum_{\gamma_0 : x \in \gamma_0} \| \bar{v}_j(\gamma_0) \|_2 \leqq 1 \tag{9.19}$$

for $\lambda \leqq c$. With this definition of $\bar{v}_j(\gamma_0)$ and thanks to (9.17), it suffices to prove that

$$\sum_{\gamma_0 \subset \bar{\mathscr{S}}_n} \| \bar{v}_j(\gamma_0) \phi \|_2 \leqq \| (\tfrac{1}{4} + \tfrac{1}{2} \tilde{S}_{\bar{\mathscr{S}}_n}) \phi \|_2 . \tag{9.20}$$

By proceeding exactly as in the proof of Theorem 1.2 (iii), one can see that

$$\sum_{\gamma_0 \subset \bar{\mathscr{S}}_n} \| \bar{v}_j(\gamma_0) \phi \|_2 \leqq \tfrac{1}{4} \| S_{A_j} \phi \|_2 . \tag{9.21}$$

Hence, we are reduced to prove the following bound:

$$\| S_{A_j} \phi \|_2 \leqq \| (1 + 2\tilde{S}_{\bar{\mathscr{S}}_n}) \phi \|_2 . \tag{9.22}$$

This bound follows from the positivity of the operator

$$(1 + 2\tilde{S}_{\bar{\mathscr{S}}_n})^2 - S_{A_j}^2 \tag{9.23}$$

and the following I prove that the ground state energy for such operator is positive.

Let us remark that it is possible to assume that

$$j \leqq \left[\frac{n - n_0}{2} \right], \tag{9.24}$$

where $[\cdot]$ denotes the integer part. In fact, if

$$(c_0 \lambda^{3/4})^{[(n-n_0)/2]} | \partial \bar{\mathscr{S}}_n | < \tfrac{1}{2} \tag{9.25}$$

one can simply use a bound in $L^{2,1}$-operator norm on V_j to control (9.1). Due to the hypothesis we make that the growth of $|\partial \bar{S}_n|$ as $n \uparrow \infty$ is exponentially bounded, the condition (9.25) can be expressed in the form

$$n - n_0 \geqq c \frac{\log c |\mathscr{S}|}{|\log c\lambda|}, \tag{9.26}$$

and thus we can assume it to be fulfilled.

Let us suppose that (9.23) holds and let

$$n_2 = n_0 + \left[\frac{n - n_0}{2} \right]. \tag{9.27}$$

Let us decompose $\tilde{S}_{\bar{\mathscr{S}}_n}$ as follows:

$$\tilde{S}_{\bar{\mathscr{S}}_n} = \tilde{S}_{\bar{\mathscr{S}}_{n_2}} + \tilde{S}_{\partial \bar{\mathscr{S}}_{n_2}} + \tilde{S}_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}}, \tag{9.28}$$

where

$$\tilde{S}_{\bar{\mathscr{S}}_{n_2}} = S_{\bar{\mathscr{S}}_{n_2}} + V_{\bar{\mathscr{S}}_{n_2}} + W_{\bar{\mathscr{S}}_{n_2}} - S^{\text{in}}_{\partial \bar{\mathscr{S}}_{n_2}} - E_{0n}, \tag{9.29}$$

$$\tilde{S}_{\partial \bar{\mathscr{S}}_{n_2}} = S_{\partial \bar{\mathscr{S}}_{n_2}} + V_{\partial \bar{\mathscr{S}}_{n_2}} + W_{\partial \bar{\mathscr{S}}_{n_2}}, \tag{9.30}$$

and

$$\tilde{S}_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} = S_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} + V_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} - S^{\text{out}}_{\partial \bar{\mathscr{S}}_{n_2}}. \tag{9.31}$$

We have

$$\begin{aligned}(1 + 2\tilde{S}_{\bar{\mathscr{S}}_n})^2 - S^2_{A_j} = {} & 1 + 2\tilde{S}_{\bar{\mathscr{S}}_n} + 4S^2_{\bar{\mathscr{S}}_{n_2}} + 4\tilde{S}^2_{\partial \bar{\mathscr{S}}_{n_2}} \\ & + 4S^2_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} - S^2_{A_j} + 4\{S^2_{\bar{\mathscr{S}}_{n_2}}, S_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}}\} \\ & + 4\{\tilde{S}_{\partial \bar{\mathscr{S}}_{n_2}}, \tilde{S}_{\bar{\mathscr{S}} \setminus \bar{\mathscr{S}}_{n_2}}\} + 4\{\tilde{S}_{\bar{\mathscr{S}}_{n_2}}, \tilde{S}_{\partial \bar{\mathscr{S}}_{n_2}}\},\end{aligned} \tag{9.32}$$

where $\{\cdot,\cdot\}$ denotes the anticommutator.

We have $\tilde{S}_{\bar{\mathscr{S}}_n} \geqq 0$ by definition and $\tilde{S}^2_{\bar{\mathscr{S}}_{n_2}} \geqq 0$, $\tilde{S}^2_{\partial \bar{\mathscr{S}}} \geqq 0$ by selfadjointness. Also, the following operator is positive:

$$4\tilde{S}_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} - S^2_{A_j} = 4(S_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} - S^{\text{out}}_{\partial \bar{\mathscr{S}}_{n_2}} - S^2_{A_j}) + 4\{S_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} - S^{\text{out}}_{\partial \bar{\mathscr{S}}_{n_2}}, V_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}}\} + 4V^2_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}}. \tag{9.33}$$

In fact, for $\lambda = 0$ this operator has a nondegenerate ground state in $\mathscr{H}(\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2})$, namely $|0_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}}\rangle$ with eigenvalue 0. The ground state energy is separated from the fixed excited level by a gap $= 3$, for $\lambda = 0$. For $\lambda > 0$, the second and the third term in (9.31) do not vanish, but $|0_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}}\rangle$ is still an eigenfunction with eigenvalue 0. Thanks to Theorem 1.2 (iv), we have

$$\langle u| V^2_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} |u\rangle \leqq (c_0\lambda)^2 \langle u| S^2_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} |u\rangle \leqq (c_0\lambda)^2 \langle u| 4S^2_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} |u\rangle. \tag{9.34}$$

In a similar way, one can prove that

$$4\langle u| \{S_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} - S_{\partial \bar{\mathscr{S}}_{n_2}}, V_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}}\} |u\rangle \leqq 8(c_0\lambda)^{3/2} \langle u| S^2_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} |u\rangle. \tag{9.35}$$

Hence, the ground state energy of (9.31) is 0 and it is separated by a gap equal to $3 - 0\ (\lambda^{3/4})$ from the rest of the spectrum.

Due to Theorem 4.3, we have

$$\inf \operatorname{spec}(\tilde{S}_{\bar{\mathscr{S}}_{n_2}}) \geqq -(c_0\lambda)^{1+(1/2)n_2}. \tag{9.36}$$

Hence, we have

$$4\{\tilde{S}_{\bar{\mathscr{S}}_{n_2}}, \tilde{S}_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}}\} \geqq 4 \inf \operatorname{spec}(\tilde{S}_{\bar{\mathscr{S}}_{n_2}}) \geqq -4(c_0\lambda)^{1+(1/2)n_2}. \tag{9.37}$$

For the second anticommutator in (9.30), we have

$$\begin{aligned}\{\tilde{S}_{\partial \bar{\mathscr{S}}_{n_2}}, \tilde{S}_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}}\} = {} & 2S_{\partial \bar{\mathscr{S}}_{n_2}}(S_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} - S^{\text{out}}_{\partial \bar{\mathscr{S}}_{n_2}}) + \{V_{\partial \bar{\mathscr{S}}_{n_2}}, (S_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} - S^{\text{out}}_{\partial \bar{\mathscr{S}}_{n_2}})\} \\ & + \{S_{\partial \bar{\mathscr{S}}_{n_2}}, V_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}}\} + \{V_{\partial \bar{\mathscr{S}}_{n_2}}, V_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}}\} \\ & + \{W_{\partial \bar{\mathscr{S}}_{n_2}}, (S_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}} + V_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}})\}.\end{aligned} \tag{9.38}$$

The second, third and fourth term have a relative bound of order $(c_0\lambda)^{1/2}$ with respect to the first term. Moreover, the last term has an L^2-operator norm

$$\leqq (c_0\lambda)^{n_2} |\mathscr{S}| c_0^n |s| = (c_0\lambda)^{n_2} |\mathscr{S}| c_0^{2n_2 - n_0} |s|$$

which is smaller than $(c_0\lambda)^{1/2}$ for a suitable choice of the constants in (7.19). Hence, we have

$$\{\tilde{S}_{\partial \bar{\mathscr{S}}_{n_2}}, \tilde{S}_{\bar{\mathscr{S}}_n \setminus \bar{\mathscr{S}}_{n_2}}\} \geqq -(c\lambda)^{1/2}. \tag{9.39}$$

Finally, to control the last term in (9.30), let us split $\tilde{S}_{\bar{\mathscr{S}}_{n_2}}$ as follows:

$$\tilde{S}_{\bar{\mathscr{S}}_{n_2}} = (S_{\bar{\mathscr{S}}_{no}} + V_{\bar{\mathscr{S}}_{no}} + W_{\bar{\mathscr{S}}_{no}} - S^{\text{in}}_{\partial\bar{\mathscr{S}}_{no}} - E_{0_n}) + (S_{\partial\bar{\mathscr{S}}_{no}} + V_{\partial\bar{\mathscr{S}}_{no}} + W_{\partial\bar{\mathscr{S}}_{no}})$$
$$+ (S_{\bar{\mathscr{S}}_{n_2}\backslash\bar{\mathscr{S}}_{no}} + V^{\text{out}}_{\partial\bar{\mathscr{S}}_{no}} - S^{\text{in}}_{\partial\bar{\mathscr{S}}_{n_2}}) \equiv \tilde{S}_{\bar{\mathscr{S}}_{no}} + \tilde{S}_{\partial\bar{\mathscr{S}}_{no}} + \tilde{S}_{\bar{\mathscr{S}}_{n_2}\backslash\bar{\mathscr{S}}_{no}}, \tag{9.40}$$

where I introduce self explanatory notations. Since, up to exponentially small terms, $\tilde{S}_{\bar{\mathscr{S}}_{no}} + \tilde{S}_{\partial\bar{\mathscr{S}}_{no}}$ and $\tilde{S}_{\partial\bar{\mathscr{S}}_{n_2}}$ have disconnected supports, we have

$$\{\tilde{S}_{\bar{\mathscr{S}}_{no}} + \tilde{S}_{\partial\bar{\mathscr{S}}_{no}}, \tilde{S}_{\partial\bar{\mathscr{S}}_{n_2}}\} \geqq -(c_0\lambda)^{(1/2)n_2 - no}$$
$$\geqq -(c_0\lambda)^{(1/2)n_2 - no} + \inf\operatorname{spec}(\tilde{S}_{\bar{\mathscr{S}}_{no}} + \tilde{S}_{\partial\bar{\mathscr{S}}_{no}})$$
$$+ \inf\operatorname{spec}(\tilde{S}_{\partial\bar{\mathscr{S}}_{n_2}}) \geqq -(c_0\lambda)^{(1/2)n_2 - no} - (c_0\lambda)^{(1/2)no + 1}. \tag{9.41}$$

Moreover, the bound

$$\{\tilde{S}_{\bar{\mathscr{S}}_{n_2}\backslash\bar{\mathscr{S}}_{no}}, S_{\partial\bar{\mathscr{S}}_{n_2}}\} \geqq -(c_0\lambda)^{1/2} \tag{6.42}$$

can be derived as (9.37).

The bounds (9.35), (9.37), (9.39) and (9.40), and the positivity of the other terms in (9.30), permit to conclude that

$$(1 + 2\tilde{S}_{\bar{\mathscr{S}}_n})^2 - S^2_{A_j} \geqq 1 - (c\lambda)^{1/2} \geqq 0. \tag{6.43}$$

This completes the proof of Lemma 9.1. Q.E.D.

10. Recurrence Inequalities

In this section I prove a set of inequalities for the sequence

$$\bar{r}^*_j = \| \mathring{S}\bar{R}_j | u_0 \rangle \|_{2,1}. \tag{10.1}$$

Let us recall that the operators $\bar{R}_j$ have the form (7.14) and that they are uniquely determined by the following recurrence relations:

$$\bar{R}_1 | u_0 \rangle = -\Pi_0 \tilde{S}^{-1} \tilde{V}_1 | u_0 \rangle, \tag{10.2}$$

$$\bar{R}_j | u_0 \rangle = -\Pi_0 \tilde{S}^{-1} \Bigg\{ \sum_{\substack{i_1 + \dots + i_k = j \\ k \geqq 2}} \frac{1}{k!} [\cdots [\tilde{S}, \bar{R}_{i_1}] \cdots \bar{R}_{i_k}]$$
$$+ \sum_{l=1}^{j} \sum_{i_1 + \dots + i_k = j - l} \frac{1}{k!} [\cdots [\tilde{V}_l, \bar{R}_{i_1}] \cdots \bar{R}_{i_k}] \Bigg\} | u_0 \rangle \tag{10.3}$$

for all $j > 1$.

Notations. Let us introduce some notations to be used here and in the following section. Let

$$\bar{r}^*(\beta) \sim \sum_{j=1}^{\infty} \bar{r}^*_j \beta^j. \tag{10.4}$$

Until the convergence of the series (10.4) is established, we have to treat $r^*(\beta)$ as a formal power series. If

$$f(\beta) \sim \sum_{j=1}^{\infty} f_j \beta^j \tag{10.5}$$

is any formal power series, let us denote with $\{f(\beta)\}_j$ the coefficient f_j. Let $F(\beta)$ be the function

$$F(\beta) := e^{4\beta} - 1 - 4\beta. \tag{10.6}$$

Lemma 10.1. *We have*

$$\bar{r}_1^* \leqq g^{-1}(c_0\lambda)^{(1/2)(n-n_0)} \tag{10.7}$$

and

$$\bar{r}_j^* \leqq \{(1 + 8g^{-1}(c_0\lambda)^{(1/3)(n-n_0)})\beta(1-\beta)^{-1}\bar{r}^*(\beta) + g^{-1}(1-\beta)^{-1}F(\bar{r}^*(\beta))\}_j. \tag{10.8}$$

Proof. Thanks to Lemma 8.1 (ii), we have

$$\bar{r}_1^* = \| \mathring{S}\bar{R}_1 |u_0\rangle \|_{2,1} = \| \mathring{S}\tilde{S}^{-1}\tilde{V}_1 |u_0\rangle \|_{2,1} \leqq g^{-1}(c_0\lambda)^{(1/3)(n-n_0)} \tag{10.9}$$

so that (10.7) is verified.

In order to establish (10.8), it is necessary to consider separately the terms in (10.3) containing one and more than one commutators. We have

$$\| \mathring{S}\tilde{S}^{-1}\Pi_0[\tilde{V}_l, \bar{R}_{j-l}] |u_0\rangle \|_{2,1} \leqq \| \mathring{S}\tilde{S}^{-1}\Pi_0\bar{R}_{j-l}\tilde{V}_l |u_0\rangle \|_{2,1}, \tag{10.10}$$

$$+ \| \mathring{S}S^{-1}\Pi_0\Pi^{\geqq 1/2}\tilde{V}_l\bar{R}_{j-l} |u_0\rangle \|_{2,1}, \tag{10.11}$$

$$+ \| \mathring{S}\tilde{S}^{-1}\Pi_0\Pi^{<1/2}\tilde{V}_l\bar{R}_{j-l} |u_0\rangle \|_{2,1}. \tag{10.12}$$

Equation (10.10) is bounded from above by

$$\| \mathring{S}\tilde{S}^{-1}\Pi_0 \|_{2,1} \| \Pi_0\bar{R}_{j-l} \|_{2,1} \| \tilde{V}_l |u_0\rangle \|_{2,1}. \tag{10.13}$$

We have

$$\begin{aligned}\| \Pi_0\bar{R}_{j-l} \|_{2,1} &\leqq \sum_\gamma |\bar{r}_{j-l,\gamma}| \, \| \Pi_0 T_{j-l,\gamma} \|_2 \\ &\leqq 2\sum_\gamma |\bar{r}_{j-l,\gamma}| = 2\| \bar{R}_{j-l} |u_0\rangle \|_{2,1} \\ &\leqq 4\| \mathring{S}\bar{R}_{j-l} |u_0\rangle \|_{2,1} = 4\bar{r}_{j-l}^*, \end{aligned} \tag{10.14}$$

where I used again (9.5). Hence,

$$(10.10) \leqq 4g^{-1}(c_0\lambda)^{(1/3)(n-n_0)}\bar{r}_{j-l}^*. \tag{10.15}$$

For the second term, we have

$$(10.11) \leqq \| \mathring{S}\tilde{S}^{-1}\Pi^{\geqq 1/2} \|_{2,1} \| \tilde{V}_l\mathring{S}^{-1} \|_{2,1} \| \mathring{S}\bar{R}_{j-l} |u_0\rangle \|_{2,1} \leqq \bar{r}_{j-l}^*. \tag{10.16}$$

Finally, we have

$$\begin{aligned}(10.12) &\leqq \| \mathring{S}\tilde{S}^{-1}\Pi_0\Pi^{<1/2} \|_{2,1} \| \Pi^{<1/2}\tilde{V}_l \|_{2,1} \| \bar{R}_{j-l} |u_0\rangle \|_{2,1} \\ &\leqq 2g^{-1}(c_0\lambda)^{(1/3)(n-n_0)}\bar{r}_{j-l}^*, \end{aligned} \tag{10.17}$$

where I used Lemma (8.1), (ii) and the following estimate deriving from the self adjointness of $\tilde{V}_l$:

$$\begin{aligned}\| \Pi^{<1/2}\tilde{V}_l \|_{2,1} &= \sup_{\|u\|_{2,1}=1} \| \Pi^{<1/2}\tilde{V}_l |u\rangle \|_{2,1} \\ &= \sup_{\|u\|_{2,1}=1} \| \Pi^{<1/2}\tilde{V}_l |u\rangle \|_2 \leqq \sup_{\|u\|_2=1} \| \Pi^{<1/2}\tilde{V}_l |u\rangle \|_2 \\ &= \| \Pi^{<1/2}\tilde{V}_l \|_2 = \| (\Pi^{<1/2}\tilde{V}_l)^* \|_2 = \| \tilde{V}_l\Pi^{<1/2} \|_2 \leqq (c_0\lambda)^{(1/3)(n-n_0)}. \end{aligned} \tag{10.18}$$

Finally, let us pass to the terms containing more than one commutator. We have

$$\Big\| \mathring{S}\tilde{S}^{-1}\Pi_0 \Big\{ \sum_{\substack{i_1+\cdots+i_k=j\\ k\geqq 2}} \frac{1}{k!}[\cdots[S,\bar{R}_{i_1}]\cdots\bar{R}_{i_k}]$$

$$+ \sum_{l=1}^{j} \sum_{\substack{i_1+\cdots+i_k=j-l\\ k\geqq 2}} \frac{1}{k!}[\cdots[\tilde{V}_l,\bar{R}_{i_1}]\cdots\bar{R}_{i_k}] \Big\} |u_0\rangle \Big\|_{2,1}$$

$$\leqq \|\mathring{S}\tilde{S}^{-1}\Pi_0\|_{2,1} \Big\{ \sum_{\substack{i_1+\cdots+i_k=j\\ k\geqq 2}} \frac{1}{k!} \sum_{t=1}^{k-1} \binom{k}{t} \|\bar{R}_{i_1}\|_{2,1}\cdots\|\bar{R}_{i_t}\|_{2,1}$$

$$\cdot \|\tilde{S}\bar{R}_{i_{t+1}}\cdots\bar{R}_{i_k}|u_0\rangle\|_{2,1} + \sum_{l=1}^{j} \sum_{\substack{i_1+\cdots+i_k=j-l\\ k\geqq 2}} \sum_{t=1}^{k-1} \binom{k}{t}$$

$$\cdot \bar{R}_{i_1}\|_{2,1}\cdots\|\bar{R}_{i_t}\|_{2,1}\|\tilde{V}_l\bar{R}_{i_{t+1}}\cdots\bar{R}_{i_k}|u_0\rangle\|_{2,1} \Big\}. \tag{10.19}$$

We have

$$\|\tilde{S}\bar{R}_{i_{t+1}}\cdots\bar{R}_{i_k}|u_0\rangle\|_{2,1}$$

$$= \sum_{s(\gamma_{t+1})\cdots s(\gamma_k)\subset\sim\mathscr{S}_n} |\bar{r}_{i_{t+1}\gamma_{t+1}}|\cdots|\bar{r}_{i_k\gamma_k}|\,\|\tilde{S}\bar{\tau}_{\gamma_{t+1}}\cdots\bar{\tau}_{\gamma_k}|u_0\rangle\|_{2,1}$$

$$\leqq \sum_{s(\gamma_{t+1}),\ldots,s(\gamma_{k+1})\subset\sim\mathscr{S}_n} |\bar{r}_{i_{t+1}\gamma_{t+1}}|\cdots|\bar{r}_{i_k\gamma_k}|(\|\tilde{S}\bar{\tau}_{\gamma_{t+1}}|u_0\rangle\|_{2,1}+\cdots+\|\tilde{S}\bar{\tau}_{\gamma_k}|u_0\rangle\|_{2,1})$$

$$\leqq \sum_{s(\gamma_{t+1}),\ldots,s(\gamma_k)\subset\sim\mathscr{S}_n} |\bar{r}_{i_{t+1}\gamma_{t+1}}|\cdots|\bar{r}_{i_k\gamma_k}|(\|\mathring{S}\bar{\tau}_{\gamma_{t+1}}|u_0\rangle\|_{2,1}+\cdots+\|\mathring{S}\bar{\tau}_{\gamma_k}|u_0\rangle\|_{2,1})$$

$$\leqq \sum_{s(\gamma_{t+1})\cdots s(\gamma_k)\subset\sim\mathscr{S}_n} |\bar{r}_{i_{t+1}\gamma_{t+1}}|\cdots|\bar{r}_{i_k\gamma_k}|2^{k-t-1}\|\mathring{S}\bar{\tau}_{\gamma_{t+1}}|u_0\rangle\|_{2,1}\cdots\|\mathring{S}\bar{\tau}_{\gamma_k}|u_0\rangle\|_{2,1}$$

$$= 2^{k-t-1}\|\mathring{S}\bar{R}_{i_{t+1}}|u_0\rangle\|_{2,1}\cdots\|\mathring{S}\bar{R}_{i_k}|u_0\rangle_{2,1}$$

$$\leqq 2^{k-t-1}\bar{r}^*_{i_{t+1}}\cdots\bar{r}^*_{i_k}. \tag{10.20}$$

Similarly, we can find

$$\|\tilde{V}_l\bar{R}_{i_{t+1}}\cdots\bar{R}_{i_k}|u_0\rangle\|_{2,1}$$

$$\leqq \|\tilde{V}_l\mathring{S}^{-1}\|_{2,1}\|\mathring{S}^{-1}\|_{2,1}\|\mathring{S}\bar{R}_{i_{t+1}}\cdots\bar{R}_{i_k}|u_0\rangle\|_{2,1}$$

$$\leqq 2^{k-t-1}\bar{r}^*_{i_{t+1}}\cdots\bar{r}^*_{i_k}. \tag{10.21}$$

Moreover, we have

$$\|\bar{R}_i\|_{2,1} = \sum_{s(\gamma)\subset\sim\bar{\mathscr{S}}_n} |\bar{r}_{i\gamma}|\,\|T_{i,\gamma}\|_2 \leqq 2 \sum_{s(\gamma)\subset\sim\bar{\mathscr{S}}_n} |\bar{r}_{i\gamma}|$$

$$= 2\sum_{s(\gamma)\subset\sim\bar{\mathscr{S}}_n} |\bar{r}_{i\gamma}|\,\|\tau_\gamma|u_0\rangle\|_{2,1}$$

$$= 4\|\bar{R}_i|u_0\rangle\|_{2,1} \leqq 4\bar{r}^*_i. \tag{10.22}$$

Hence, we have

$$(10.19) \leqq \sum_{l=0}^{j-2} \sum_{\substack{i_1+\cdots+i_k=j-l\\ k\geqq 2}} \frac{4^k}{k!}\bar{r}^*_{i_1}\cdots\bar{r}^*_{i_k}$$

$$= g^{-1}\{(1-\beta)^{-1}F(\bar{r}^*(\beta))\}_j. \quad \text{Q.E.D.} \tag{10.23}$$

11. Convergence of the Perturbative Expansion

In this section, I prove that if $\bar{r}_j^*$ is a series of positive numbers such that the bounds in Lemma 10.1 hold, i.e.

$$\bar{r}_1^* \leqq g^{-1}(c_0\lambda)^{(1/2)(n-n_0)} \tag{11.1}$$

and

$$\bar{r}_j^* \leqq \{(1+8g^{-1}(c_0\lambda)^{(1/3)(n-n_0)})\beta(1-\beta)^{-1}\bar{r}^*(\beta)+g^{-1}(1-\beta)^{-1}F(\bar{r}^*(\beta))\}_j, \tag{11.2}$$

and if

$$(c_0\lambda)^{(1/3)(n-n_0)} \leqq \min\left(\frac{g^2}{200}, \frac{g^4}{16}\right), \tag{11.3}$$

then the power series expansion

$$\bar{r}^*(\beta) = \sum_{j=1}^{\infty} \bar{r}_j^* \beta^j \tag{11.4}$$

converges for

$$0 \leqq \beta \leqq \tfrac{1}{4} \tag{11.5}$$

and

$$\bar{r}^*(\beta) \leqq (c_0\lambda)^{(1/4)(n-n_0)}. \tag{11.6}$$

This concludes the proof of Theorem 7.1.

It is easy to argue that the series (11.4) converges for some $\beta > 0$. In fact, let $a(\beta)$ be the function implicitly defined as the solution to the following equation:

$$a(\beta) = g^{-1}(c_0\lambda)^{(1/3)(n-n_0)}\beta + (1+8g^{-1}(c_0\lambda)^{(1/3)(n-n_0)})\beta(1-\beta)^{-1}a(\beta) + g^{-1}(1-\beta)^{-1}F(a(\beta)). \tag{11.7}$$

Due to the implicit function theorem, $a(\beta)$ is analytic in a neighborhood of $\beta = 0$. Moreover, the coefficients of the power series expansion

$$a(\beta) = \sum_{j=1}^{\infty} a_j \beta^j \tag{11.8}$$

for $a(\beta)$ are all positive and such that

$$\bar{r}_j^* \leqq a_j. \tag{11.9}$$

Hence, also the series (11.4) converges for β small.

Let β_0 be the maximum positive number such that the series (11.4) converges and

$$\bar{r}^*(\beta) \leqq \min\left(\frac{1}{4}, \frac{1}{100}g, (c_0\lambda)^{(1/4)(n-n_0)}\right). \tag{11.10}$$

Our aim is to show that

$$\beta_0 \geqq \tfrac{1}{4}. \tag{11.11}$$

Let us remark that, if $0 \leqq x \leqq \frac{1}{4}$, we have

$$F(x) \equiv e^{4x} - 4x - 1 \leqq 20x^2. \tag{11.12}$$

Hence, if $\beta \in [0, \min(\beta_0, 1/4)]$, we have

$$\begin{aligned}\bar{r}^*(\beta) &\leqq g^{-1}(c_0\lambda)^{(1/3)(n-n_0)}\beta + (1 + 8g^{-1}(c_0\lambda)^{(1/3)(n-n_0)})\beta(1-\beta)^{-1}\bar{r}^*(\beta) \\ &\quad + \tfrac{1}{5}(1-\beta)^{-1}\bar{r}^*(\beta) \leqq \tfrac{1}{4}g^{-1}(c_0\lambda)^{(1/3)(n-n_0)} + (\tfrac{1}{2} + 8g^{-1}(c_0\lambda)^{(1/3)(n-n_0)})\bar{r}^*(\beta)\end{aligned} \tag{11.13}$$

that gives the bound

$$\begin{aligned}\bar{r}^*(\beta) &\leqq \tfrac{1}{4}g^{-1}(c_0\lambda)^{(1/3)(n-n_0)}(\tfrac{1}{2} - 8g^{-1}(c_0\lambda)^{(1/3)(n-n_0)})^{-1} \\ &\leqq \frac{1}{2}\min\left(\frac{1}{4}, \frac{g}{100}, (c_0\lambda)^{(1/4)(n-n_0)}\right).\end{aligned} \tag{11.14}$$

Since this bound is 1/2 of the bound in (11.10) used to define β_0, we see that β_0 must be larger than 1/4 because, otherwise, it could not be maximal. Q.E.D.

Part IV. The Ground State Problem in the Presence of a Finite Density of Singular Sites

12. Introduction, Notations and Results

Let Λ be a large connected graph and let us consider a quantum spin system on Λ with Hamiltonian operator

$$H_\lambda = \sum_{x\in\Lambda} s_x + \sum_{\gamma_0 \subset \Lambda} \lambda^{|\gamma_0|_c - 1} t_{\gamma_0}, \tag{12.1}$$

where the notations are as in Part I. The operators s_x are assumed to have a gap $\geqq 1$ for $x \in \Lambda \backslash \mathcal{S}$. In this Part, the set $\mathcal{S} \subset \Lambda$ on which the gap of s_x can be < 1, is assumed to consist of the union of a finite density of small clusters separated by a large distance. Each one of these clusters has the property that the Hamiltonian operator obtained by restricting (12.1) to a large neighborhood of it, has a gap $\geqq g$, g being a constant $\leqq 1$ independent of the cluster. I consider the problem of constructing perturbatively the ground state of H_λ by starting from the ground state of the operator obtained from H_λ by removing the couplings among large and non-intersecting neighbourhoods of the clusters of $\mathcal{S}$. This is a problem of many-body perturbation theory involving a perturbation whose relative bound with respect to the main part of the Hamiltonian is proportional to the number of clusters. Since $\mathcal{S}$ is a set of finite density, in the infinite volume limit such relative bound diverges and it is necessary to use a dressing transformation. The main goal of this part is to construct such a transformation. As applications, the stability of the gap and the exponential decay of truncated correlations are established.

In this section, I introduce some notations and state the main results. Section 13 contains the preliminary constructions that are needed to reduce the problem to a conjugacy problem similar to the one considered in Part I. We define a representation in which the operator (12.1) has the form

$$(\tilde{S}(\lambda) + V_\Gamma(\lambda)) + W_\Gamma(\lambda), \tag{12.2}$$

$W_\Gamma(\lambda)$ being the perturbation. $\tilde{S}(\lambda)$ is unitarily equivalent to the operator (12.1) with the couplings among the different clusters of $\mathcal{S}$ removed and $V_\Gamma(\lambda)$ is an operator relatively bounded with respect to $\tilde{S}(\lambda)$. Here, two additional difficulties are met that are not present in Part I. First, one has to prove that $(\tilde{S}(\lambda)+V_\Gamma(\lambda))$ has a gap $\geqq \frac{1}{2}g, g$ being the gap of $\tilde{S}(\lambda)$. Second, a decay estimate on a pole-subtracted Green's function of $(\tilde{S}(\lambda)+V_\Gamma(\lambda))$ is needed because this operator is not diagonal in the basis of local excitations we use. Sections 14 and 15 are dedicated to these two problems, respectively. In Sect. 16, there are recurrence inequalities for a numerical sequence that permit us in Sect. 17 to establish the convergence of our perturbation scheme. Section 17 contains also the proof of two corollaries concerning the existence of a gap $\geqq \frac{1}{4}g$ for the operators (12.1) and (12.2) and the exponential decay of truncated correlations.

Let us formulate some hypothesis and introduce a few notations. Let us suppose that the graph Λ enjoys the following geometric property:

Condition b. *There is a constant $c_0 > 0$ such that*

$$\sup_{x\in\Lambda} \#\{y\in\Lambda \mid d(x,y)=n\} \leqq c_0^n \tag{12.3}$$

for all integers $n>0$.

Let N be the number

$$N=\max_{x\in\Lambda} N_x \tag{12.4}$$

and let us suppose that N is a constant independent of Λ.

The set $\mathscr{S}$ is supposed to be of the form

$$\mathscr{S}=\bigcup_{\alpha\in\mathscr{S}} C_\alpha, \tag{12.5}$$

where $\mathscr{I}$ is a set of indices and $\{C_\alpha\}_{\alpha\in\mathscr{I}}$ is a family of subsets of Λ. Let $2L_1$ be the minimum distance among two components of $\mathscr{S}$, i.e.

$$L_1=\tfrac{1}{2}\min_{\substack{\alpha,\beta\in\mathscr{I}\\ \alpha\neq\beta}} d(C_\alpha, C_\beta). \tag{12.6}$$

Let M be the maximum volume of the components

$$M=\max_{\alpha\in\mathscr{I}} |C_\alpha|. \tag{12.7}$$

Let $\{B_\alpha\}_{\alpha\in\mathscr{I}}$ be a partition of Λ into connected components such that $C_\alpha\subset B_\alpha$ and

$$d(C_\alpha, \Lambda\backslash B_\alpha)\geqq \tfrac{1}{2}L_1 \tag{12.8}$$

for all $\alpha\in\mathscr{I}$. If A is a subset of Λ, let $H_\lambda(A)$ denote the operator

$$H_\lambda(A)=\sum_{x\in A} s_x+\sum_{\gamma_0\subset A}\lambda^{|\gamma_0|_c-1}t_{\gamma_0} \tag{12.9}$$

and let $E_{0,\lambda}(A)$ be its ground state energy. Let g_α denote the energy gap between the ground state and the first excited state of $H_\lambda(B_\alpha)$ and let

$$g=\inf_{\alpha\in\mathscr{I}} g_\alpha. \tag{12.10}$$

The kind of statements that are relevant in this part hold under conditions of the form

$$\lambda\leqq c, \quad L_1\geqq F(g,M), \tag{12.11}$$

where the constant c and the function $F(g, M)$ do not depend on Λ.

If $A \subset \Lambda$ is a subset, let $H_\lambda^{\mathrm{reg}}(A)$ be the operator

$$H_\lambda^{\mathrm{reg}}(A) = \sum_{x \in A \setminus \mathscr{S}} s_x + \sum_{x \in A \cap \mathscr{S}} (\Lambda - P_{|0\rangle_x}) + \sum_{\gamma_0 \subset A} \lambda^{|\gamma_0|_c - 1} t_{\gamma_0}, \tag{12.12}$$

and let $E_{0,\lambda}^{\mathrm{reg}}(A)$ be its ground state energy. Let $U_\lambda(A)$ denote the unitary dressing transformation for $H_\lambda^{\mathrm{reg}}(A)$ constructed as in Part I. Let L_0 be a constant to be fixed later so that $L_0 \ll L_1$ and let

$$\bar{C}_\alpha = \{x \in \Lambda \,|\, d(x, C_\alpha) \leqq L_0\} \tag{12.13}$$

and

$$\bar{\mathscr{S}} = \bigcup_{\alpha \in \mathscr{I}} \bar{C}_\alpha.$$

Thanks to Condition b above, we have

$$\bar{M} \equiv \max_\alpha |\bar{C}_\alpha| \leqq M c_0^{L_0}. \tag{12.14}$$

Let $|0\rangle_\alpha \in \mathscr{H}(\bar{C}_\alpha)$ be the ground state of the operator

$$U_\lambda(\bar{C}_\alpha)^{-1} H_\lambda(\bar{C}_\alpha) U_\lambda(\bar{C}_\alpha). \tag{12.15}$$

Finally, let

$$|u_0\rangle = \left(\bigotimes_{\alpha \in \mathscr{I}} |0\rangle_\alpha\right) \otimes |0_{\Lambda \setminus \bar{\mathscr{S}}}\rangle. \tag{12.16}$$

The state $|u_0\rangle$ will be taken as the starting point in the construction of the ground state of the operator (12.1). More precisely, a unitary operator $\hat{U}$ is constructed such that $\hat{U}|u_0\rangle$ is the ground state of (12.1).

$\hat{U}$ enjoys some clustering properties that are important to establish its existence and that are useful in applications. To describe them, it is convenient to consider the graph $\bar{\Lambda}$ whose vertices form the set $(\Lambda \setminus \bar{\mathscr{S}}) \cup \mathscr{I}$. Let us denote with symbols like $\bar{x}, \bar{y}$ the vertices of $\bar{\Lambda}$. Two vertices $\bar{x}, \bar{y} \in \bar{\Lambda}$ are joined by a line in the following cases

(i) $\bar{x}, \bar{y} \in \Lambda \setminus \bar{\mathscr{S}}$ and $\bar{x}, \bar{y}$ are joined by a line in Λ;

(ii) $\bar{x} \in \Lambda \setminus \mathscr{S}$, $\bar{y} \in \mathscr{I}$ and $\bar{x}$ is joined by a line in Λ to some sites of $\bar{C}_{\bar{y}}$.

One can define a quantum spin system on $\bar{\Lambda}$ that is equivalent to our quantum spin system originally defined on Λ. To this end, let us associated to each site $\bar{x} \in \bar{\Lambda}$ the Hilbert space $\mathscr{H}_{\bar{x}} = \mathbb{C}^{N_x + 1}$ if $\bar{x} = x \in \bar{\Lambda} \setminus \bar{\mathscr{S}}$, and the space $\mathscr{H}_{\bar{x}} = \mathscr{H}(\bar{C}_{\bar{x}})$ if $\bar{x} \in \mathscr{I}$. We have

$$\mathscr{H}(\Lambda) = \bigotimes_{x \in \Lambda} \mathbb{C}^{N_x + 1} = \bigotimes_{\bar{x} \in \bar{\Lambda}} \mathscr{H}_{\bar{x}}. \tag{12.17}$$

Let us fix a basis

$$|0\rangle_{\bar{x}}, \ldots, |N_{\bar{x}}\rangle_{\bar{x}} \tag{12.18}$$

of $\mathscr{H}_{\bar{x}}$ for all $\bar{x} \in \bar{\Lambda}$. If $\bar{x} = x \in \bar{\Lambda} \setminus \bar{\mathscr{S}}$, then (12.18) is the same basis (1.3) in which $s_{\bar{x}}$ is diagonal. If $\bar{x} \in \mathscr{I}$, the vectors (12.18) are the eigenstates of the operator

$$U_\lambda(\bar{C}_x)^{-1} H_\lambda(\bar{C}_x) U_\lambda(\bar{C}_x) \tag{12.19}$$

arranged in order of increasing energy. Let us remark that

$$|u_0\rangle = \bigotimes_{\bar{x}\in\bar{\Lambda}} |0\rangle_{\bar{x}}. \tag{12.20}$$

Let $\|\cdot\|_{2,1}$ denote the norm such that, if a state u is written in the form

$$u = \sum_{\gamma: s(\gamma)\subset \sim\bar{\mathscr{S}}} \phi_\gamma \otimes |\gamma\rangle, \tag{12.21}$$

then

$$\|u\|_{2,1} = \sum_{\gamma: s(\gamma)\subset \sim\bar{\mathscr{S}}} \|\phi_\gamma\|_2. \tag{12.22}$$

We suppose that

$$L_0 = |\log c\lambda|^{-1}(c\log cM + c|\log cg|), \tag{12.23}$$

and $L_0 < \frac{1}{2}L_1$, so that by virtue of Theorem 7.1 there exists a unitary operator $\exp(\bar{R}_\alpha)$ in $\mathscr{H}(B_\alpha)$, for all $\alpha\in\mathscr{I}$, such that $|0\rangle_\alpha$ is the ground state of

$$e^{-\bar{R}_\alpha} U_\lambda(B_\alpha)^{-1} H_\lambda(B_\alpha) U_\lambda(B_\alpha) e^{\bar{R}_\alpha}. \tag{12.24}$$

Let us remark that if $\lambda \leqq c$, we have

$$L_0 \leqq (\log c_0)^{-1}(\log cM + |\log cg|), \tag{12.25}$$

where c_0 is the constant in (12.14). Hence, since $g \leqq 1$, we have

$$\bar{M} \leqq cg^{-1}M^2. \tag{12.26}$$

The operator $\hat{U}$ we construct has the form

$$\hat{U} = U_\lambda(\Lambda)\left(\prod_{\alpha\in\mathscr{I}} e^{\bar{R}_\alpha}\right) \lim_{v\to\infty} e^{\hat{R}^1}\cdots e^{\hat{R}^v}. \tag{12.27}$$

If

$$\hat{R}^v = \sum_{\gamma_0\subset\hat{\Lambda}} \hat{r}^v(\hat{\gamma}_0) \tag{12.28}$$

is the expansion of the operator $\hat{R}^v$ into a sum of operators $\hat{r}^v(\bar{\gamma}_0)$ with support $\bar{\gamma}_0$, let us introduce the following quantity:

$$\hat{r}^*(Z) = \sup_{\bar{x}\in\bar{\Lambda}} \sum_{\bar{\gamma}_0: \bar{x}\in\bar{\gamma}_0} \sum_{v=1}^{\infty} Z^{|\bar{\gamma}_0|_c} |\bar{\gamma}_0|_c \|\hat{r}^v(\bar{\gamma}_0)|u_0\rangle\|_{2,1}; \tag{12.29}$$

$\hat{r}^*(Z)$ measures the size of the local deviations of the ground state of (12.1) from the unperturbed ground state of the partially decoupled problem which is given by

$$U_\lambda(\Lambda)\left(\sum_{\alpha\in\mathscr{I}} e^{\bar{R}_\alpha}\right)|u_0\rangle. \tag{12.30}$$

We have

Theorem 12.1. *If* $\lambda < c$,

$$L_1 - L_0 \geqq |\log c\lambda|^{-1}(c + |\log g| + \log(\bar{M}N^{\bar{M}})), \tag{12.31}$$

where $\bar{M}$ is defined in (12.14) and satisfies (12.26), and

$$1 \leqq Z \leqq c\lambda^{-1/16}, \tag{12.32}$$

then there exists a unitary operator $\hat{U}$ of the form (12.27) such that $\hat{U}|u_0\rangle$ is the ground state of the operator (12.1) and

$$\not{r}^*(Z) \leqq (c\lambda)^{(1/24)(L_1 - L_0)}. \tag{12.33}$$

The ground state energy is separated by a gap $\geqq \frac{1}{4}g$ from the rest of the spectrum. Moreover, if $\mathcal{O}_{\bar{\gamma}_1}$ and $\mathcal{O}_{\bar{\gamma}_2}$ are two operators of $L^{2,1}$-operator norm 1 and with support $\bar{\gamma}_1$ and $\bar{\gamma}_2$, respectively, then we have

$$|\langle \hat{U}u_0|\mathcal{O}_{\bar{\gamma}_1}\mathcal{O}_{\bar{\gamma}_2}|\hat{U}u_0\rangle - \langle \hat{U}u_0|\mathcal{O}_{\bar{\gamma}_1}|\hat{U}u_0\rangle\langle \hat{U}u_0|\mathcal{O}_{\bar{\gamma}_2}|\hat{U}u_0\rangle| \leqq (c\lambda)^{(1/16)d_c(\bar{\gamma}_1,\bar{\gamma}_2)}. \tag{12.34}$$

13. Preliminary Constructions

In this section, I perform some preliminary constructions that reduce the ground state problem to a conjugacy problem for a unitary operator. This problem is solved by a unitary dressing transformation with good clustering properties that is formally defined in this section and whose existence is proven in Sects. 15, 16 and 17.

Thanks to Theorem 7.1, if

$$L_0 \geqq c\frac{\log cM}{|\log c\lambda|} + c\frac{|\log cg|}{|\log c\lambda|} \tag{13.1}$$

and $L_0 < \frac{1}{2}L_1$, for all $\alpha \in \mathcal{T}$ we can construct a skew-symmetric operator $\bar{R}_\alpha(\lambda)$ acting on $\mathcal{H}(B_\alpha)$ such that

$$U_\lambda(B_\alpha)e^{\bar{R}_\alpha(\lambda)}(|0\rangle_\alpha \otimes |0_{B_\alpha \setminus \bar{C}_\alpha}\rangle) \tag{13.2}$$

is the ground state of $H_\lambda(B_\alpha)$. Here, $|0\rangle_\alpha$ is the ground state of $H_\lambda(\bar{C}_\alpha)$. Moreover, we have

$$\|\bar{R}_\alpha(\lambda)\|_{2,1} \leqq (C_0\lambda)^{(1/8)L_0}. \tag{13.3}$$

Let $\bar{U}_\lambda$ be the unitary operator

$$\bar{U}_\lambda = \prod_{\alpha \in \mathcal{I}} e^{\bar{R}_\alpha(\lambda)}. \tag{13.4}$$

It is convenient to study our problem in the representation in which the Hamiltonian is the following selfadjoint operator:

$$\bar{U}_\lambda^{-1}U_\lambda(\Lambda)^{-1}H_\lambda U_\lambda(\Lambda)\bar{U}_\lambda - \sum_\alpha E_{0,\lambda}^{\mathrm{reg}}(B_\alpha). \tag{13.5}$$

The operator (13.5) has to be split into the sum of a main part having $|u_0\rangle$ as ground state, plus a "small" perturbation. The relative bound of the perturbation with respect to the main part is proportional to the number of clusters C_α contained in $\mathcal{S}$. Hence, the relative bound of the perturbation with respect to the main part diverges in the infinite volume limit. However, it is possible to organize the terms of the Hamiltonian so that the perturbation is an operator given by a cluster expansion such that the sum of all the clusters containing one site $\bar{x} \in \bar{\Lambda}$ is exponentially small in the parameter L_1 in (12.8), uniformly in $\bar{x}\in\bar{\Lambda}$. This suffices to set up a convergent perturbative expansion for the ground state.

Let us introduce the operators S_α, $V_\alpha(\lambda)$ and $W_\alpha(\lambda)$ so that

$$U_\lambda(B_\alpha)^{-1}H_\lambda(B_\alpha)U_\lambda(B_\alpha)-E^{\mathrm{reg}}_{0,\lambda}(B_\alpha)=S_\alpha+V_\alpha(\lambda)+W_\alpha(\lambda), \tag{13.6}$$

where

$$S_\alpha=\sum_{x\in B_\alpha}s_x, \tag{13.7}$$

$$V_\alpha(\lambda)=U_\lambda(B_\alpha)^{-1}H^{\mathrm{reg}}_\lambda(B_\alpha)U_\lambda(B_\alpha)-E^{\mathrm{reg}}_{0,\lambda}(B_\alpha)-\sum_{x\in B_\alpha\backslash C_\alpha}s_x-\sum_{x\in C_\alpha}(1-P_{|0\rangle_x}) \tag{13.8}$$

and $W_\alpha(\lambda)$ is the remainder. Let $\overline{W_\alpha}(\lambda)$ be the operator such that

$$e^{-\bar{R}_\alpha(\lambda)}(S_\alpha+V_\alpha(\lambda)+W_\alpha(\lambda))e^{\bar{R}_\alpha(\lambda)}=S_\alpha+V_\alpha(\lambda)+\bar{W}_\alpha(\lambda), \tag{13.9}$$

and let us introduce the following notations:

$$\tilde{S}_\alpha(\lambda)\equiv S_\alpha+V_\alpha(\lambda)+\bar{W}_\alpha(\lambda) \tag{13.10}$$

and

$$\tilde{S}(\lambda)=\sum_{\alpha\in\mathscr{I}}\tilde{S}_\alpha(\lambda). \tag{13.11}$$

We have

$$\tilde{S}_\alpha(\lambda)|0\rangle_\alpha=0. \tag{13.12}$$

Let $V_{\sim\bar{\mathscr{S}}}(\lambda)$ be the operator

$$V_{\sim\bar{\mathscr{S}}}(\lambda)=U_\lambda(\sim\bar{\mathscr{S}})^{-1}H_\lambda(\sim\bar{\mathscr{S}})U_\lambda(\sim\bar{\mathscr{S}})-E_{0,\lambda}(\sim\bar{\mathscr{S}})-\sum_{x\in\sim\bar{\mathscr{S}}}s_x, \tag{13.13}$$

and let

$$V_{B_\alpha\backslash\bar{C}_\alpha}(\lambda)=U_\lambda(B_\alpha\backslash\bar{C}_\alpha)^{-1}H_\lambda(B_\alpha\backslash\bar{C}_\alpha)U_\lambda(B_\alpha\backslash\bar{C}_\alpha)-E_{0,\lambda}(B_\alpha\backslash\bar{C}_\alpha)-\sum_{x\in B_\alpha\backslash\bar{C}_\alpha}s_x. \tag{13.14}$$

Let us define the boundary operator $V_\Gamma(\lambda)$ as follows:

$$V_\Gamma(\lambda)=V_{\sim\bar{\mathscr{S}}}(\lambda)-\sum_{\alpha\in\mathscr{I}}V_{B_\alpha\backslash\bar{C}_\alpha}(\lambda). \tag{13.15}$$

The first basic property of $V_\Gamma(\lambda)$ is that it is equal to the boundary V-operator on the full space

$$V(\lambda)-\sum_{\alpha\in\mathscr{I}}V_{B_\alpha}(\lambda) \tag{13.16}$$

up to corrections exponentially small in (L_1-L_0). The second property is that, unlike the operator (13.16), $V_\Gamma(\lambda)$ is zero on all states with support in $\bar{\mathscr{S}}$. In particular, we have

$$V_\Gamma(\lambda)|u_0\rangle=0. \tag{13.18}$$

The operator (13.5) is equal to the following operator for $\beta=1$:

$$(\tilde{S}(\lambda)+V_\Gamma(\lambda))+\sum_{n_0=L_1-L_0}\beta^{[n_0/(L_1-L_1)]}W_{\Gamma,n_0}(\lambda) \tag{13.19}$$

where the operators $W_{\Gamma,n_0}(\lambda)$ are such that

$$\sum_{n_0=L_1-L_2}^{\infty} W_{\Gamma,n_0}(\lambda) = \bar{U}_\lambda^{-1}\left[W(\lambda) - \sum_\alpha W_\alpha(\lambda)\right]\bar{U}_\lambda + \bar{U}_\lambda^{-1}\left[V(\lambda) - \sum_\alpha V_\alpha(\lambda)\right]\bar{U}_\lambda - V_\Gamma(\lambda). \tag{13.20}$$

The definition of $W_{\Gamma,n_0}(\lambda)$ can be given as follows. The operator on the right hand side of (13.20) can be expressed as the sum of commutators of the form

$$[\cdots[[\cdots s_x, R_{i_1}]\cdots R_{i_k}], \bar{R}_{j_1}], \cdots \bar{R}_{j_l}] \tag{13.21}$$

and of the form

$$[\cdots[[\cdots[t_{\gamma_0}, R_{i_1}], \cdots R_{i_k}]\bar{R}_{j_1}], \cdots \bar{R}_{j_l}]. \tag{13.22}$$

$W_{\Gamma,n_0}(\lambda)$ is the sum of all the terms given by commutators of the form (13.21) with

$$i_1 + \cdots + i_k + j_1 + \cdots + j_l = n_0, \tag{13.23}$$

or by commutators of the form (13.22) with

$$|\gamma_0|_c + i_1 + \cdots + i_k + j_1 + \cdots + j_l = n_0. \tag{13.24}$$

The sum starts from $(L_1 - L_0)$ because this is the lowest possibly nonvanishing order in the expansion of the second term of the right-hand side of (13.20). The first term gives contribution only to terms of order $n_0 \geqq L_1$.

Let us expand the operator $W_{\Gamma,n_0}(\lambda)$ as follows

$$W_{\Gamma,n_0}(\lambda) = \sum_{\bar{\gamma}_0 \subset \bar{\Lambda}} w_{\Gamma,n_0}(\bar{\gamma}_0), \tag{13.25}$$

where $w_{\Gamma,n_0}(\bar{\gamma}_0)$ is an operator with support $\bar{\gamma}_0 \subset \bar{\Lambda}$. If $n_0 < |\bar{\gamma}_0|_c$, we have $w_{\Gamma,n_0}(\bar{\gamma}_0) = 0$. Moreover, thanks to the estimates in Parts I and III, we have

$$\sup_{\bar{x}\in\bar{\Lambda}} \sum_{\bar{\gamma}_0:\bar{x}\in\bar{\gamma}_0} \| w_{\Gamma,n_0}(\bar{\gamma}_0)\|_{2,1} \leqq (c\lambda)^{(1/4)n_0}. \tag{13.26}$$

Hence, the β-dependent part of (13.19) is a perturbation locally small with respect to g^{-1} and MN^M that, as discussed in the following, are the two large factors to be killed. To construct the ground state of (13.19), one can start from $|u_0\rangle$ that, as proven in Sect. 14, is the ground state of $(\tilde{S}(\lambda) + V_\Gamma(\lambda))$. Since the relative bound of the perturbation in (13.19) with respect to $(\tilde{S}(\lambda) + V_\Gamma(\lambda))$ is proportional to the volume $|\Lambda|$, a technique based on dressing transformations is necessary.

Let us consider the following conjugacy problem:

$$\hat{U}_\lambda(\beta)^{-1}\left[(\tilde{S}(\lambda) + V_\Gamma(\lambda)) + \sum_{j=1}^{\infty} \sum_{n_0=j(L_1-L_0)}^{(j+1)(L_1-L_0)-1} \beta^j W_{\Gamma,n_0}(\lambda)\right]\hat{U}_\lambda(\beta)|u_0\rangle = \hat{E}_{0,\lambda}(\beta)|u_0\rangle, \tag{13.27}$$

where the unknown operator $\hat{U}_\lambda(\beta)$ has to be unitary for β real and analytic for $|\beta| \leqq 1$. It is convenient to study this problem on the lattice $\bar{\Lambda}$ introduced in Sect. 12. One can obtain a unique solution by imposing some restrictions on the form of

$\hat{U}_\lambda(\beta)$. Let us choose to look for an operator of the form

$$\hat{U}_\lambda(\beta) = \lim_{v\to\infty} e^{\hat{R}^v_\lambda(\beta)} \cdots e^{\hat{R}^1_\lambda(\beta)}, \tag{13.28}$$

where the operators $\hat{R}^v_\lambda(\beta)$, $v = 1, 2, \ldots$, are skew-adjoint for β real and analytic for $|\beta| \leqq 1$. They can be expanded in power series in β

$$\hat{R}^v_\lambda(\beta) = \sum_{n=1}^{\infty} \hat{R}^v_n(\lambda)\beta^n, \tag{13.29}$$

where

$$\hat{R}^v_n = \sum_{|s(\bar{\gamma})|=v} \hat{r}_{n\bar{\gamma}}(\lambda)\tau_{\bar{\gamma}}. \tag{13.30}$$

Here and in the following, all sums over $\bar{\gamma}$ run over all excitations in $\bar{\Lambda}$. The operators $\tau_{\bar{\gamma}}$ for excitations $\bar{\gamma}$ in $\bar{\Lambda}$ are defined exactly as the operators τ_γ for excitations γ in Λ are defined in Sect. 2.

The coefficients $\hat{r}_{n\bar{\gamma}}(\lambda)$ are uniquely determined by the condition

$$\sum_{v=1}^{\infty} \hat{R}^v_1(\lambda)|u_0\rangle = K(\lambda)W_\Gamma(\lambda)|u_0\rangle \tag{13.31}$$

if $n = 1$, and by the following recurrence relations if $n \geqq 2$:

$$\begin{aligned}\sum_{v=1}^{\infty} \hat{R}^v_n(\lambda)|u_0\rangle = K(\lambda)\Bigg\{ &\sum_{\substack{v_1 \leqq \cdots \leqq v_k \\ i_1+\cdots+i_k=n \\ k\geqq 2}} \frac{1}{(v)!}[\cdots[(\tilde{S}(\lambda)+V_\Gamma(\lambda)), \hat{R}^{v_1}_{i_1}(\lambda)]\cdots\hat{R}^{v_k}_{i_k}(\lambda)] \\ &+ \sum_{\substack{v_1 \leqq \cdots \leqq v_k \\ i_1+\cdots+i_k=n-1}} \frac{1}{(v)!}[\cdots[W_\Gamma(\lambda), \hat{R}^{v_1}_{i_1}(\lambda)]\cdots\hat{R}^{v_k}_{i_k}(\lambda)]\Bigg\}|0\rangle. \end{aligned} \tag{13.32}$$

The operator $K(\lambda)$ in (13.31) and (13.32) is the pole-subtracted Green's function defined as the analytic continuation to $z=0$ of the operator

$$(\tilde{S}(\lambda) + V_\Gamma(\lambda) - z)^{-1}P_{|u_0\rangle}. \tag{13.33}$$

In Sect. 14, it is proven that $|u_0\rangle$ is the ground state of $(\tilde{S}(\lambda) + V(\lambda))$. Its energy is zero and it is separated by a gap $\geqq \frac{1}{2}g$ from the rest of the spectrum of $(\tilde{S}(\lambda) + V(\lambda))$. In particular, the operator $K(\lambda)$ exists and its L^2-operator norm is $\leqq 2g^{-1}$. Section 15 is dedicated to the study of the kernel of the operator $K(\lambda)$. It is shown that, in the basis of the excitations $\bar{\gamma}$ on $\bar{\Lambda}$, the following decay estimate in $L^{2,1}$ norm holds:

$$\sup_{\bar{\gamma}} \left\| \sum_{\bar{\gamma}^1 : \bar{d}_c(\bar{\gamma}', \bar{\gamma}) = k} |\bar{\gamma}'\rangle\langle\bar{\gamma}'|\bar{S}K|\bar{\gamma}\rangle \right\|_{2,1} \leqq g^{-1}(c_0\lambda)^{(1/8)k}, \tag{13.34}$$

where k is any integer $\geqq 0$ and $\bar{S}$ is the operator such that

$$\bar{S}|\bar{\gamma}\rangle = |s(\bar{\gamma})||\bar{\gamma}\rangle. \tag{13.35}$$

Let us introduce also the operator $\bar{S}_c$ such that

$$\bar{S}_c|\bar{\gamma}\rangle = |s(\bar{\gamma})|_c|\bar{\gamma}\rangle \tag{13.36}$$

for all excitations $|\bar{\gamma}\rangle$ in $\bar{\Lambda}$, and the operator

$$Q = \prod_{\alpha \in \mathscr{I}} Q_\alpha, \tag{13.37}$$

where Q_α is such that

$$Q_\alpha |\gamma\rangle = \max(1, N^M (c\lambda)^{(1/16) d(\bar{C}_\alpha, s(\gamma))}) \tag{13.38}$$

for all excitations γ in Λ. N is defined in (12.4) and the constant c appearing in (13.32) is fixed as in Corollary 15.2. Since we are going to assume that $(L_1 - L_0) \geqq cM \log N$, we have

$$s(Q_\alpha) \subset \bar{B}_\alpha. \tag{13.39}$$

In order to control the expansion (13.26), it is convenient to use the following sequence:

$$\hat{r}_n^*(Z) = \sup_{\bar{x} \in \bar{\Lambda}} \| P_{\bar{x}} Q Z^{\bar{S}_c} \bar{S} \hat{R}_n |u_0\rangle \|_{2,1}. \tag{13.40}$$

In Sect. 16, recurrence inequalities for this sequence are proved that allow us in Sect. 17 to conclude the proof of Theorem 12.1.

14. The Gap of $\tilde{S}(\lambda) + V_\Gamma(\lambda)$

The ground state $|u_0\rangle$ of $\tilde{S}(\lambda)$ has support in $\bar{\mathscr{S}}$ and, thus, it is annihilated by $V_\Gamma(\lambda)$. In this section, we show that $|u_0\rangle$ is also the ground state of $\tilde{S}(\lambda) + V_\Gamma(\lambda)$. Namely, we have

Lemma 14.1. *$|u_0\rangle$ is the nondegenerate ground state of $\tilde{S}(\lambda) + V_\Gamma(\lambda)$ and it is separated by a gap $\geqq \frac{1}{2} g$ from the rest of the spectrum.*

Remark. The proof of this lemma is a little delicate because the relative bound of $V_\Gamma(\lambda)$ with respect to $\tilde{S}(\lambda)$ is of order λ, but we are not supposing that $\lambda \ll g$. The situation here is similar to the one met in Part III. The boundary perturbation $V_\Gamma(\lambda)$ is able to induce transitions with large probability amplitude, only among states with excitations for away from $\mathscr{S}$. But the energy of such states is of the order of the number of the excitations. Hence, these couplings can be controlled with a relative boundedness estimate and are not associated to small divisors. On the other hand, states with low energy give rise to small divisors, but, in the dressed representation, they are exponentially close to the unperturbed vacuum far from S. Hence, they are almost annihilated by $V_\Gamma(\lambda)$ and the small divisors are outweighted by factors of order $(c\lambda)^{L_1}$.

Proof of Lemma 14.1. The paper [2] contains an idea that is useful also in the present situation. Namely, let $\mathscr{C}$ be the circle

$$\mathscr{C} = \{ z \in \mathbb{C} \,|\, |z| = \tfrac{1}{2} g \} \tag{14.1}$$

and let us consider the spectral projection

$$P_\delta = \oint_{\mathscr{C}} \frac{dz}{2\pi i} (z - \tilde{S}(\lambda) - \delta V_\Gamma(\lambda))^{-1} \tag{14.2}$$

for $\delta \in [0, 1]$. It suffices to prove that

$$\dim \operatorname{Ran} P_\delta = 1 \tag{14.3}$$

for all $\delta \in [0, 1]$. In fact, $|u_0\rangle$ is an eigenstate with eigenvalue zero of $\tilde{S}(\lambda) + \delta V_\Gamma(\lambda)$ for all δ. In turn, (14.3) follows from the following estimate in L^2-operator norm:

$$\| P_\delta - P_0 \|_2 < 1, \tag{14.4}$$

holding for all $\delta \in [0, 1]$.

Let $\Pi^{<1/2}$ be the orthogonal projection onto the space spanned by the eigenfunctions of $\tilde{S}(\lambda)$ with energy $< \frac{1}{2}$ and let $\Pi^{\geqq 1/2} = 1 - \Pi^{<1/2}$. We have

$$\begin{aligned} \| P_\delta - P_0 \|_2 &\leqq \| (P_\delta - P_0)\Pi^{<1/2} \|_2 + \| \Pi^{<1/2}(P_\delta - P_0)\Pi^{\geqq 1/2} \|_2 \\ &\quad + \| \Pi^{\geqq 1/2}(P_\delta - P_0)\Pi^{\geqq 1/2} \|_2 \\ &\leqq 2 \| (P_\delta - P_0)\Pi^{<1/2} \|_2 + \| \Pi^{\geqq 1/2}(P_\delta - P_0)\Pi^{\geqq 1/2} \|_2 . \end{aligned} \tag{14.5}$$

Hence, to prove (14.4) it suffices to bound separately the two terms in (14.5).

Let us expand $(P_\delta - P_0)$ in geometric series

$$(P_\delta - P_0) = \sum_{j=1}^{\infty} \delta^j \oint_{\mathscr{C}} \frac{dz}{2\pi i} (z - \tilde{S}(\lambda))^{-1} [V_\Gamma(\lambda)(z - \tilde{S}(\lambda))^{-1}]^j . \tag{14.6}$$

We have

$$\begin{aligned} &\| (P_\delta - P_0)\Pi^{<1/2} \|_2 \\ &\leqq \sup_{z \in \mathscr{C}} \Big\{ \| (z - \tilde{S}(\lambda))^{-1} \|_2^2 \| V_\Gamma(\lambda)\Pi^{<1/2} \|_2 \\ &\quad \cdot \sum_{j=1}^{\infty} \| (z - \tilde{S}(\lambda))^{1/2} V_\Gamma(\lambda)(z - \tilde{S}(\lambda))^{-1/2} \|_2^{j-1} \Big\} \\ &\leqq 4g^{-2}(c_0\lambda)^{L_1 - L_0} \sum_{j=1}^{\infty} \sup_{z \in \mathscr{C}} \| (z - \tilde{S}(\lambda))^{-1/2} V_\Gamma(\lambda)(z - \tilde{S}(\lambda))^{-1/2} \|_2^{j-1}, \end{aligned} \tag{14.7}$$

where Lemma 8.1 is used, and

$$\begin{aligned} &\| \Pi^{\geqq 1/2}(P_\delta - P_0)\Pi^{\geqq 1/2} \|_2 \\ &\leqq \frac{1}{2} g \sup_{z \in \mathscr{C}} \Big\{ \| (z - \tilde{S}(\lambda))^{-1/2} \Pi^{\geqq 1/2} \|_2^2 \sum_{j=1}^{\infty} \| (z - \tilde{S}(\lambda))^{-1/2} V_\Gamma(\lambda)(z - \tilde{S}(\lambda))^{-1/2} \|_2^j \Big\} \\ &\leqq \sum_{j=1}^{\infty} \sup_{z \in \mathscr{C}} \| (z - \tilde{S}(\lambda))^{-1/2} V_\Gamma(\lambda)(z - \tilde{S}(\lambda))^{-1/2} \|_2^j , \end{aligned} \tag{14.8}$$

where (12.10) is used. Hence, it suffices to verify that the relative form bound of $V_\Gamma(\lambda)$ with respect to $\tilde{S}(\lambda)$ is $\leqq c\lambda$. Since

$$\begin{aligned} &\| (z - \tilde{S}(\lambda))^{-1/2} V_\Gamma(\lambda)(z - \tilde{S}(\lambda))^{-1/2} \|_2 \\ &\leqq \| \tilde{S}(\lambda)^{1/2}(z - \tilde{S}(\lambda))^{-1/2} \|_2^2 \| \tilde{S}(\lambda)^{-1/2} V_\Gamma(\lambda)\tilde{S}(\lambda)^{-1/2} \|_2 \\ &\leqq 2 \| \tilde{S}(\lambda)^{-1/2} V_\Gamma(\lambda)\tilde{S}(\lambda)^{-1/2} \|_2 , \end{aligned} \tag{14.9}$$

we are reduced to prove the following bound:

$$|\langle u|V_\Gamma(\lambda)|u\rangle| \leqq (c\lambda)\langle u|\tilde{S}(\lambda)|u\rangle. \tag{14.10}$$

This result can be established by means of the techniques developed in Sect. 6 and the details are omitted.

15. Decay of the Pole-Subtracted Green's Function K

This section in dedicated to the study of the decay of the kernel of the pole-subtracted Green's function.

$$K(\lambda) = (\tilde{S}(\lambda) + V_\Gamma(\lambda))^{-1}\Pi_0. \tag{15.1}$$

Hence

$$\Pi_0 = 1 - P_{|u_0\rangle} \tag{15.2}$$

and $P_{|u_0\rangle}$ is the orthogonal projection along $|u_0\rangle$, the ground state of $\tilde{S}(\lambda) + V_\Gamma(\lambda)$. Since in Sect. 14 it is shown that $|u_0\rangle$ is a nondegenerate eigenstate, $K(\lambda)$ is well defined by (15.1). Moreover, from Lemma 14.1 follows that the L^2-operator norm of $K(\lambda)$ is $\leqq 2g^{-1}$. This section contains two bounds on the exponential decay of the kernel of $K(\lambda)$ with respect to the $L^{2,1}$-norm.

Lemma 15.1. *For all integers $k \geqq 0$, we have*

$$\sup_{\bar{\gamma}_0 \subset \bar{\Lambda}} \sup_{\substack{u \in \mathscr{H}(\bar{\gamma}_0) \\ \|u\|_{2,1}=1}} \left\| \sum_{\bar{\gamma}' : d_c(\bar{\gamma}_0, S(\bar{\gamma}'))=k} |\bar{\gamma}'\rangle\langle\bar{\gamma}'|\tilde{S}K(\lambda)(u \otimes |\bar{0}_{\sim\bar{\gamma}_0}\rangle \right\|_{2,1} \leqq g^{-1}(c\lambda)^{(1/8)k}. \tag{15.3}$$

Corollary 15.2. *If the constant c in the definition (13.38) is chosen to be equal to the constant c in (15.3), then for all $|Z \geqq 1$ and all integers $k \geqq 0$, we have*

$$\sup_{\bar{\gamma}_0 \subset \bar{\Lambda}} \sup_{\substack{u \in \mathscr{H}(\bar{\gamma}_0) \\ \|u\|_{2,1}=1}} \left\| \sum_{\bar{\gamma}' : d_c(\bar{\gamma}_0, S(\bar{\gamma}'))=k} |\bar{\gamma}'\rangle\langle\bar{\gamma}'|Z^{\bar{S}_c}Q\tilde{S}K(\lambda)Q^{-1}Z^{-\bar{S}_c}(u \otimes |\bar{0})_{\sim\bar{\gamma}_0}\rangle) \right\|_{2,1}$$
$$\leqq g^{-1}Z^k(c\lambda)^{(1/16)k}. \tag{15.4}$$

Notation. In the following, the dependency of operators on λ is not explicitly denoted.

Proof of Lemma 15.1. Let us split the operator $\tilde{S} + V_\Gamma$ as follows:

$$\tilde{S} + V_\Gamma = \tilde{S}_{\mathscr{S}} + S_{\sim\mathscr{S}} + \sum_\alpha V_{\partial\bar{C}_\alpha} + \sum_\alpha W_\alpha + V_\Gamma, \tag{15.5}$$

where

$$V_{\partial\bar{C}_\alpha} = V_{B_\alpha} - V_{\bar{C}_\alpha} - V_{B_\alpha \setminus \bar{C}_\alpha}, \tag{15.6}$$

and

$$\bar{W}_\alpha = \bar{U}_\lambda^{-1}(\tilde{S}_{\bar{C}_\alpha} + S_{B_\alpha\setminus\bar{C}_\alpha} + V_{\partial\bar{C}_\alpha})\bar{U}_\lambda - (\tilde{S}_{\bar{C}_\alpha} + S_{B_\alpha\setminus\bar{C}_\alpha} + V_{\partial\bar{C}_\alpha}) + \bar{U}_\lambda^{-1}(W_{B_\alpha} - W_{\bar{C}_\alpha})\bar{U}_\lambda. \tag{15.7}$$

By construction, we have

$$(V_{\partial\bar{C}_\alpha} + \bar{W}_\alpha)|0_\alpha\rangle = 0, \tag{15.8}$$

where $|0_\alpha\rangle$ is the ground state of $\tilde{S}_{\bar{C}_\alpha} + S_{B_\alpha \setminus \bar{C}_\alpha}$.

Let us expand $V_{\partial\bar{C}_\alpha}$ in sum of operators $v_{\partial\bar{C}_\alpha}(\bar{\gamma}_0)$ with support $\bar{\gamma}_0 \subset (B_\alpha \setminus \bar{C}_\alpha) \cup \{\alpha\} \subset \bar{\Lambda}$:

$$V_{\partial\bar{C}_\alpha} = \sum_{\bar{\gamma}_0} v_{\partial\bar{C}_\alpha}(\bar{\gamma}_0), \tag{15.9}$$

and let us define $\overline{\mathrm{ad}}\, v_{\partial\bar{C}_\alpha}(\bar{\gamma}_0)$ as the operator acting as follows on the basis of excitations $|\bar{\gamma}\rangle$:

$$\overline{\mathrm{ad}}\, v_{\partial\bar{C}_\alpha}(\bar{\gamma}_0)|\bar{\gamma}\rangle = \begin{cases} [v_{\partial\bar{C}_\alpha}(\bar{\gamma}_0), \tau_{\bar{\gamma}}]|0_\alpha\rangle & \text{if} \quad \alpha \notin s(\bar{\gamma}), \\ v_{\partial\bar{C}_\alpha}(\bar{\gamma}_0)|\bar{\gamma}\rangle & \text{if} \quad \alpha \in s(\bar{\gamma}). \end{cases} \tag{15.10}$$

Analogously one can define $\bar{w}_\alpha(\bar{\gamma}_0)$ and $\overline{\mathrm{ad}}\, \bar{w}_\alpha(\bar{\gamma}_0)$. Thanks to (15.8), we have

$$V_{\partial\bar{C}_\alpha} + \bar{W}_\alpha = \sum_{\bar{\gamma}_0} \overline{\mathrm{ad}}\, v_{\partial\bar{C}_\alpha}(\bar{\gamma}_0) + \overline{\mathrm{ad}}\, \overline{w_\alpha}(\bar{\gamma}_0). \tag{15.11}$$

Let us remark that the operators $\overline{\mathrm{ad}}\, v_{\partial\bar{C}_\alpha}(\bar{\gamma}_0)$ and $\overline{\mathrm{ad}}\, \overline{w_\alpha}(\bar{\gamma}_0)$ have nonvanishing matrix elements only among excitations $|\bar{\gamma}\rangle, |\bar{\gamma}'\rangle$ with

$$d_c(s(\bar{\gamma}), s(\bar{\gamma}')) \leqq |\bar{\gamma}_0|_c. \tag{15.12}$$

Let $T^{(k)}$ be the operator

$$T^{(k)} = \sum_{\substack{\gamma_0 \subset \Lambda \setminus \mathscr{S} \\ |\gamma_0|_c = k}} \overline{\mathrm{ad}}\, v_\Gamma(\gamma_0) + \sum_\alpha \sum_{|\bar{\gamma}_0|_c = k} \overline{\mathrm{ad}}\, v_{\partial\bar{C}_\alpha}(\bar{\gamma}_0) + \overline{\mathrm{ad}}\, \overline{w_\alpha}(\bar{\gamma}_0), \tag{15.13}$$

where $k \geqq 2$ is an integer.

Lemma 15.3. *We have*

$$\| T^{(k)} u \|_{2,1} \leqq (c\lambda)^{(1/4)\bar{k}} \| (\tilde{S} + S_{\sim\mathscr{S}}) u \|_{2,1} \tag{15.14}$$

for all integers $k \geqq 0$ and all states $u \in \mathscr{H}(\Lambda)$ orthogonal to u_0. Here $\bar{k} = \max(k, 1)$.

Proof of Lemma 15.3. Let us expand $u \in \mathscr{H}(\Lambda)$ as follows:

$$u = \sum_{\bar{\gamma}} u_{\bar{\gamma}} |\bar{\gamma}\rangle. \tag{15.15}$$

Since we have

$$\| (\tilde{S} + S_{\sim\mathscr{S}}) u \|_{2,1} = \sum_{\bar{\gamma}} |u_{\bar{\gamma}}| \, \| (\tilde{S} + S_{\sim\mathscr{S}}) |\bar{\gamma}\rangle \|_{2,1}, \tag{15.16}$$

it is enough to prove (15.14) in case u is an excitation $|\bar{\gamma}\rangle$.

Let $\Pi_\alpha^{<1/2}$ be the orthogonal projection onto the space spanned by the eigenstates of $\tilde{S}_{\bar{C}_\alpha} + S_{B_\alpha \setminus \bar{C}_\alpha}$ with energy $< 1/2$ and let $\Pi_\alpha^{\geqq 1/2} = 1 - \Pi_\alpha^{<1/2}$. Thanks to Lemma 8.1 we have

$$\begin{aligned} \sum_{|\bar{\gamma}_0|_c = k} \| \overline{\mathrm{ad}}\, v_{\partial\bar{C}_\alpha}(\bar{\gamma}_0) \Pi_\alpha^{<1/2} u \|_{2,1} &\leqq (c\lambda)^{(1/4)(L_0 + k)} \\ &\leqq g(c\lambda)^{(1/4)k} \leqq (c\lambda)^{(1/4)k} \| (\tilde{S}_{\bar{C}_\alpha} + S_{B_\alpha \setminus \bar{C}_\alpha}) \Pi_\alpha^{<1/2} u \|_{2,1} \end{aligned} \tag{15.17}$$

for all states $u\in\mathscr{H}(B_\alpha)$ with $u\perp|0_\alpha\rangle$. Moreover, by virtue of Lemma 9.1, we have

$$\sum_{|\bar{\gamma}_0|_c=k} \|\overline{\mathrm{ad}}\, v_{\partial\bar{C}_\alpha}(\bar{\gamma}_0)\Pi_\alpha^{\geqq 1/2}u\|_{2,1} \leqq (c\lambda)^{(1/4)k}\|(\tilde{S}_{\bar{C}_\alpha}+S_{B_\alpha\setminus\bar{C}_\alpha})\Pi_\alpha^{\geqq 1/2}u\|_{2,1}. \quad (15.18)$$

Hence, we have

$$\sum_{\alpha\in s(\bar{\gamma})\cup\mathscr{I}}\sum_{|\gamma_0|_c=k} \|\overline{\mathrm{ad}}\, v_{\partial\bar{C}_\alpha}(\bar{\gamma}_0)|\bar{\gamma}\rangle\|_{2,1} \leqq (c\lambda)^{(1/4)k}\sum_{\alpha\in s(\bar{\gamma})}\|(\tilde{S}_{\bar{C}_\alpha}+S_{B_\alpha\setminus\bar{C}_\alpha})|\bar{\gamma}\rangle\|)_{2,1}$$

$$= (c\lambda)^{(1/4)k}\|(\tilde{S}_{\bar{\mathscr{S}}}+S_{\sim\bar{\mathscr{S}}})|\bar{\gamma}\rangle\|_{2,1} \quad (15.19)$$

for all excitations $|\bar{\gamma}\rangle$.

Due to Theorem 7.1, we have

$$\sum_{|\bar{\gamma}_0|_c=k}\|\overline{\mathrm{ad}}\,\overline{w_\alpha}(\bar{\gamma}_0)u\|_{2,1}\leqq (c\lambda)^{(1/4)(L_0+k)}$$

$$\leqq g(c\lambda)^{(1/4)k}\leqq (c\lambda)^{(1/4)k}\|(\tilde{S}_{\bar{C}_\alpha}+S_{B_\alpha\setminus\bar{C}_\alpha})u\|_{2,1}. \quad (15.20)$$

For all $u\in\mathscr{H}(B_\alpha)$ with $u\perp|0_\alpha\rangle$. Hence

$$\sum_{\alpha\in s(\bar{\gamma})}\sum_{|\bar{\gamma}_0|_c=k}\|\overline{\mathrm{ad}}\,\overline{w_\alpha}(\bar{\gamma}_0)|\bar{\gamma}\rangle\|_{2,1}\leqq (c\lambda)^{(1/4)k}\|(\tilde{S}_{\bar{\mathscr{S}}}+S_{\sim\bar{\mathscr{S}}})|\bar{\gamma}\rangle\|_{2,1}. \quad (15.21)$$

To bound the first term in the expression (15.13) for $T^{(k)}$, one can use Theorem 1.2 and complete the proof of Lemma 15.13. Q.E.D.

The operator $T^{(k)}$ defined in (15.13) has non-vanishing matrix elements only among excitations $\bar{\gamma},\bar{\gamma}'$ with $d_c(s(\bar{\gamma}),s(\bar{\gamma}'))\leqq k$. Moreover, we have

$$\tilde{S}+V_\Gamma=\tilde{S}_{\bar{\mathscr{S}}}+S_{\sim\bar{\mathscr{S}}}+\sum_{k=1}^{\infty}T^{(k)}. \quad (15.22)$$

Thanks to Lemma 15.3, the geometric series expansion for K

$$K=(\tilde{S}+V_\Gamma)^{-1}\Pi_0$$

$$=\sum_{j=0}^{\infty}(\tilde{S}_{\bar{\mathscr{S}}}+S_{\sim\bar{\mathscr{S}}})^{-1}\left[\left(\sum_{k=1}^{\infty}T^{(k)}\right)(\tilde{S}_{\bar{\mathscr{S}}}+S_{\sim\bar{\mathscr{S}}})^{-1}\right]^j\Pi_0 \quad (15.23)$$

is convergent in $L^{2,1}$-operator norm. By using (15.14), we find

$$\sup_{\bar{\gamma}_0\subset\bar{\Lambda}}\ \sup_{\substack{u\in\mathscr{H}(\bar{\gamma}_0)\\ \|u\|_{2,1}=1}}\left\|\sum_{\bar{\gamma}'\cdot d_c(\bar{\gamma}_0,S(\bar{\gamma}'))=k}|\bar{\gamma}'\rangle\langle\bar{\gamma}'|\bar{S}K(u\otimes|\bar{0}\sim\bar{\gamma}_0\rangle\right\|_{2,1}$$

$$\leqq\sum_{r=1}^{\infty}\sum_{\substack{k_1+\cdots+k_r\geqq k\\ k_1,\dots,k_r\geqq 0}}\|(\tilde{S}_{\bar{\mathscr{S}}}+S_{\sim\bar{\mathscr{S}}})^{-1}\Pi_0\|_{2,1}\prod_{i=1}^{r}\|T^{(k_i)}(\tilde{S}_{\bar{\mathscr{S}}}+S_{\sim\bar{\mathscr{S}}})^{-1}\Pi_0\|_{2,1}$$

$$\leqq\sum_{r=1}^{\infty}\sum_{\substack{\bar{k}_1+\cdots+\bar{k}_r\geqq k\\ k_1,\dots,k_r\geqq 0}}g^{-1}(c\lambda)^{1/4(\bar{k}_1+\cdots+\bar{k}_r)}. \quad (15.24)$$

If $F(\beta)=\sum_{k=0}^{\infty}f_k\beta^k$ is a power series, let us denote with $\{F(\beta)\}_k$ the coefficient f_k. The right-hand side of (5.24) can be written as follows:

$$\left\{\sum_{r=1}^{\infty}\sum_{k_1,\ldots,k_r\geqq 0} g^{-1}(c\lambda\beta)^{(1/4)(\bar{k}_1+\cdots+\bar{k}_r)}\right\}_k$$
$$=\left\{g^{-1}\sum_{r=1}^{\infty}\left(\sum_{j=1}^{\infty}(c\lambda\beta)^{(1/4)j}\right)^r\right\}_k$$
$$=g^{-1}\{c\lambda\beta(1+(1-c\lambda\beta)^{-1})[1-c\lambda\beta(1+(1-c\lambda\beta)^{-1})]^{-1}\}_k$$
$$\leqq g^{-1}(c\lambda)^k$$

for $\lambda\leqq c$. This completes the proof of Lemma 15.1. Q.E.D.

Proof of Corollary 15.2. Let $\bar{\gamma},\bar{\gamma}'$ be two excitations in $\bar{\Lambda}$ with $d_c(s(\bar{\gamma}),s(\bar{\gamma}'))=k$. Then, we have

$$\frac{\langle\bar{\gamma}'|Z^{\bar{S}_c}|\bar{\gamma}'\rangle}{\langle\bar{\gamma}|Z^{\bar{S}_c}|\bar{\gamma}\rangle}\leqq Z^k \tag{15.26}$$

and

$$\frac{\langle\bar{\gamma}'|Q|\bar{\gamma}'\rangle}{\langle\bar{\gamma}|Q|\bar{\gamma}\rangle}\leqq(c\lambda)^{-(1/16)k}, \tag{15.27}$$

where c is the constant in the definition (13.32). If this constant is fixed to be equal to the constant c in (15.3), then (15.4) follows from (15.3), (15.26) and (15.27). Q.E.D.

16. Recurrence Inequalities

This section contains the derivation of a set of inequalities for the sequence $\hat{r}_n^*(Z)$ defined in (13.40) and used in the following section to control the convergence of our perturbative expansion. We have

$$\begin{aligned}\hat{r}_n^*(Z)&=\sup_{\bar{x}\in\Lambda}\|P_{\bar{x}}QZ^{\bar{S}_c}\bar{S}\hat{R}_n|u_0\rangle\|_{2,1}\\&=\sup_{\bar{x}\in\bar{\Lambda}}\sum_{\bar{\gamma}:\bar{x}\in s(\bar{\gamma})}Q(s(\bar{\gamma}))Z^{|s(\bar{\gamma})|_c}|s(\bar{\gamma})|\,|\hat{r}_{n\bar{\gamma}}|,\end{aligned} \tag{16.1}$$

where, if $\bar{\gamma}_0\subset\bar{\Lambda}$, we define

$$Q(\bar{\gamma}_0)=\prod_{\alpha\in\mathscr{I}}\max(1,N^M(c\lambda)^{d(\bar{\gamma}_0,\bar{c}_\alpha)}). \tag{16.2}$$

We have

Lemma 16.1. *If* $\lambda<c$,

$$(L_1-L_0)\geqq cM\log N \tag{16.3}$$

and

$$1\leqq Z\leqq c\lambda^{-1/16}, \tag{16.4}$$

we have

$$\hat{r}_1^*(Z)\leqq g^{-1}(c\lambda)^{(1/12)(L_1-L_0)} \tag{16.5}$$

and

$$\hat{r}_n^*(Z) \leqq \sum_{\substack{i_1+\cdots+i_k=n \\ k\geqq 2}} 8^k g^{-1} M N^M \prod_{j=1}^{k} \hat{r}_{i_j}^*(Z)$$
$$+ \sum_{j=1}^{\infty} \sum_{n_0=j(L_1-L_0)}^{(j+1)(L_1-L_0)-1} \sum_{i_1+\cdots+i_k+j=n} 8^k (c\lambda)^{(1/12)n_0} g^{-1} \sum_{j=1}^{k} \hat{r}_{i_j}^*(Z). \quad (16.6)$$

Proof. We have

$$\hat{r}_1^*(Z) = \sup_{\bar{x}\in\bar{\Lambda}} \left\| P_{\bar{x}} Q Z^{\bar{S}_c} \bar{S} K \sum_{n_0=L_1-L_0}^{2(L_1-L_0)-1} W_{\Gamma,n_0} |u_0\rangle \right\|_{2,1}. \quad (16.7)$$

Let us decompose K as follows:

$$K = \sum_{m=0}^{\infty} K_m,$$

where K_m has matrix elements

$$\langle \bar{y}|K_m|\bar{y}'\rangle = \begin{cases} \langle \bar{y}|K|\bar{y}'\rangle & \text{if} \quad d_c(s(\bar{y}), s(\bar{y}') = m \\ 0 & \text{ottherwise,} \end{cases} \quad (16.8)$$

By virtue of Corollary 15.2, we have

$$\| Q Z^{\bar{S}_c} \bar{S} K_m Z^{-\bar{S}_c} Q^{-1} \|_{2,1} \leqq g^{-1} (c\lambda)^{(1/16)m}. \quad (16.9)$$

Hence,

$$\hat{r}_1^*(Z) \leqq \sup_{\bar{x}\in\bar{\Lambda}} \sum_{m=0}^{\infty} \sup_{\bar{y}:d(\bar{x},\bar{y})=m} \left\| (Q Z^{\bar{S}_c} K_m Z^{-\bar{S}_c} Q^{-1}) \left(P_{\bar{y}} Z^{\bar{S}_c} Q \sum_{n_0=L_1-L_0}^{2(L_1-L_0)-1} W_{\Gamma,n_0} \right) |u_0\rangle \right\|_{2,1}$$
$$\leqq \sum_{m=0}^{\infty} g^{-1} Z^m (c\lambda)^{(1/16)m} (M^{[m/(L_1-L_0)]} c^m) \sup_{\bar{x}\in\bar{\Lambda}} \left\| P_{\bar{x}} Z^{\bar{S}_c} Q \sum_{n_0=L_1-L_0}^{2(L_1-L_0)} W_{\Gamma,n_0} |u_0\rangle \right\|_{2,1}, \quad (16.10)$$

where the estimate

$$\sup_{\bar{x}\in\bar{\Lambda}} \#\{\bar{y}\in\bar{\Lambda} \text{ such that } d(\bar{x},\bar{y}) = m\} \leqq M^{[m/(L_1-L_0)]} c^m \quad (16.11)$$

is used. Thus, if $\lambda \leqq c$ and $Z \leqq c\lambda^{-1/16}$, we have

$$\hat{r}_1^*(Z) \leqq g^{-1} \sup_{\bar{x}\in\bar{\Lambda}} \left\| P_{\bar{x}} Z^{\bar{S}_c} Q \sum_{n_0=L_1-L_0}^{2(L_1-L_0)} W_{\Gamma,n_0} |u_0\rangle \right\|_{2,1}. \quad (16.12)$$

Due to Lemma 13.1, we have

$$\sup_{\bar{x}\in\bar{\Lambda}} \| P_{\bar{x}} Q Z^{\bar{S}_c} W_{\Gamma,n_0} |u_0\rangle \|_{2,1} \leqq \sup_{\bar{x}\in\bar{\Lambda}} \sum_{\bar{\gamma}_0:\bar{x}\in\bar{\gamma}_0} Z^{|\bar{\gamma}_0|_c} \| Q w_{\Gamma,n_0}(\bar{\gamma}_0) \|_{2,1}$$
$$\leqq Z^{n_0} N^{[(n_0/(L_1-L_2))]M} \sup_{\bar{x}\in\bar{\Lambda}} \sum_{\bar{\gamma}_0:\bar{x}_0\in\bar{\gamma}_0} \| w_{\Gamma,n_0}(\bar{\gamma}_0) \|_{2,1}$$
$$\leqq Z^{n_0} N^{[(n_0/(L_1-L_0))]M} (c\lambda)^{(1/4)n_0}$$
$$\leqq (c N^{M/(L_1-L_0)} \lambda^{1/12})^{n_0} (c Z \lambda^{1/12})^{n_0} (c\lambda)^{(1/12)n_0} \leqq (c\lambda)^{(1/12)n_0}. \quad (16.13)$$

This bound implies (16.5).

To prove (16.6), one can find the following bound with arguments similar to the ones above:

$$\hat{r}_n^*(Z) = \sup_{\bar{x}\in\bar{\Lambda}} \Bigg\| P_{\bar{x}} Q Z^{\bar{S}_c} \bar{S} K \Bigg\{ \sum_{\substack{i_1+\cdots+i_k=n\\ k\geqq 2}} \sum_{v_1\leqq\cdots\leqq v_k} \frac{1}{(v)!} [\cdots[(\tilde{S}+V_\Gamma), \hat{R}_{i_1}^{v_1}], \ldots, \hat{R}_{i_k}^{v_k}]$$

$$+ \sum_{j=1}^{\infty} \sum_{n_0=j(L_1-L_0)}^{(j+1)(L_1-L_0)-1} \sum_{i_1+\cdots+i_k+j=n} \sum_{v_1\leqq\cdots\leqq v_k}$$

$$\cdot \frac{1}{(v)!} [\cdots[W_{\Gamma,n_0}, \hat{R}_{i_1}^{v_1}], \ldots, \hat{R}_{i_k}^{v_k}] \Bigg\} |u_0\rangle \Bigg\|_{2,1}$$

$$\leqq g^{-1} \sum_{\substack{i_1+\cdots+i_k=n\\ k\geqq 2}} \sum_{v_1\leqq\cdots\leqq v_k} \frac{1}{(v)!} \sup_{\bar{x}\in\bar{\Lambda}} \| P_{\bar{x}} Q Z^{\bar{S}_c} [\cdots[(\tilde{S}+V_\Gamma), \hat{R}_{i_1}^{v_1}], \ldots, \hat{R}_{i_k}^{v_k}] |u_0\rangle \|_{2,1}$$

$$+ g^{-1} \sum_{j=1}^{\infty} \sum_{n_0=j(L_1-L_0)}^{(j+1)(L_1-L_0)-1} \sum_{i_1+\cdots+i_k+j=n} \sum_{v_1\leqq\cdots\leqq v_k}$$

$$\cdot \frac{1}{(v)!} \sup_{\bar{x}\in\bar{\Lambda}} \| P_{\bar{x}} Q Z^{\bar{S}_c} [\cdots[W_{\Gamma,n_0}, \hat{R}_{i_1}^{v_1}], \ldots, \hat{R}_{i_k}^{v_k}] |u_0\rangle \|_{2,1}. \tag{16.14}$$

Let k be an integer $\geqq 1$ and let us fix the integers $v_1, \ldots, v_k, i_1, \ldots, i_k$ so that

$$1 \leqq v_1 \leqq \cdots \leqq v_k \tag{16.15}$$

and

$$i_1, \ldots, i_k \geqq 1. \tag{16.16}$$

Let $\hat{r}_i(\bar{\gamma}_2)$ be the operator with support $\bar{\gamma}_2 \subset \bar{\Lambda}$ such that

$$\hat{r}_i(\bar{\gamma}_2) = \sum_{\bar{\gamma}: s(\bar{\gamma})=\bar{\gamma}_2} r_{i\bar{\gamma}} \hat{t}_{\bar{\gamma}}. \tag{16.17}$$

For all integers j such that $0 \leqq j \leqq k$, let us define inductively the operators $\mathscr{T}^j(\bar{\gamma}_0)$, with $\bar{\gamma}_0 \subset \bar{\Lambda}$, such that $\mathscr{T}^0(\bar{\gamma}_0) = w_{\Gamma,n_0}(\bar{\gamma}_0)$ and, if $j \geqq 1$,

$$\mathscr{T}^j(\bar{\gamma}_0) = \sum_{\bar{\gamma}_1 \cup \bar{\gamma}_2 = \bar{\gamma}_0} [\mathscr{T}^{j-1}(\bar{\gamma}_1), \hat{r}_{i_j}(\bar{\gamma}_2)]. \tag{16.18}$$

Let us remark that $\mathscr{T}^j(\bar{\gamma}_0)$ contains at most $n_0 + j$ centers of noncommutativity. Moreover $\mathscr{T}^j(\bar{\gamma}_0) = 0$ if

$$|\bar{\gamma}_0| > n_0 + v_1 + \cdots + v_j. \tag{16.19}$$

If $\Delta \subset \mathscr{I}, \mathscr{I}$ being the set of indices α for the clusters C_α, let us introduce the following family of pseudonorms for operators $\mathcal{O}_{\bar{\gamma}_0}$ with support $\bar{\gamma}_0 \subset \bar{\Lambda}$:

$$\|\mathcal{O}\|_{2,1}^{\Delta} \equiv \sup_{u_\Delta \in \mathscr{H}(\Delta)} \| Q Z^{\bar{S}_c} \mathcal{O}_{\bar{\gamma}_0} (u_\Delta \otimes |\bar{0}_{\sim\Delta}\rangle) \|_{2,1} \tag{16.20}$$

and

$$\|\mathcal{O}_{\bar{\gamma}_0}\|_{2,1}^{*} = \sup_{\Delta\in\mathscr{I}} (N^{-M|\Delta\cap\bar{\gamma}_0|} \|\mathcal{O}_{\bar{\gamma}_0}\|_{2,1}^{\Delta}). \tag{16.21}$$

If $\bar{\gamma}_2 \subset \bar{\Lambda}$, we have

$$\| \hat{r}_i(\bar{\gamma}_2) \|_{2,1}^{\Delta} \leqq N^{M|\bar{\gamma}_2 \cap \Delta|} \| \hat{r}_i(\bar{\gamma}_0) | u_0 \rangle \|_{2,1}. \tag{16.22}$$

Moreover, thanks to the bounds in the proof of (16.5), we have

$$\sup_{\bar{x} \in \bar{\Lambda}} \sum_{\bar{\gamma}_0 : \bar{x} \in \bar{\gamma}_0} \| \mathcal{T}^0(\bar{\gamma}_0) \|_{2,1}^{*} \leqq (c\lambda)^{(1/12)n_0}. \tag{16.23}$$

We have

Lemma 16.2. *If* $j \geqq 1$, *then*

$$\| \mathcal{T}^j(\bar{\gamma}_0) \|_{2,1}^{*} \leqq 2 \sum_{\bar{\gamma}_1 \cup \bar{\gamma}_2 = \bar{\gamma}_0} q(\bar{\gamma}_1, \bar{\gamma}_2) \| \mathcal{T}^{j-1}(\bar{\gamma}_1) \|_{2,1}^{*} \| QZ^{\bar{S}_c} \hat{r}_{i_j}(\bar{\gamma}_2) | u_0 \rangle \|_{2,1}, \tag{16.24}$$

where $q(\bar{\gamma}_1, \bar{\gamma}_2) = 1$ *if* $\bar{\gamma}_2$ *contains one of the centers of noncommutativity of* $\bar{\gamma}_1$, *and*

$$q(\bar{\gamma}_1, \bar{\gamma}_2) = \frac{|\bar{\gamma}_1 \cap \bar{\gamma}_2|}{|\bar{\gamma}_2|} \tag{16.25}$$

otherwise.

Proof of Lemma 16.2. Let us fix a set $\Delta \subset \mathcal{I}$ and a state

$$|u\rangle = |u_\Delta\rangle \otimes |0_{\sim\Delta}\rangle \tag{16.26}$$

with $|u_\Delta\rangle \in \mathcal{H}(\Delta)$. Let $\bar{\gamma}_1$ and $\bar{\gamma}_2$ be two subsets of $\bar{\Lambda}$ with $\bar{\gamma}_1 \cup \bar{\gamma}_2 = \bar{\gamma}_0$. We have

$$\begin{aligned} &\| QZ^{\bar{S}_c}[\mathcal{T}^{j-1}(\bar{\gamma}_1), r_{i_j}(\bar{\gamma}_2)] | u \rangle \|_{2,1} \\ &\leqq q(\bar{\gamma}_1, \bar{\gamma}_2) Q(\bar{\gamma}_0) Z^{|\gamma_0|_c} \{ \| \mathcal{T}^{j-1}(\bar{\gamma}_1) \|_{2,1}^{\Delta \cup (\bar{\gamma}_2 \cap \mathcal{I})} \| r_{i_j}(\bar{\gamma}_2) \|_{2,1}^{\Delta} \\ &\quad + \| \mathcal{T}^{j-1}(\bar{\gamma}_1) \|_{2,1}^{\Delta} \| r_{i_j}(\bar{\gamma}_2) \|_{2,1}^{\Delta \cup (\bar{\gamma}_1 \cap \mathcal{I})} \}. \end{aligned} \tag{16.27}$$

We have

$$|(\Delta \cup (\bar{\gamma}_2 \cap \mathcal{I})) \cap \bar{\gamma}_1)| + |\bar{\gamma}_2 \cap \Delta| = |\Delta \cap \bar{\gamma}_0| + |\bar{\gamma}_1 \cap \bar{\gamma}_2 \cap \mathcal{I}|. \tag{16.28}$$

Hence, by using (16.6) and the definition (16.15), we find

$$\begin{aligned} (16.27) &\leqq 2q(\bar{\gamma}_1, \bar{\gamma}_2) Q(\bar{\gamma}_0) Q(\bar{\gamma}_1)^{-1} Q(\bar{\gamma}_2)^{-1} Z^{|\bar{\gamma}_0|_c - |\bar{\gamma}_1|_c - |\bar{\gamma}_2|_c} \\ &\quad \cdot N^{M|\Delta \cap \bar{\gamma}_0|} N^{M|\bar{\gamma}_1 \cap \bar{\gamma}_2 \cap \mathcal{I}|} \| \mathcal{T}^{j-1}(\bar{\gamma}_1) \|_{2,1}^{*} \| QZ^{\bar{S}_c} \hat{r}_{i_j}(\bar{\gamma}_2) | u_0 \rangle \|_{2,1}. \end{aligned} \tag{16.29}$$

Since $|\bar{\gamma}_0|_c < |\bar{\gamma}_1|_c + |\bar{\gamma}_2|_c$ and

$$Q(\bar{\gamma}_0) Q(\bar{\gamma}_1)^{-1} Q(\bar{\gamma}_2)^{-1} \leqq N^{-M|\bar{\gamma}_1 \cap \bar{\gamma}_2 \cap \mathcal{I}|}, \tag{16.30}$$

we have

$$(16.27) \leqq 2q(\bar{\gamma}_1, \bar{\gamma}_2) N^{M|\Delta \cap \bar{\gamma}_0|} \| \mathcal{T}^{j-1}(\bar{\gamma}_1) \|_{2,1}^{*} \| QZ^{\bar{S}_c} \hat{r}_{i_j}(\bar{\gamma}_2) | u_0 \rangle \|_{2,1}. \tag{16.31}$$

This proves Lemma 16.2. Q.E.D.

By iterating (16.24) and using (16.23), we find

$$\begin{aligned} &\sup_{\bar{x} \in \bar{\Lambda}} \sum_{v_1 \leqq \cdots \leqq v_k} \frac{1}{(v)!} \| P_{\bar{x}} QZ^{\bar{S}_c}[\cdots[W_{\Gamma, n_0}, \hat{R}_{i_1}^{v_1}], \ldots, \hat{R}_{i_k}^{v_k}] | u_0 \rangle \|_{2,1} \\ &\leqq 2^k (c_0 \lambda)^{(1/12)n_0} \sum_{v_1 \leqq \cdots \leqq v_k} \frac{1}{(v)!} \prod_{j=1}^{k} \left[\left(n_0 + j - 1 + \frac{v_1 + \cdots + v_{j-1}}{v_j} + v_j \right) \right. \end{aligned}$$

$$\left.\cdot\sup_{\bar{x}\in\bar{\Lambda}}\sum_{\substack{\bar{\gamma}_0:\bar{x}\in\bar{\gamma}_0\\|\bar{\gamma}_0|=v_j}}\|QZ^{\bar{S}_c}\hat{r}_{i_j}(\bar{\gamma}_0)|u_0\rangle\|_{2,1}\right]$$

$$\leqq 8^k(c_0\lambda)^{(1/12)n_0}\prod_{j=1}^{k}\left[\sup_{\bar{x}\in\bar{\Lambda}}\sum_{\substack{\bar{\gamma}_0:\bar{x}\in\bar{\gamma}_0\\|\bar{\gamma}_0|=v_j}}\|QZ^{\bar{S}_c}\bar{S}\hat{r}_{i_j}(\bar{\gamma}_0)|u_0\rangle\|_{2,1}\right]$$

$$= 8^k(c_0\lambda)^{(1/12)n_0}\prod_{j=1}^{k}\hat{r}^*_{i_j}(Z). \tag{16.32}$$

To complete the proof of (16.14), we still have to bound the first term. Let us expand $\tilde{S}+V_\Gamma$ into a sum of operators $\tilde{S}(\bar{\gamma}_0)+v_\Gamma(\bar{\gamma}_0)$ with support $\bar{\gamma}_0$, i.e.

$$\tilde{S}+V_\Gamma=\sum_{\bar{\gamma}_0\subset\bar{\Lambda}}\tilde{s}(\bar{\gamma}_0)+v_\Gamma(\bar{\gamma}_0). \tag{16.33}$$

Let

$$\tilde{S}_{n_0}+V_{\Gamma,n_0}=\sum_{\bar{\gamma}_0:|\bar{\gamma}_0|_c=n_0}\tilde{s}(\bar{\gamma}_0)+v_\Gamma(\bar{\gamma}_0). \tag{16.34}$$

For all $n_0\geqq 1$, we have

$$\sup_{\bar{x}\in\bar{\Lambda}}\sum_{\bar{\gamma}_0:\bar{x}\in\bar{\gamma}_0}\|\tilde{s}(\bar{\gamma}_0)+v_\Gamma(\bar{\gamma}_0)\|^*_{2,1}$$

$$\leqq\sup_{\bar{x}\in\bar{\Lambda}}\sum_{\substack{\bar{\gamma}_0:|\bar{\gamma}_0|_c=n_0\\\bar{x}\in\bar{\gamma}_0}}Q(\bar{\gamma}_0)Z^{|\bar{\gamma}_0|_c}\|\tilde{s}(\bar{\gamma}_0)+v_\Gamma(\bar{\gamma}_0)\|_{2,1}$$

$$\leqq Z^{n_0}N^{(1+[n_0/L_1-L_0])M}\sup_{\bar{x}\in\bar{\Lambda}}\sum_{\substack{\bar{\gamma}_0:\bar{x}\in\bar{\gamma}\\|\bar{\gamma}_0|_c=n_0}}\|\tilde{S}(\bar{\gamma}_0)+v_\Gamma(\bar{\gamma}_0)\|_{2,1}$$

$$\leqq Z^{n_0}N^{(1+[n_0/L_1-L_0])M}(c\lambda)^{n_0}M$$

$$\leqq MN^M(c\lambda)^{(1/12)n_0}. \tag{16.36}$$

Fixed a $k\geqq 2$ and two k-tuples $i_1,\ldots,i_k$, $v_1,\ldots,v_k$ of integers $\geqq 1$ such that $v_1\leqq\cdots\leqq v_k$, one can define the operators $\mathscr{T}^j(\bar{\gamma}_0)$ so that

$$\mathscr{T}^0(\bar{\gamma}_0)=\tilde{s}(\bar{\gamma}_0)+v_\Gamma(\bar{\gamma}_0) \tag{16.37}$$

and (16.18) holds for all $j\geqq 1$. By using Lemma 16.2 that is still valid and by replacing the bound (16.23) with the bound (16.36), we find

$$\sum_{n_0=1}^{\infty}\sup_{\bar{x}\in\bar{\Lambda}}\sum_{v_1\leqq\cdots\leqq v_k}\frac{1}{(v)!}\|P_{\bar{x}}QZ^{\bar{S}_c}[\cdots[\tilde{S}_{n_0}+V_{\Gamma,n_0},\hat{R}^{v_1}_{i_1}],\cdots\hat{R}^{v_k}_{i_k}]u_0\rangle\|_{2,1}$$

$$\leqq\sum_{n_0=1}^{\infty}MN^M8^k(c_0\lambda)^{(1/12)n_0}\prod_{j=1}^{k}\hat{r}^*_{i_j}(Z)$$

$$\leqslant MN^M8^k\sum_{i=j}^{k}\hat{r}^*_{i_j}(Z). \tag{16.38}$$

Due to (16.14), this implies (16.6). Q.E.D.

17. Convergence of the Perturbative Expansion

In this section, the proof of Theorem 12.1 is completed.

To prove the convergence of the cluster expansions giving the operators $\hat{R}^v_\lambda(\beta)$ in (13.22), one can show that if $\lambda < c$ and

$$L_1 - L_0 \geqq |\log c\lambda|^{-1}(c + \log g + \log(\bar{M}N^{\bar{M}})), \tag{17.1}$$

then the series

$$\hat{r}^*(\beta, Z) \equiv \sum_{n=1}^{\infty} \hat{r}^*_n(Z)\beta^n \tag{17.2}$$

converges for $\beta \in [0, 1]$ and Z such that

$$1 \leqq Z \leqq c\lambda^{-1/16}. \tag{17.3}$$

Moreover, we have to show that

$$\hat{r}^*(1, Z) \leqq (c\lambda)^{(1/24)(L_1 - L_0)}. \tag{17.4}$$

One can rewrite the inequalities (16.5) and (16.6) in the form

$$\begin{aligned}\{\hat{r}^*(\beta, Z)\}_n \leqq \{&g^{-1}(c\lambda)^{(1/12)(L_1 - L_0)}\beta + 64g^{-1}\bar{M}N^{\bar{M}}\hat{r}^*(\beta, Z)^2(1 - 8\hat{r}(\beta, Z))^{-1}\\ &+ g^{-1}(c\lambda)^{(1/12)(L_1 - L_0)}(1 - (c\lambda)^{(1/12)(L_1 - L_0)})^{-1}(1 - 8\hat{r}^*(\beta, Z))^{-1}\}_n.\end{aligned} \tag{17.5}$$

Let us consider the function $a(\beta)$ defined as the function analytic near $\beta = 0$ that solves the equation

$$\begin{aligned}a(\beta) = {}&g^{-1}(c\lambda)^{(1/12)(L_1 - L_0)}\beta + 64g^{-1}\bar{M}N^{\bar{M}}a(\beta)^2(1 - 8a(\beta))^{-1}\\ &+ g^{-1}(c\lambda)^{(1/12)(L_1 - L_0)}(1 - (c\lambda)^{(1/12)(L_1 - L_0)})^{-1}(1 - 8a(\beta))^{-1}.\end{aligned} \tag{17.6}$$

If

$$a(\beta) = \sum_{n=1}^{\infty} a_n\beta^n \tag{17.7}$$

is the power series expansion for $a(\beta)$, we have

$$r^*_n(Z) \leqq a_n \tag{17.8}$$

for all $n \geqq 1$. Hence, it suffices to show that, under the conditions above, the function $a(\beta)$ is analytic for $|\beta| \leqq 1$ and

$$a(1) \leqq A \equiv (c\lambda)^{(1/24)(L_1 - L_0)}. \tag{17.9}$$

Let $[0, \beta_0]$ be the largest interval such that the function $a(\beta)$ is analytic for $\beta \in [0, \beta_0]$ and fulfills (17.9). For all $\beta \in [0, \beta_0] \cap [0, 1]$, we have

$$a(\beta) \leqq g^{-1}(c\lambda)^{(1/12)(L_1 - L_0)} + [128g^{-1}\bar{M}N^{\bar{M}}A]\cdot A + 4g^{-1}(c\lambda)^{(1/12)(L_1 - L_0)}, \tag{17.10}$$

where λ is assumed to be so small that $(c\lambda)^{(1/12)(L_1 - L_0)} \leqq \frac{1}{2}$ and $A \leqq \frac{1}{2}$. Hence, under a condition of the form (17.1), we have

$$a(\beta) \leqq \tfrac{1}{2}(c\lambda)^{(1/24)(L_1 - L_0)} + \tfrac{1}{3}(c\lambda)^{(1/24)(L_1 - L_0)} < A. \tag{17.11}$$

This implies that $\beta_0 > 1$. The proof of convergence is thus completed.

To prove the stability of the gap, let us introduce the operator $\hat{V}$ such that

$$\hat{U}^{-1}((\tilde{S}+V_\Gamma)+W_\Gamma)\hat{U}=(\tilde{S}+V_\Gamma)+\hat{V}. \tag{17.12}$$

Since $\hat{U}$ solves the conjugacy problem (13.27) for $\beta=1$, we have

$$\hat{V}|u_0\rangle=0. \tag{17.13}$$

The stability of the gap is a consequence of the following relative boundedness result:

Lemma 17.1. *We have*

$$|\langle u|\hat{V}|u\rangle|\leqq(c\lambda)^{(1/24)(L_1-L_0)}\langle u|\tilde{S}+V_\Gamma|u\rangle. \tag{17.14}$$

Proof of Lemma 17.1. Let us expand u as follows:

$$u=\sum_{\bar{\gamma}_0\subset\bar{\Lambda}}\ \sum_{\gamma:s(\gamma)=\bar{\gamma}_0\backslash\mathscr{I}}\phi_{\gamma,\bar{\gamma}_0}\otimes(\tau_\gamma|0_{\sim\bar{\mathscr{G}}}\rangle) \tag{17.15}$$

where $\phi_{\gamma,\bar{\gamma}_0}\in\mathscr{H}(\bar{\gamma}_0\cap\mathscr{I})$. Thanks to (17.13) and to the decay estimates in $L^{2,1}$-operator norm for $\hat{R}$, we have

$$\begin{aligned}
&|\langle u|\hat{V}|u\rangle|\\
&\leqq 2\sum_{\bar{\gamma}_0\subset\bar{\Lambda}}\ \sum_{\gamma:s(\gamma)=\bar{\gamma}_0\backslash\mathscr{I}}\left(\sum_{\bar{\gamma}'_0\subset\bar{\Lambda}}\ \sum_{\gamma':s(\gamma')=\bar{\gamma}'_0\backslash\mathscr{I}}\langle\phi_{\gamma',\bar{\gamma}'_0}\otimes(\tau_\gamma|0_{\sim\bar{\mathscr{G}}}\rangle)|\hat{V}|\phi_{\gamma,\bar{\gamma}_0}\otimes(\tau_\gamma|0_{\sim\bar{\mathscr{G}}}\rangle\rangle\right)\\
&\leqq 2\sum_{\bar{\gamma}_0\subset\bar{\Lambda}}\ \sum_{\gamma:s(\gamma)=\bar{\gamma}_0\backslash\mathscr{I}}|\bar{\gamma}_0|(c\lambda)^{(1/12)(L_1-L_0)}\|\phi_{\gamma,\bar{\gamma}_0}\|_2.
\end{aligned} \tag{17.16}$$

On the other hand, we have

$$\begin{aligned}
\langle u|\tilde{S}+V_\Gamma|u\rangle&=\frac{1}{2}\left\langle u|S+\sum_{\alpha\in\mathscr{I}}(V_{\bar{c}_\alpha}+W_{\bar{c}_\alpha})|u\right\rangle\\
&\quad+\left\langle u\left|\frac{1}{2}S+\sum_{\alpha\in\mathscr{I}}\left(\frac{1}{2}V_{\bar{c}_\alpha}+\frac{1}{2}\bar{W}_{\bar{c}_\alpha}+V_{\partial\bar{c}_\alpha}+\bar{W}_{\partial\bar{c}_\alpha}+V_{B_\alpha\backslash\bar{c}_\alpha}\right)+V_\Gamma\right|u\right\rangle\\
&\geqq\frac{1}{2}\langle u|S+\sum_\alpha(V_{\bar{c}_\alpha}+W_{\bar{c}_\alpha})|u\rangle\\
&\geqq\sum_{\bar{\gamma}_0}\ \sum_{\gamma:s(\gamma)=\bar{\gamma}_0\backslash\mathscr{I}}\frac{1}{2}g|\bar{\gamma}_0|\|\phi_{\gamma,\bar{\gamma}_0}\|_2,
\end{aligned} \tag{17.17}$$

where the first inequality follows from a positivity argument similar to the one used in Sect. 6. Equations (17.16) and (17.17) imply (17.14), under a condition of the form (17.1). Q.E.D.

Finally, we have to prove the following decay estimate for the truncated expectation value in the ground state of the product of two operators $\mathcal{O}_{\bar{\gamma}_0},\mathcal{O}_{\bar{\gamma}'_0}$, of $L^{2,1}$-operator norm 1 and with supports $\bar{\gamma}_0,\bar{\gamma}'_0\subset\bar{\Lambda}$, respectively:

$$|\langle\hat{U}u_0|\mathcal{O}_{\bar{\gamma}_0}\mathcal{O}_{\bar{\gamma}'_0}|\hat{U}u_0\rangle-\langle\hat{U}u_0|\mathcal{O}_{\bar{\gamma}_0}|\hat{U}u_0\rangle\langle\hat{U}u_0|\mathcal{O}_{\bar{\gamma}'_0}|\hat{U}u_0\rangle\leqq(c\lambda)^{(1/16)d_c(\bar{\gamma}_0,\bar{\gamma}'_0)}. \tag{17.18}$$

This follows from the unitarity of $\hat{U}$ and the fact that the cluster expansions for the functions

$$\langle u_0|\hat{U}^{-1}\mathcal{O}_{\bar{\gamma}_0}\mathcal{O}_{\bar{\gamma}'_0}\hat{U}|u_0\rangle \tag{17.19}$$

and

$$\langle u_0 | \hat{U}^{-1} \mathcal{O}_{\bar{\gamma}_0} \hat{U} | u_0 \rangle \langle u_0 | \hat{U}^{-1} \mathcal{O}_{\bar{\gamma}'_0} \hat{U} | u_0 \rangle \tag{17.20}$$

differ only by terms involving commutators of operators whose supports connect $\bar{\gamma}_0$ to $\bar{\gamma}'_0$. Due to the decay estimates in Sect. 2, Sect. 11 and in this section, (17.18) follows. Q.E.D.

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