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Hydrodynamics of Stationary Non-Equilibrium States for Some Stochastic Lattice Gas Models*

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Dedicated to Res Jost and Arthur Wightman

Abstract. We consider discrete lattice gas models in a finite interval with stochastic jump dynamics in the interior, which conserve the particle number, and with stochastic dynamics at the boundaries chosen to model infinite particle reservoirs at fixed chemical potentials. The unique stationary measures of these processes support a steady particle current from the reservoir of higher chemical potential into the lower and are non-reversible. We study the structure of the stationary measure in the hydrodynamic limit, as the microscopic lattice size goes to infinity. In particular, we prove as a law of large numbers that the empirical density field converges to a deterministic limit which is the solution of the stationary transport equation and the empirical current converges to the deterministic limit given by Fick's law.

1. Introduction

As a common experience, the large scale properties of a system in a non-equilibrium steady state are determined by the stationary solution of the relevant macroscopic equation with appropriate boundary conditions. Just to recall a familiar example: Let us consider a Rayleigh-Bénard cell consisting of a liquid between two plates at different temperatures, T_1 and T_2. The temperature difference is assumed to be sufficiently small so that heat is transported only diffusively and that the velocity field vanishes. In such a situation the hydrodynamic equations have a unique stationary solution with density $\varrho(z)$, velocity $\mathbf{v}=0$ and temperature $T(z)$, $0 \leqq z \leqq h$, $T(0)=T_1$ and $T(h)=T_2$, where z is the direction of the temperature gradient.

From a microscopic point of view we may model the liquid as a collection of a huge number of hard spheres (with a diameter of 1 Å, say), whose time evolution is governed by Newton's equation of motion. Within this framework, the steady state is described by a probability *measure* on phase space. In principle, we know how

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such a measure has to be defined. We impose thermal boundary conditions at the upper and lower plates. This means that a particle forgets its incoming velocity upon hitting the plate and is emitted instantaneously with outgoing velocity distributed according to a Maxwellian with a temperature characteristic of that plate. We have to find then a stationary solution of the Liouville equation satisfying the thermal boundary conditions. On physical grounds we expect, for a fixed number of particles, this stationary solution to be unique. Only for vanishing temperature difference, $T_1 = T_2 = T$, we know the solution. It is the familiar canonical distribution

$$\frac{1}{Z} \exp\left[-H/k_B T\right] \tag{1.1}$$

with H the energy (kinetic + potential) of the system.

The stationary nonequilibrium measure in our example has a feature which is of a general nature. We note that the density and the temperature vary *slowly* on the scale measured in units of a typical interparticle distance. Therefore, we can pick a fluid element which is so small that $\varrho(z)$ and $T(z)$ can be considered as constant across the element and, at the same time, so large that it still contains a huge number of particles (say 10^{12}). The positions and the velocities of the particles in the fluid element under consideration have a certain probability distribution. To an excellent approximation this probability distribution should be given by the grand canonical ensemble with temperature $T(z)$ and chemical potential $\mu(z)$. Here, $\mu(z)$ is adjusted in such a way as to produce $\varrho(z)$, and $\varrho(z)$, $T(z)$ is the stationary solution of the hydrodynamic equations. Of course, this cannot be the full story. After all, energy is transported through the fluid. Therefore, the velocity distribution of the particles must be a slightly distorted Maxwellian. Also, the diffusive transport gives rise to correlations (which we ignored) on a macroscopic scale [Sch, Sp1]. However, these are effects of higher order in the gradients. To lowest order the fluid is locally in equilibrium with parameters determined by the steady solution of the macroscopic equation.

Unfortunately, we are very far from being able to establish any detail of this picture for a realistic model of a fluid, such as the Boltzmann-Gibbs model of hard spheres. The simple models for which non-equilibrium properties can be computed, e.g. the non-interacting gas and the perfect harmonic crystal corresponding to an ideal fluid and an ideal solid, do not obey any macroscopic kinetic laws, such as Fourier's law of heat conduction [SL]. We do not understand at present the dynamical properties responsible for real systems obeying hydrodynamic laws. Even when it is believed with certainty that the hydrodynamical laws are obeyed, as for the hard sphere fluid, we have too little knowledge about the stationary nonequilibrium measure to establish them from first principles.

One of the developments of recent years is the rigorous derivation of hydrodynamical laws for lattice gases with stochastic dynamics. Although these models are certainly caricatures of the models we really wish to understand, they are accurate caricatures. They have a basic structure qualitatively similar to some real systems and exhibit in a precise mathematical form a surprisingly rich variety of the interesting phenomena observed in real systems. In this paper, we investigate stationary nonequilibrium measures for such stochastic lattice gas models.

These lattice gases have as their only locally conserved field the particle number. Therefore, we seek to verify Fick's law of particle transport rather than Fourier's law and, rather than temperature, we impose the chemical potential (or particle density) at the boundaries. To be precise, the models we consider are continuous-time Markov processes on the finite state space $\Omega=\{0,1\}^{\Lambda}$ where $\Lambda=([-M,M]\cap\mathbb{Z})^d$ is a lattice of $(2M+1)^d$ sites. The components η_x, $x\in\Lambda$ of the state vector $\eta\in\Omega$ denote the occupation numbers of the sites x (1 = occupied, 0 = unoccupied). In all cases we consider, the generator of the process has the form

$$(Lf)(\eta)=\tfrac{1}{2}\sum_{\substack{x,y\in\Lambda\\|x-y|=1}} c(x,y;\eta)[f(\eta^{x,y})-f(\eta)] + \sum_{x:x_1=\pm M} c(x,\eta)[f(\eta^{x})-f(\eta)] \ . \tag{1.2}$$

For simplicity, we have chosen the exchange dynamics to allow only nearest neighbor jumps. More essential restrictions are

(a) *finite range:* $c(x,y;\eta)$ depends on η only through $\{\eta_z|\ |x-z|\leqq R, |y-z|\leqq R\}$.

(b) *translation invariance:* Let τ_a be the shift by a on $\mathbb{Z}^d$, $\tau_a\eta_x=\eta_{x-a}$, $a\in\mathbb{Z}^d$. Then for all $x,y\in\Lambda$, $\eta\in\Omega$, $a\in\mathbb{Z}^d$

$$\begin{aligned} &c(x,y;\eta)=c(x+a,y+a;\tau_a\eta) \quad \text{for} \\ &|(x+a)_1\pm M|>R\ , \qquad |(y+a)_1\pm M|>R\ . \end{aligned} \tag{1.3}$$

We adopt periodic boundary conditions except in the 1-direction.

(c) *detailed balance:* There exists a Hamiltonian $H(\eta)$, which is translation-invariant (up to boundary effects) and has finite range (R) interactions, so that

$$c(x,y;\eta)=c(x,y;\eta^{x,y})e^{-(\Delta_{x,y}H)(\eta)} \ . \tag{1.4}$$

($\eta^{x,y}$ denotes η with the occupancies at x,y interchanged, and $(\Delta_{x,y}H)(\eta) = H(\eta^{x,y})-H(\eta)$.)

(d) *non-degeneracy:*

$$\inf_{\substack{\eta_x\neq\eta_y\\|x-y|=1}} c(x,y;\eta)>0 \ . \tag{1.5}$$

It is known, under these conditions, that the bulk diffusion coefficient D defined by the Green-Kubo formula is finite and nonnegative [DIPP, Sp], so we expect good transport properties for these systems. Note that the exchange rates $c(x,x+e_\mu;\eta)$ in the boundary regions $|x_1\pm M|\leqq R$ may be chosen arbitrarily (i.e. (b) is not required) subject to conditions (a), (c), (d). The boundary rates $c(x,\eta)$ correspond to particle creation and annihilation at the sites x: $x_1=\pm M$. They represent in an idealized way the interaction of the system with infinite particle reservoirs in equilibrium at chemical potentials $\lambda_\pm$, and are thus required to satisfy the detailed balance conditions

$$c(x,\eta)=c(x,\eta^x)e^{-(\Delta_x H)(\eta)+\lambda_x(1-2\eta_x)} \ ; \tag{1.6}$$

here η^x is the configuration η with occupation switched at site x and $\lambda_{(\pm M, x_\perp)} = \lambda_\pm$. We also assume for these rates the finite range condition analogous to (a) above, and non-degenracy as $\inf_{\eta, x} c(x, \eta) > 0$. However, we make no further assumptions or special choice for the boundary rates. With these assumptions on the rates there is, for each fixed M, a unique stationary measure $\mu_{ss}^{(M)}$ for the process which is approached exponentially quickly starting from any arbitrary state. We wish to study the large-scale (hydrodynamic) structure of this measure.

To carry out the proofs, we must make for the bulk exchange rates one further assumption, namely:

(e) *gradient condition:* there is a bounded, local function $h(\eta)$ (of range R) so that the particle current

$$\begin{aligned} j_{x,y}(\eta) &\equiv c(x, y; \eta)(\eta_x - \eta_y) \\ &= h_x(\eta) - h_y(\eta) \ . \end{aligned} \tag{1.7}$$

(Here, $h_x(\eta) \equiv \tau_x h_0(\eta)$, etc.)

This assumption is of a more technical nature, but is unfortunately necessary at present. Notice it states that the microscopic current is a gradient of a local function, which is already close to the macroscopic transport law. Nontrivial examples of rates which satisfy all of our conditions simultaneously, particularly (c) and (e), are rare in more than one dimension. Therefore, we restrict ourselves to one-dimensional models. Our proof actually carries over, with only minor modifications, to a somewhat more general case, namely one-dimensional models in which the current is a "spatial gradient" plus a "time-derivative":

$$j_{x,x+1}(\eta) = h_x(\eta) - h_{x+1}(\eta) + (Lg_x)(\eta) \ , \tag{1.8}$$

for bounded, local functions h, g. A simple example which nontrivially realizes this structure is the "alternating rates" model of Wick [W]. Furthermore, a decomposition of this sort is in some sense generically true (see [DPSW]), but in a weaker form than (1.8) above. The extension of the results of the present paper to the general (non-gradient) case seems to us of some importance as a necessary step in the long road to an understanding of the physically realistic models.

Although most previous work on the hydrodynamics of stochastic lattice gases has been for the time-dependent case, without reservoirs, there has been some prior work on the stationary nonequilibrium case. The previous results may be summarized as follows:

1. For some models the stationary measure can be computed explicitly. Typically it has the form of a Gibbs measure with a linearly varying chemical potential. This is the case for Ginzburg-Landau models, interacting Brownian particles, and the zero-range process [DF]. The stationary measures can be studied by standard equilibrium methods.

2. For the symmetric exclusion process there is no explicit formula for the stationary measure. However, because of duality, the n^{th} correlation function of the

stationary measure can be expressed as an expectation for n random walks with exclusion. This yields enough information to cover the questions of interest [GKMP].

In the lattice gases studied here there are no such simplifying features.

For our models we prove the weak convergence of the density field in the hydrodynamic scaling limit to a deterministic density profile obtained as the solution of the stationary transport equation. This provides exactly the justification required for that equation. The argument, in fact, yields further the weak convergence of an arbitrary extensive field to a deterministic limit which is an appropriate function of the local density. As a consequence, we establish both the convergence in probability of the empirical current field to the deterministic limit given by Fick's law and a local form of the normal transport property. The strongest result of the argument is an L^2 version of the local equilibrium property (LEP). By this, we mean that for any bounded local function g_0 depending only on the occupations in some neighborhood of the origin, that

$$\lim_{M\to\infty} \mu_{ss}^{(M)}(g_{[\![Mq]\!]}) = \langle g_0 \rangle_{\bar{\varrho}(q)} , \tag{1.9}$$

where $q \in [-1, 1]$, $\bar{\varrho}$ is the solution of the stationary transport equation, $\langle \cdot \rangle_\varrho$ is the expectation with respect to the Gibbs measure for the Hamiltonian H at density ϱ, and the limit is in the L^2 sense. This property is a precise statement of our earlier intuitive considerations. We postpone the proof of this result, however, to a second paper [ELS2]. There also we study, by closely related arguments, the relaxation of initial, local equilibrium measures to the steady state on a hydrodynamic time scale. In that case, we establish a deterministic weak limit for the time-dependent, empirical density field to that density field which is the solution of the initial-boundary value problem for the time-dependent non-linear diffusion equation.

Our proof is by the entropy production method of Guo, Papanicolaou and Varadhan, adapted to the present situation [GPV]. Our paper advances the previous work since it allows many new cases to be treated and gives a unified treatment of all the models. Furthermore, the present proof is robust in being independent of any specific choice of boundary dynamics. The original proofs for the specific models made specific choices (albeit natural) for the boundary dynamics, whereas the details of the boundary dynamics should be irrelevant to the bulk, hydrodynamical properties of the steady state, subject only to the requirement of their satisfying local detailed balance: in physical terms, the structure of the steady state should be identical whether water, champagne, or vinegar is used for the thermal reservoir, so long as the temperature (here, the chemical potential) is the same. It is gratifying to be able to verify such independence in our case.

2. Entropy Production and Hydrodynamics with Stochastic Reservoirs

In our context, the entropy production is defined as a function σ on the set $\mathscr{D}$ of nonnegative measures on $\{0,1\}^{\Lambda}$ by

$$\sigma[\mu]=\frac{1}{4}\sum_{\substack{x,y\in\Lambda\\|x-y|=1}}\sum_{\eta}c(x,y;\eta)[e^{\Delta_{x,y}H(\eta)}\mu(\eta^{x,y})-\mu(\eta)]\log\left[\frac{e^{\Delta_{x,y}H(\eta)}\mu(\eta^{xy})}{\mu(\eta)}\right]$$
$$+\frac{1}{2}\sum_{x\in\partial\Lambda}\sum_{\eta}c(x,\eta)[e^{\Delta_x H(\eta)+\lambda_x(2\eta_x-1)}\mu(\eta^x)-\mu(\eta)]$$
$$\times\log\left[\frac{e^{\Delta_x H(\eta)+\lambda_x(2\eta_x-1)}\mu(\eta^x)}{\mu(\eta)}\right]. \tag{2.1}$$

The motivation for the definition of this function from macroscopic thermodynamics and a further discussion of its properties is contained in a separate paper [ELS]. Here, we simply point out that, since the function $F(z)=F(x,y)=(x-y)\log\left(\frac{x}{y}\right)$ satisfies $F(z)\geqq 0$, $F(\lambda z)=\lambda F(z)$ for $\lambda\geqq 0$ and $F(z)$ is convex, σ inherits these properties:

(i) (positivity) $\sigma[\mu]\geqq 0$, (2.2)

(ii) (homogeneity) $\sigma[\lambda\mu]=\lambda\sigma[\mu]$, $\lambda\geqq 0$, (2.3)

(iii) (convexity) $\sigma[\lambda\mu_1+(1-\lambda)\mu_2]\leqq\lambda\sigma[\mu_1]+(1-\lambda)\sigma[\mu_2]$,
for $0\leqq\lambda\leqq 1$. (2.4)

The corresponding properties of "marginal entropy production" functions, defined below, prove essential in the arguments we present.

The essential, technical role of entropy production in the GPV method is that, for measures whose entropy production is "small," it may be shown that in a suitable sense an arbitrary extensive field becomes a nonlinear function of the density field in the hydrodynamic limit. This achieves the fundamental goal of closure of the hydrodynamic equations or correlation hierarchy equations in terms of the conserved density. More precisely, let us define a set, $S(\varepsilon)$, of measures on $\{0,1\}^{\Lambda}$, or, equivalently, of densities f relative to the finite-volume (grand canonical) Gibbs measure $\nu_{\varrho}^{\varepsilon}$ by

$$S(\varepsilon)=\{f\,|\,\sigma^{\varepsilon}[f]\leqq c\cdot\varepsilon,\ f\geqq 0,\ \langle f\rangle_{\varrho}^{\varepsilon}=1\}\ . \tag{2.5}$$

Here, σ^{ε} denotes the entropy production function in (2.1) for $\Lambda=\mathbb{Z}\cap[-\varepsilon^{-1},\varepsilon^{-1}]$. Then, we have the following:

Proposition 1. *For any bounded, local function $g_0(\eta)$ (with finite range R) and smooth function $\phi\in C_0^{\infty}[-1,1]$,*

$$\text{a)}\quad \lim_{l\to 0}\lim_{\varepsilon\to 0}\sup_{f\in S(\varepsilon)}\left\langle f\cdot\left[\varepsilon\sum_{x=-\varepsilon^{-1}+R}^{\varepsilon^{-1}(1-l)}\phi(\varepsilon x)\left(g_x(\eta)-\hat{g}\left(\frac{\varepsilon}{l}\sum_{y=x}^{x+\varepsilon^{-1}l}\eta_y\right)\right)\right]^2\right\rangle_{\varrho}^{\varepsilon}=0\ , \tag{2.6}$$

$$\text{b)}\quad \lim_{l\to 0}\lim_{\varepsilon\to 0}\sup_{f\in S(\varepsilon)}\left\langle f\cdot\left[\frac{\varepsilon}{l}\sum_{y=-\varepsilon^{-1}(1-l)}^{\varepsilon^{-1}-R} g_y(\eta)-\hat{g}(\varrho_+)\right]^2\right\rangle_\varrho^\varepsilon=0\ , \tag{2.7}$$

$$\text{c)}\quad \lim_{l\to 0}\lim_{\varepsilon\to 0}\sup_{f\in S(\varepsilon)}\left\langle f\cdot\left[\frac{\varepsilon}{l}\sum_{y=-\varepsilon^{-1}+R}^{-\varepsilon^{-1}(1-l)} g_y(\eta)-\hat{g}(\varrho_-)\right]^2\right\rangle_\varrho^\varepsilon=0\ . \tag{2.8}$$

Here, $\hat{g}$ is the nonlinear function of the density

$$\hat{g}(\varrho)=\langle g_0\rangle_\varrho\ , \tag{2.9}$$

with expectation in the infinite-volume Gibbs measure at density ϱ. For both the static and dynamic law of large numbers in the following sections, the limits (2.6–2.8) are the central steps in the argument. We now turn to the proof of Proposition 1. Since the proof of this technical proposition is not necessary to an understanding of the arguments in the following sections, the reader might wish to skip the remainder of this section at a first reading.

Proof of Proposition 1. The method is by now standard but a few modifications are necessary for lack of translation invariance, etc. To set notation, let

$$A(g,[a,b])=\frac{1}{\#[\Lambda_{[a,b]}]}\sum_{x\in\Lambda_{[a,b]}} g_x(\eta)\ , \tag{2.10}$$

with $\Lambda_{[a,b]}=\Lambda\cap[a,b]$ and $\#(\cdot)$ counting measure. Up to an error of order $O(l^2)$, we have the inequality

$$\begin{aligned}
&\left\langle f\left|\varepsilon\sum_{x=-\varepsilon^{-1}+R}^{\varepsilon^{-1}(1-l)}\varphi(\varepsilon x)\left(g_x(\eta)-\hat{g}\left(\frac{\varepsilon}{l}\sum_{y=x}^{x+\varepsilon^{-1}l}\eta_y\right)\right)\right|^2\right\rangle_\varrho^\varepsilon\\
&\leqq 2\left\langle f\left|\varepsilon\sum_{x=-\varepsilon^{-1}+R}^{\varepsilon^{-1}(1-l)}\varphi(\varepsilon x)\left(g_x(\eta)-A\left(g,[x,x+\varepsilon^{-1}l]\right)\right)\right|^2\right\rangle_\varrho^\varepsilon\\
&\quad+2\left\langle f\left|\varepsilon\sum_{x=-\varepsilon^{-1}+R}^{\varepsilon^{-1}(1-l)}\varphi(\varepsilon x)\left(A(g,[x,x+\varepsilon^{-1}l])-\hat{g}\left(\frac{\varepsilon}{l}\sum_{y=x}^{x+\varepsilon^{-1}l}\eta_y\right)\right)\right|^2\right\rangle_\varrho^\varepsilon\\
&\leqq(2+O(\varepsilon))\left\langle f\left|\varepsilon\sum_{x=-\varepsilon^{-1}(1-l)}^{\varepsilon^{-1}(1-l)} g_x(\eta)\left(\varphi(\varepsilon x)-\frac{\varepsilon}{l}\sum_{y=x-\varepsilon^{-1}l}^{x}\varphi(\varepsilon y)\right)\right|^2\right\rangle_\varrho^\varepsilon\\
&\quad+2(\|\varphi\|_2^2+O(\varepsilon))\left\langle f\cdot\varepsilon\sum_{x=-\varepsilon^{-1}-R}^{\varepsilon^{-1}(1-l)}\left(A(g,[x,x+\varepsilon^{-1}l])\right.\right.\\
&\quad\left.\left.-\hat{g}(A(\eta,[x,x+\varepsilon^{-1}l]))\right)^2\right\rangle_\varrho^\varepsilon\\
&\leqq 4\|g\|_\infty^2\left\langle f\cdot\varepsilon\sum_{x=-\varepsilon^{-1}(1-l)}^{\varepsilon^{-1}(1-l)}\left(\varphi(\varepsilon x)-\frac{\varepsilon}{l}\sum_{y=x-\varepsilon^{-1}l}^{x}\varphi(\varepsilon y)\right)^2\right\rangle_\varrho^\varepsilon\\
&\quad+4\|\varphi\|_2^2\left\langle f\cdot\varepsilon\sum_{x=-\varepsilon^{-1}+R}^{\varepsilon^{-1}(1-l)}\left(A(g,[x,x+\varepsilon^{-1}l])\right.\right.\\
&\quad\left.\left.-\hat{g}(A(\eta,[x,x+\varepsilon^{-1}l]))\right)^2\right\rangle_\varrho^\varepsilon\ .
\end{aligned}\tag{2.11}$$

The first term vanishes as $\varepsilon\to 0$, $l\to 0$ $\left(\text{note } \|g\|_\infty \equiv \sup_\eta |g_0(\eta)|\right)$.

The result (a) is a consequence of the inequality (2.11) above and

Lemma 1.

$$\lim_{l\to 0}\lim_{\varepsilon\to 0}\sup_{f\in S(\varepsilon)} \cdot\left\langle f\cdot\left(\varepsilon\sum_{x=-\varepsilon^{-1}+R}^{-\varepsilon^{-1}(1-l)} |A(g,[x,x+\varepsilon^{-1}l])-\hat{g}(A(\eta,[x,x+\varepsilon^{-1}l]))|^2\right)\right\rangle_\varrho = 0 \ . \tag{2.12}$$

Proof. We partition

$$[x,x+\varepsilon^{-1}l]=\bigcup_{j=1}^{J} B_j(x) \ , \tag{2.13}$$

where the $B_j(x)$'s are disjoint intervals of length k, except $B_J(x)$ whose length is (possibly) less than k. Let us suppress the x-dependence for awhile. We have

$$\begin{aligned}
&|A(g,[x,x+\varepsilon^{-1}l])-\hat{g}(A(\eta,[x,x+\varepsilon^{-1}l]))|^2\\
&\leqq\left|\frac{1}{J}\sum_{j=1}^{J} A(g,B_j)-\hat{g}(A(\eta,[x,x+\varepsilon^{-1}l]))\right|^2\\
&\leqq\left|\frac{1}{J}\sum_{j=1}^{J} |A(g,B_j)-\hat{g}(A(\eta,B_j))|\right.\\
&\qquad\left.+\frac{1}{J}\sum_{j=1}^{J} |\hat{g}(A(\eta,B_j))-\hat{g}(A(\eta,[x,x+\varepsilon^{-1}l]))|\right|^2\\
&\leqq 2\left(\frac{1}{J}\sum_{j=1}^{J} |A(g,B_j)-\hat{g}(A(\eta,B_j))|\right)^2\\
&\qquad+2\left(\frac{1}{J}\sum_{j=1}^{J} |\hat{g}(A(\eta,B_j))-\hat{g}(A(\eta,[x,x+\varepsilon^{-1}l]))|\right)^2\\
&\leqq\frac{2}{J}\sum_{j=1}^{J} |A(g,B_j)-\hat{g}(A(\eta,B_j))|^2\\
&\qquad+\frac{2}{J}\sum_{j=1}^{J} |\hat{g}(A(\eta,B_j))-\hat{g}(\eta,[x,x+\varepsilon^{-1}l]))|^2 \ .
\end{aligned} \tag{2.14}$$

In the last term, we Taylor expand to first order

$$\begin{aligned}
&|\hat{g}(A(\eta,B_j))-\hat{g}(A(\eta,[x,x+\varepsilon^{-1}l]))|^2\\
&\qquad\leqq\|\hat{g}'\|_\infty^2\,|A(\eta,B_j)-A(\eta,[x,x+\varepsilon^{-1}l])|^2\\
&\qquad\leqq\|\hat{g}'\|_\infty^2\left(\frac{1}{J}\sum_{i=1}^{J} |A(\eta,B_j)-A(\eta,B_i)|\right)^2\\
&\qquad\leqq\|\hat{g}'\|_\infty^2\cdot\frac{1}{J}\sum_{i=1}^{J} |A(\eta,B_j)-A(\eta,B_i)|^2 \ .
\end{aligned} \tag{2.15}$$

It suffices then to prove the following two results:
(one block estimate)

$$\lim_{k\to\infty}\lim_{\varepsilon\to 0}\sup_{1\leqq j\leqq J}\sup_{f\in S(\varepsilon)}\left\langle f\cdot\varepsilon\sum_{x=-\varepsilon^{-1}+R}^{\varepsilon^{-1}(1-l)}|A(g,B_j(x))-\hat{g}(A(\eta,B_j(x))|^2\right\rangle_\varrho^\varepsilon=0 \tag{2.16}$$

(two block estimate)

$$\lim_{k\to\infty}\lim_{l\to 0}\lim_{\varepsilon\to 0}\sup_{1\leqq i,j\leqq J}\sup_{f\in S(\varepsilon)} \cdot\left\langle f\cdot\varepsilon\sum_{x=-\varepsilon^{-1}+R}^{\varepsilon^{-1}(1-l)}|A(\eta,B_j(x))-A(\eta,B_i(x))|^2\right\rangle_\varrho^\varepsilon=0\ . \tag{2.17}$$

A few words of explanation may be helpful here. (2.12) says that a spatial average of a function, depending only on a finite number of occupation variables, over a "small macroscopic" block is close to a certain function of the density in the same block. The proof of this result is then divided into two steps. In the first step, we prove the analogous statement for a "large microscopic" block. In the second step, we show that the densities in two "large microscopic" blocks are almost the same when these blocks are "macroscopically close."

Proof of the One-Block Estimate. Without loss of generality, we may take the range of g to be also R. Then only the marginal of f^ε in an interval of the form $[x-R,x+k+R]$ is required. Denote this by f_1^ε. Now we observe that the entropy production σ is subadditive in the following sense: for any partition of Λ into intervals $\Lambda=A+B$, it follows that

$$\sigma_A[f_A]+\sigma_B[f_B]\leqq\sigma[f]\ , \tag{2.18}$$

where f_A is the density with respect to $\nu_{\varrho,A}$ (the grand canonical distribution at density ϱ in the finite block A) of the marginal of the measure $\mu=f\nu_\varrho$ in the block A, given by

$$f_A(\eta_A)\equiv\sum_{\eta_B}f(\eta)\nu_{\varrho,B|A}(\eta_B|\eta_A)\ , \tag{2.19}$$

and σ_A is defined as in (2.1) but with the summations over $\Lambda\cup\partial\Lambda$ restricted to those lattice sites x for which $N_x\subset A$ (with N_x the set of lattice sites within distance R of site x). The inequality (2.18) is obtained from the convexity properties of the "marginal entropies" σ_A, σ_B and Jensen's inequality. As a direct consequence, there follows the monotonicity property

$$\sigma_A[f_A]\leqq\sigma[f] \tag{2.20}$$

for any sub-block $A\subset\Lambda$. For the present case, denoting the "marginal entropy production" on the fixed block $[x-R,x+k+R]$ by σ_1, one has the formula

$$\sigma_1[f_1]=\frac{1}{2}\sum_{y=x}^{x+k}\sum_{\eta_1}c(y,y+1;\eta_1)[f_1(\eta_1^{y,y+1}) -f_1(\eta_1)]\log\left[\frac{f_1(\eta_1^{y,y+1})}{f_1(\eta_1)}\right]\nu_{1,\varrho}(\eta_1)\ , \tag{2.21}$$

where $\nu_{1,\varrho}$ is the marginal on the block $[x-R, x+k+R]$ of the (grand canonical) Gibbs measure ν_λ at chemical potential $\lambda(\varrho)$, as above. This marginal entropy production inherits the essential properties of σ: it is non-negative and strictly convex when restricted to f_1's supported by configurations with fixed occupancies η_x for x within R of the boundary of $[x-R, x+k+R]$ and fixed number of particles in the interior region $[x, x+k]$. It therefore has a unique minimum when restricted to that class. In fact, as a consequence of the detailed balance condition

$$c(y, y+1; \eta_1) = c(y, y+1; \eta_1^{y,y+1}) e^{-(\Delta_{y,y+1}H)(\eta_1)} , \tag{2.22}$$

this minimum is the finite marginal of a grand canonical Gibbs measure for H, conditioned to have specified occupancies outside the interior region $[x, x+k]$ and a specified total particle number inside $[x, x+k]$. It is thus a *canonical* Gibbs measure for the finite block. Note the minimum value of the entropy production for each such subclass is zero.

Now, as a consequence of monotonicity and the bound in (2.5) it follows that

$$\sigma_1[f_1^\varepsilon] \leqq c \cdot \varepsilon . \tag{2.23}$$

Therefore, any weak limit point f_1^* of f_1^ε as $\varepsilon \to 0$ must have

$$\sigma_1[f_1^*] = 0 , \tag{2.24}$$

i.e. it must be a convex combination of the canonical Gibbs measures $\mathscr{G}_c$ on $[x-R, x+k+R]$ described above. The estimate

$$\begin{aligned} &\overline{\lim_{\varepsilon\to 0}} \left\langle f_1^\varepsilon \cdot |A(g, [x, x+k]) - \hat{g}(A(\eta, [x, x+k]))|^2 \right\rangle_{1,\varrho} \\ &\quad \leqq \sup_{\nu\in\mathscr{G}_c} \nu(|A(g, [1, k]) - \hat{g}(A(\eta, [1, k]))|^2) , \end{aligned} \tag{2.25}$$

uniform in x, and the limit

$$\overline{\lim_{k\to 0}} \sup_{\nu\in\mathscr{G}_c} \nu(|A(g, [1, k]) - \hat{g}(A(\eta, [1, k]))|^2) = 0 , \tag{2.26}$$

provided by the law of large numbers for canonical Gibbs measures and the equivalence of ensembles, yield then the result (2.16).

For the two block estimate we first isolate two essential estimates. Let us choose an interval I, of the form $I = \{x, \ldots, x+\varepsilon^{-1}l\}$. Then let B_i, $B_j \subset I$ be two disjoint intervals of length k. Points in $B_i \cup B_j$ are separated at most by a distance $\varepsilon^{-1}l$. Let f_I be the marginal density of f onto I and let f_{ij} be that onto $B_i \cup B_j$. Consider a process of the symmetric exclusion with speed-change type with only "long-range" jumps between the two blocks B_i, B_j. For specificity, we choose the process with generator

$$(L_{ij}f)(\eta) = \frac{1}{2k^2} \sum_{x\in B_i} \sum_{y\in B_j} \tilde{c}(x, y : \eta)[f(\eta^{xy}) - f(\eta)] , \tag{2.27}$$

where

$$\tilde{c}(x, y; \eta) \equiv e^{-(\Delta_{x,y}H)(\eta)/2} \tag{2.28}$$

obeys detailed balance with respect to the Gibbs measures for the Hamiltonian H. Notice that there are constants $\tilde{c}_1, \tilde{c}_2 > 0$ such that

$$\tilde{c}_1 \leqq \inf_{\substack{x \in B_i, \eta_x \neq \eta_y \\ y \in B_j}} e^{-(\Delta_{x,y} H)(\eta)/2} \tag{2.29}$$

$$\sup_{\substack{x \in B_i \\ y \in B_j}} e^{-(\Delta_{x,y} H)(\eta)/2} \leqq \tilde{c}_2 , \tag{2.30}$$

since the Hamiltonian is finite-ranged, translation invariant and the exchange of occupancies at x, y alters the value of only a finite number of bounded interactions. The corresponding Dirichlet form is defined by

$$D_{ij}(f) = \frac{1}{2k^2} \sum_{x \in B_i} \sum_{y \in B_j} \langle \tilde{c}(x, y : \eta)[f(\eta^{xy}) - f(\eta)]^2 \rangle_\varrho . \tag{2.31}$$

Lemma 2.

$$D_{ij}(\sqrt{f_{ij}}) \leqq \text{const}\, \varepsilon^{-1} l \sigma_I[f_I] . \tag{2.32}$$

Proof. We define the exchange operator by

$$(T_{x,y} f)(\eta) = f(\eta^{x,y}) . \tag{2.33}$$

An exchange between x and y, $x < y$, can be written as

$$T_{x,y} = T_{x,x+1} \cdots T_{y-1,y} T_{y-1,y-2} \cdots T_{x+1,x} . \tag{2.34}$$

By adding and subtracting terms and applying the elementary inequality

$$\left[\sum_{i=1}^{n-1} (x_i - x_{i+1})\right]^2 \leqq (n-1) \sum_{i=1}^{n-1} (x_i - x_{i+1})^2 , \tag{2.35}$$

it follows that

$$\begin{aligned} \langle (T_{xy}\sqrt{f} - \sqrt{f})^2 \rangle_\varrho \leqq \varepsilon^{-1} l \sum_{z=x}^{y-1} \langle [(T_{x,x+1} \cdots T_{z-1,z})(T_{z,z+1}\sqrt{f} - \sqrt{f})]^2 \rangle_\varrho \\ + \varepsilon^{-1} l \sum_{z=x+1}^{y-1} \langle [(T_{x,x+1} \cdots T_{y-1,y} T_{y-1,y-2} \cdots T_{z+1,z}) \\ \times (T_{z,z-1}\sqrt{f} - \sqrt{f})]^2 \rangle_\varrho \end{aligned} \tag{2.36}$$

or

$$\begin{aligned} \langle (T_{xy}\sqrt{f} - \sqrt{f})^2 \rangle_\varrho \leqq \varepsilon^{-1} l \sum_{z=x}^{y-1} \langle e^{-(T_{z,z-1} \cdots T_{x+1,x} H - H)} (T_{z,z+1}\sqrt{f} - \sqrt{f})^2 \rangle_\varrho \\ + \varepsilon^{-1} l \sum_{z=x+1}^{y-1} \langle e^{-(T_{z,z+1} \cdots T_{y-2,y-1} T_{y-1,y} \cdots T_{x+1,x} H - H)} \\ \times (T_{z-1,z}\sqrt{f} - \sqrt{f})^2 \rangle_\varrho . \end{aligned} \tag{2.37}$$

Each of the energy differences receives contributions from the reversal of only a fixed, finite number of interactions, independent of k, ε and l. Thus,

$$\langle (T_{x,y}\sqrt{f} - \sqrt{f})^2 \rangle_\varrho \leqq \text{const}\, \varepsilon^{-1} l \sum_{z=x}^{y-1} \langle (T_{z,z+1}\sqrt{f} - \sqrt{f})^2 \rangle_\varrho . \tag{2.38}$$

Exploiting then (2.30) and the similar non-degeneracy condition

$$0 < c_1 \leqq \inf_{\substack{x,\eta \\ \eta_x \neq \eta_{x+1}}} c(x, x+1; \eta) , \tag{2.39}$$

it follows that

$$\begin{aligned} &\langle \tilde{c}(x, y; \eta)(T_{x,y}\sqrt{f} - \sqrt{f})^2 \rangle_\varrho \\ &\leqq \mathrm{const}\, \varepsilon^{-1} l \sum_{z=x}^{y-1} \langle c(z, z+1; \eta)(T_{z,z+1}\sqrt{f} - \sqrt{f})^2 \rangle_\varrho . \end{aligned} \tag{2.40}$$

By the inequality

$$2(\sqrt{u} - \sqrt{v})^2 \leqq (u - \mathrm{v}) \log\left(\frac{u}{v}\right) , \tag{2.41}$$

$$\begin{aligned} &\langle \tilde{c}(x, y; \eta)(T_{x,y}\sqrt{f} - \sqrt{f})^2 \rangle_\varrho \\ &\leqq \mathrm{const}\, \varepsilon^{-1} l \sum_{z=x}^{y-1} \left\langle c(z, z+1; \eta)(T_{z,z+1} f - f) \log\left(\frac{T_{z,z+1} f}{f}\right)\right\rangle_\varrho \\ &\leqq \mathrm{const}\, \varepsilon^{-1} l \sigma_I [f_I] . \end{aligned} \tag{2.42}$$

Summing over $x \in B_i$, $y \in B_j$ yields then (2.32). □

Lemma 3. *There exists a constant c, independent of* ε, *l and k such that*

$$\langle f_{ij}(A(\eta, B_i) - A(\eta, B_j))^2 \rangle_\varrho \leqq c\left(D_{ij}(\sqrt{f_{ij}}) + \frac{1}{k}\right) . \tag{2.43}$$

Proof. By a well-known variational characterization of the Dirichlet form (see [Str]), we have for any u with $u > 0$

$$-\langle f_{ij} \cdot u^{-1} L_{ij} u \rangle_\varrho \leqq D_{ij}(\sqrt{f_{ij}}) . \tag{2.44}$$

Denoting

$$(L_{x,y} f)(\eta) = \tilde{c}(x, y; \eta)[f(\eta^{x,y}) - f(\eta)] , \tag{2.45}$$

for $x \in B_i$, $y \in B_j$, and choosing

$$u(\eta) = \exp[k(A(\eta, B_i)^2 + A(\eta, B_j)^2)] , \tag{2.46}$$

we observe that

$$\begin{aligned} \left(\frac{L_{x,y} u}{u}\right)(\eta) &= \tilde{c}(x, y; \eta) 2\eta_x (1 - \eta_y)\left\{\exp\left[2(A(\eta, B_j) - A(\eta, B_i)) + \frac{2}{k}\right] - 1\right\} \\ &\quad + 2\tilde{c}(x, y; \eta)(1 - \eta_x)\eta_y \left\{\exp\left[2(A(\eta, B_i) - A(\eta, B_j)) + \frac{2}{k}\right] - 1\right\} . \end{aligned} \tag{2.47}$$

Summing now over $x \in B_i$, $y \in B_j$ and exploiting (2.29) gives

$$\begin{aligned} -\left(\frac{L_{ij} u}{u}\right)(\eta) \geqq \mathrm{const} &\left\{ A(\eta, B_i)(1 - A(\eta, B_j))\left(1 - \exp\left[2(A(\eta, B_j) - A(\eta, B_i)) + \frac{2}{k}\right]\right)\right. \\ &\left. + (1 - A(\eta, B_i)) A(\eta, B_j)\left(1 - \exp\left[2(A(\eta, B_i) - A(\eta, B_j)) + \frac{2}{k}\right]\right)\right\} , \end{aligned} \tag{2.48}$$

so that

$$-\left(\frac{L_{ij}u}{u}\right)(\eta) \geqq \text{const}\left[-\frac{1}{k}+\left(A(\eta,B_i)-A(\eta,B_j)\right)^2\right], \tag{2.49}$$

and thus, by (2.44),

$$\langle f_{ij}(A(\eta,B_i)-A(\eta,B_j))^2\rangle_\varrho \leqq \text{const}\left(\frac{1}{k}+D_{ij}(\sqrt{f_{ij}})\right). \quad \square \tag{2.50}$$

Proof of the Two Block Estimate. Let $f \in S(\varepsilon)$. Out of the intervals $\{x,\dots,x+\varepsilon^{-1}l\}$ which appear in (2.17) we produce covers of $\{-M,\dots,M\}$ except for border intervals of length less than $\varepsilon^{-1}l$. There are at most $\varepsilon^{-1}l$ such covers labelled, say, by the leftmost lattice site y in that interval of the cover containing the origin. ($y=-\varepsilon^{-1}l+1,\dots,-1,0$). Let $I(m)$, $m=-1/l,\dots,0,\dots,1/l$ be one such cover. $f^{(m)}$ is the marginal of f onto $I(m)$ and $f_{ij}^{(m)}$ is the marginal of f onto $B_i(m)\cup B_j(m)\subseteq I(m)$, for $B_i(m)$, $B_j(m)$ containing k lattice sites. Combining Lemmas 2 and 3, we have

$$\begin{aligned}
&l\sum_{m=-1/l}^{1/l}\langle f_{ij}^{(m)}\cdot(A(\eta,B_i(m))-A(\eta,B_j(m)))^2\rangle_\varrho^\varepsilon\\
&\leqq \text{const}\left(\frac{2}{k}+l\sum_{m=-1/l}^{1/l}D_{ij}(\sqrt{f_{ij}^{(m)}})\right)\\
&\leqq \text{const}\left(\frac{2}{k}+2\varepsilon^{-1}l^2\sum_{m=-1/l}^{1/l}\sigma_{I(m)}[f^{(m)}]\right)\\
&\leqq \text{const}\left(\frac{1}{k}+\varepsilon^{-1}l^2\sigma[f]\right)\\
&\leqq \text{const}\left(\frac{1}{k}+l^2\right).
\end{aligned} \tag{2.51}$$

In the next to the last line we used the subadditivity of the entropy production $\sigma[f]$. Clearly, the bound is independent of ε, the choice of the pair i,j and the choice of the cover. Since we may bound the sum in (2.17) by an average over the $\varepsilon^{-1}l$ covers:

$$\begin{aligned}
&\varepsilon\sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-l)}|A(\eta,B_j(x))-A(\eta,B_i(x))|^2\\
&\leqq\frac{1}{\varepsilon^{-1}l}\sum_{y=-(\varepsilon^{-1}l-1)}^{0}\left[l\sum_{m=-1/l}^{1/l}|A(\eta,B_j(m+y))-A(\eta,B_i(m+y))|^2\right]
\end{aligned} \tag{2.52}$$

the estimate follows.

Now, from the combination of the one and two-block estimates, and the inequalities (2.14), (2.15), Lemma 1 follows, and, in particular, the limit in (a). $\square$

The remaining inequalities of Proposition 1 follow in a similar fashion. We consider only the estimate (b), for the right boundary: the treatment of the left boundary is entirely identical. From the inequality

$$\begin{aligned}
&|A(g,[\varepsilon^{-1}(1-l),\varepsilon^{-1}])-\hat{g}(\varrho_+)|\\
&\leqq\frac{1}{J+1}\sum_{j=0}^{J}|A(g,B_j)-\hat{g}(\varrho_+)|\\
&=\frac{1}{J+1}\sum_{j=1}^{J}|A(g,B_j)-\hat{g}(A(\eta,B_j))+\hat{g}(A(\eta,B_j))-\hat{g}(A(\eta,B_0))\\
&\quad+\hat{g}(A(\eta,B_0))-\hat{g}(\varrho_+)|+\frac{1}{J+1}|A(g,B_0)-\hat{g}(\varrho_+)|\\
&\leqq\frac{1}{J}\sum_{j=1}^{J}\{|A(g,B_j)-\hat{g}(A(\eta,B_j))|+\|\hat{g}'\|_\infty|A(\eta,B_j)-A(\eta,B_0)|\}\\
&\quad+\|\hat{g}\|_\infty|A(\eta,B_0)-\varrho_+|+\frac{1}{J}|A(g,B_0)-\hat{g}(\varrho_+)|\ ,
\end{aligned}\tag{2.53}$$

where B_j are blocks of $k+2R$ spins counted successively from the right boundary and $J=[\![\varepsilon^{-1}l/k+2R]\!]$, it follows further by Cauchy-Schwartz that

$$\begin{aligned}
&(A(g,[\varepsilon^{-1}(1-l),\varepsilon^{-1}])-\hat{g}(\varrho_+))^2\\
&\leqq4\left\{O\left(\frac{\varepsilon^2k^2}{l^2}\right)+\frac{1}{J}\sum_{j=1}^{J}(A(g,B_j)-\hat{g}(A(\eta,B_j)))^2\right.\\
&\qquad\qquad+\|\hat{g}'\|_\infty^2\frac{1}{J}\sum_{j=1}^{J}(A(\eta,B_j)-A(\eta,B_0))^2\\
&\qquad\qquad\left.+\|\hat{g}\|_\infty^2(A(\eta,B_0)-\varrho_+)^2\right\}\ .
\end{aligned}\tag{2.54}$$

It appears that it is sufficient to show that

(one block estimates)

$$\lim_{k\to\infty}\lim_{\varepsilon\to0}\sup_{f\in S(\varepsilon)}\langle f\cdot(A(g,B_0)-\hat{g}(\varrho_+))^2\rangle_\varrho^\varepsilon=0\ , \tag{2.55}$$

$$\lim_{k\to\infty}\lim_{\varepsilon\to0}\sup_{j}\sup_{f\in S(\varepsilon)}\langle f\cdot(A(g,B_j)-\hat{g}(A(\eta,B_j))^2\rangle_\varrho^\varepsilon=0 \tag{2.56}$$

(two block estimate)

$$\lim_{k\to\infty}\lim_{l\to0}\lim_{\varepsilon\to0}\sup_{j}\sup_{f\in S(\varepsilon)}\langle f\cdot(A(\eta,B_j)-A(\eta,B_0))^2\rangle_\varrho^\varepsilon=0\ . \tag{2.57}$$

Proof of the One-Block Estimates. Now the marginal of $f^\varepsilon\nu_l$ in the interval $[\varepsilon^{-1}-k-R,\varepsilon^{-1}]$ is required. Denote the density of this marginal by f_+^ε, considered as a measure on configurations in a fixed interval $[-k-R,0]$. With

$$\sigma_+[f_+]=\frac{1}{2}\sum_{x=-k+1}^{0}\left\langle c(x,x-1;\eta)[f_+(\eta^{x,x-1})-f_+(\eta)]\log\left[\frac{f_+(\eta^{x,x-1})}{f_+(\eta)}\right]\right\rangle_\varrho$$
$$+\frac{1}{2}\Big\langle c_+(0,\eta)[e^{(\lambda_+-\lambda)(2\eta_0-1)}f_+(\eta^0)$$
$$-f_+(\eta)]\log\left[\frac{e^{(\lambda_+-\lambda)(2\eta_0-1)}f_+(\eta^0)}{f_+(\eta)}\right]\Big\rangle_\varrho\ , \tag{2.58}$$

it follows that

$$\sigma_+[f_+^\varepsilon]\leqq c\cdot\varepsilon \tag{2.59}$$

and thus any weak limit f_+^* along a subsequence has $\sigma_+[f_+^*]=0$. As previously, σ_+ is a non-negative, convex functional, which, now, is strictly convex on sets of measures with fixed occupancy in the boundary interval $[-k-R,\ -k-1]$. It is easy to see that the unique minimum on this class is the finite version of the (grand canonical) Gibbs measure at chemical potential $\lambda(\varrho_+)$. By the L^2-law of large numbers for such ensembles it therefore follows that

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{\varepsilon\to 0}}\langle f^\varepsilon\cdot(A(g,B_0)-\hat{g}(\varrho_+))^2\rangle\leqq\overline{\lim_{k\to\infty}}\ \langle(A(g,B_0)-\hat{g}(\varrho_+))^2\rangle_{+,\varrho_+}=0\ . \tag{2.60}$$

The second one-block estimate is obtained exactly as in (a), by using the L^2-law of large numbers for the canonical Gibbs measure.

Proof of the Two Block Estimate. Let I_+ denote the block $\{\varepsilon^{-1}(1-l),\dots,\varepsilon^{-1}\}$, $I_+=\bigcup_{j=0}^{J}B_j$. Exactly as in the proof of the previous two-block estimate, we introduce the exclusion process with only "long-range" jumps between B_0 and B_j with the generator

$$(L_{0j}f)(\eta)=\frac{1}{2k^2}\sum_{x\in B_0}\sum_{y\in B_j}\tilde{c}(x,y;\eta)[f(\eta^{xy})-f(\eta)] \tag{2.61}$$

and corresponding Dirichlet form

$$D_{0j}(f)=\frac{1}{2k^2}\sum_{x\in B_0}\sum_{y\in B_j}\langle\tilde{c}(x,y;\eta)[f(\eta^{xy})-f(\eta)]^2\rangle_\varrho\ . \tag{2.62}$$

It follows as previously that

$$\langle f^\varepsilon(A(\eta,B_j)-A(\eta,B_0))^2\rangle\leqq\text{const}\left(\frac{1}{k}+D_{0j}(\sqrt{f_{0j}})\right)\quad\text{from Lemma 3} \tag{2.63}$$

$$\leqq\text{const}\left(\frac{1}{k}+\varepsilon^{-1}l\sigma_{I_+}[f_{I_+}]\right)\quad\text{from Lemma 2} \tag{2.64}$$

$$\leqq\text{const}\left(\frac{1}{k}+l\right)\ , \tag{2.65}$$

by the monotonicity of the entropy production. This yields the two block estimate (2.57). This completes the proof of Proposition 1. □

Suppose that $(\mu^\varepsilon|\varepsilon>0)$ is a sequence of probability measures, $\mu^\varepsilon\in S(\varepsilon)$ for each $\varepsilon>0$. Let $\mathcal{M}_1$ be the set of all measurable functions ϱ on $[-1,1]$ such that $0\leqq\varrho(q)\leqq 1$ almost everywhere, equipped with the topology of weak convergence. For a microscopic configuration $\eta\in 2^\Lambda$ set

$$\varrho^\varepsilon(q)\equiv\eta_{[\![\varepsilon^{-1}q]\!]}\ , \tag{2.66}$$

so that $\varrho^\varepsilon\in\mathcal{M}_1$ ($[\![\cdot]\!]$ denotes the integer part). The measure μ^ε induces via this identification a measure P^ε on $\mathcal{M}_1$. Consider now any weak limit P^* of P^ε as $\varepsilon\to 0$ (as we see later, such limits always exist by compactness). From the arguments above we can infer some regularity in the support of P^* which shall be required in the proof of the following section. We state the result as:

Lemma 4. *For all* $d\in(0,d_0)$,

$$E^*\left[\int_{-1}^{1-d} dq(\varrho(q+d)-\varrho(q))^2\right]\leqq \text{const}\, d^2\ . \tag{2.67}$$

Consequently, $\varrho\in H^1$ P^*-a.s. *and, in fact,*

$$E^*\left[\int_{-1}^{1} dq(\varrho'(q))^2\right]\leqq \text{const}\ . \tag{2.68}$$

Also,

$$\lim_{l\to 0} E^*\left[\left(\frac{1}{l}\int_{1-l}^{1} dq\varrho(q)-\varrho_+\right)^2\right]=0 \tag{2.69}$$

and

$$\lim_{l\to 0} E^*\left[\left(\frac{1}{l}\int_{-1}^{-1+l} dq\varrho(q)-\varrho_-\right)^2\right]=0\ . \tag{2.70}$$

Hence,

$$\varrho(\pm 1)=\varrho_\pm\ P^*\text{-a.s.} \tag{2.71}$$

Proof. It follows from the proof of the two-block estimate that

$$\sup_{f\in S(\varepsilon)}\left\langle f^\varepsilon\left[\varepsilon\sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-d)}(A(\eta,[x,x+k])-A(\eta,[x+\varepsilon^{-1}d-k,x+\varepsilon^{-1}d]))^2\right]\right\rangle$$
$$\leqq \text{const}\left(\frac{1}{k}+d^2\right)\ . \tag{2.72}$$

[See especially Eqs. (2.51–52).] Now we write

$$\left\langle f\varepsilon\sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-d)}(A(\eta,[x,x+\varepsilon^{-1}l])-A(\eta,[x+\varepsilon^{-1}(d-l),x+\varepsilon^{-1}d]))^2\right\rangle$$
$$\leqq\left\langle f\varepsilon\sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-l)}(A(\eta,[x,x+\varepsilon^{-1}l])-A(\eta,[x,x+k]))^2\right\rangle$$
$$+\left\langle f\varepsilon\sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-d)}(A(\eta,[x,x+k])-A(\eta,[x+\varepsilon^{-1}d-k,x+\varepsilon^{-1}d]))^2\right\rangle$$
$$+\left\langle f\varepsilon\sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-d)}(A(\eta,[x+\varepsilon^{-1}d-k,x+\varepsilon^{-1}d])\right.$$
$$\left.-A(\eta,[x+\varepsilon^{-1}(d-l),x+\varepsilon^{-1}d]))^2\right\rangle\ . \tag{2.73}$$

As a consequence of the two-block estimate (2.17) [see also (2.15)] and (2.72), we have

$$\overline{\lim_{l\to 0}}\ \overline{\lim_{\varepsilon\to 0}} \left\langle f\cdot\varepsilon \sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-d)} (A(\eta,[x,x+\varepsilon^{-1}l]) - A(\eta,[x+\varepsilon^{-1}(d-l),x+\varepsilon^{-1}d]))^2 \right\rangle$$
$$\leqq \mathrm{const}\, d^2 \ . \tag{2.74}$$

We may replace step-function averages by averages with the smooth functions $\psi_{q,l}$ and the discrete sum $\varepsilon \sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-d)}$ by an integral [see the discussion in the following section, after (3.30)] to obtain finally

$$\overline{\lim_{l\to 0}}\ \overline{\lim_{\varepsilon\to 0}} E^{\varepsilon}\left[\varepsilon \int_{-1}^{1-d} dq(X(\psi_{q,l})) - X(\psi_{q+d,l}))^2\right] \leqq \mathrm{const}\, d^2 \ . \tag{2.75}$$

Then it follows, taking the limits, that for all $d\in(0,d_0)$

$$E^*\left[\int_{-1}^{1-d} dq(\varrho(q)-\varrho(q+d))^2\right] \leqq \mathrm{const}\, d^2 \ , \tag{2.76}$$

which is (2.67). We next observe that

$$\sup_{\substack{\varphi\in C_0^\infty[-1,1]\\ \|\varphi\|_2=1}} \left|\int_{-1}^{1} dq\,\varphi'(q)\varrho(q)dq\right|^2$$
$$= \sup_{\varphi:\|\varphi\|_2=1} \lim_{d\to 0} \left|\int_{-1+d}^{1} dq\left(\frac{\varphi(q)-\varphi(q-d)}{d}\right)\varrho(q)\right|^2$$
$$= \sup_{\varphi:\|\varphi\|_2=1} \lim_{d\to 0} \left|\int_{-1}^{1-d} dq\,\varphi(q)\left(\frac{\varrho(q+d)-\varrho(q)}{d}\right)\right|^2$$
$$\leqq \underline{\lim_{d\to 0}} \int_{-1}^{1-d} dq\left(\frac{\varrho(q+d)-\varrho(q)}{d}\right)^2 \quad \text{by Cauchy-Schwartz} \ . \tag{2.78}$$

Since

$$\int_{-1}^{1-d} dq\left(\frac{\varrho(q+d)-\varrho(q)}{d}\right)^2 = \sup_{\varphi:\|\varphi\|_2=1} \left|\int_{-1}^{1-d} dq\,\varphi(q)\left(\frac{\varrho(q+d)-\varrho(q)}{d}\right)\right|^2 \tag{2.79}$$

is a measurable function of ϱ (as a supremum of continuous functions), Fatou's lemma may be applied to obtain

$$E^*\left[\sup_{\varphi:\|\varphi\|_2=1} \left|\int_{-1}^{1} dq\,\varphi'(q)\varrho(q)\right|^2\right]$$
$$\leqq E^*\left[\underline{\lim_{d\to 0}} \int_{-1}^{1-d} dq\left(\frac{\varrho(q+d)-\varrho(q)}{d}\right)^2\right]$$
$$\leqq \underline{\lim_{d\to 0}} E^*\left[\int_{-1}^{1-d} dq\left(\frac{\varrho(q+d)-\varrho(q)}{d}\right)^2\right]$$
$$\leqq \mathrm{const} \ , \tag{2.80}$$

by the bound (2.67). From (2.80) and the Riesz theorem we infer that $\varrho \in H^1$ P^*-a.s. In that case, further,

$$\|\varrho'\|_2^2 = \sup_{\varphi:\|\varphi\|_2=1} \left| \int_{-1}^{1} dq \varphi'(q)\varrho(q) \right|^2 P^*\text{-a.s.} \tag{2.81}$$

and (2.68) follows. The Eqs. (2.69–70) follow in a very similar manner from (2.7–8) [for $g_0(\eta)=\eta_0$], replacing step function averages by averages with respect to $\psi_{\pm 1,l}$, taking $\varepsilon \to 0$ and then returning to step functions. From these it follows that along a subsequence $(l_n | n \in \omega)$, $l_n \downarrow 0$,

$$\lim_{n\to\infty} \frac{1}{l_n} \int_{1-l_n}^{1} dq \varrho(q) = \varrho_+ \, P^*\text{-a.s.} \ , \tag{2.82}$$

and

$$\lim_{n\to\infty} \frac{1}{l_n} \int_{-1}^{-1+l_n} dq \varrho(q) = \varrho_- \, P^*\text{-a.s.} \ , \tag{2.83}$$

by the Borel-Cantelli theorem. By the P^*-a.s. continuity of ϱ, the boundary conditions (2.71) follow. □

3. Stationary Hydrodynamics (Hydrostatics)

The main result of the present section is a hydrodynamic law of large numbers for the stationary state of the models presented in the Introduction. However, the method of proof should give the same result for the steady state of all gradient models with local reversibility. We have observed already in the previous section that there is a unique measure μ_{SS}^{ε} on 2^{Λ}, $\Lambda = \mathbb{Z} \cap [-M, M]$ with $M^{-1} = \varepsilon$, stationary under the dynamics. Let $\varphi \in C_0^{\infty}[-1,1]$. The density field is defined by

$$X^{\varepsilon}(\varphi) = \varepsilon \sum_{x=-M}^{M} \varphi(\varepsilon x)\eta_x \ . \tag{3.1}$$

We shall prove that for every $\delta > 0$

$$\lim_{\varepsilon\to 0} \mu_{SS}^{\varepsilon}\left(\left| X^{\varepsilon}(\varphi) - \int_{-1}^{1} dq \varphi(q)\bar{\varrho}(q) \right| > \delta \right) = 0 \ , \tag{3.2}$$

where $\bar{\varrho}(q)$ is the solution of the stationary hydrodynamic equation

$$\partial_q [D(\varrho(q))\partial_q \varrho(q)] = 0 \tag{3.3}$$

with boundary conditions

$$\varrho(\pm) = \varrho_{\pm} \ . \tag{3.4}$$

(Here, $\varrho_{\pm}$ are the density in the global equilibrium state at temperature β and chemical potential $\lambda_{\pm}$.) In (3.3), $D(\varrho)$ is the bulk diffusion coefficient calculated from the Green-Kubo formula:

$$D(\varrho) = \hat{h}'(\varrho) = \langle c(0, e_1)(\eta_0 - \eta_{e_1})^2 \rangle_{\varrho} \ . \tag{3.5}$$

We state the theorem and then sketch its proof. As in the preceding section, let $\mathcal{M}_1$ be the set of all measurable functions ϱ on $[-1,1]$ such that $0 \leqq \varrho(q) < 1$ almost everywhere, equipped with the topology of weak convergence. For a microscopic configuration $\eta \in 2^{\Lambda}$ set

$$\varrho^{\varepsilon}(q) \equiv \eta_{[\![\varepsilon^{-1}q]\!]} \ , \tag{3.6}$$

so that $\varrho^{\varepsilon} \in \mathcal{M}_1$. The measure μ^{ε}_{SS} induces via this identification a measure P^{ε} on $\mathcal{M}_1$ and a corresponding random field

$$X^{\varepsilon}(\varphi) \equiv \int_{-1}^{1} dq\, \varphi(q) \varrho^{\varepsilon}(q) \ . \tag{3.7}$$

Note that $X(\varphi) \equiv \int_{-1}^{1} dq\, \varphi(q) \varrho(q)$ for all $\varphi \in C_0^{\infty}[-1,1]$ uniquely determines ϱ. Now consider the deterministic density field

$$X(\varphi) \equiv \int_{-1}^{1} dq\, \varphi(q) \bar{\varrho}(q) \ , \tag{3.8}$$

where $\bar{\varrho}(q)$ is the (unique) solution of the boundary-value problem (3.3–4) and let P be the delta-distribution of that field on $\mathcal{M}_1$. Then:

Theorem 1. *P is the weak limit of P^{ε} as $\varepsilon \to 0$.*

We first outline the proof and then sketch the details. The proof requires the verification of two statements:

(1) Tightness of $(P^{\varepsilon} | \varepsilon > 0)$.

(2) For any weak limit point P^* of the sequence $(P^{\varepsilon} | \varepsilon > 0)$ and $\varphi, \psi \in C_0^{\infty}[-1,1]$,

a) $$\int_{-1}^{1} dq\, \varphi''(q) E^*[\hat{h}(\varrho(q))] = 0 \ , \tag{3.9}$$

b) $$\int_{-1}^{1} \int_{-1}^{1} dq\, dp \{ \varphi''(q) \psi(p) E^*[\hat{h}(\varrho(q)) \varrho(p)] + \varphi(q) \psi''(p) E^*[\hat{h}(\varrho(p)) \varrho(q)] \} = 0 \ . \tag{3.10}$$

It is shown that the conditions of (2) require, in fact, that the weak limit E^* be a delta distribution $\delta_{\bar{\varrho}}$ on the unique density profile $\bar{\varrho}$ which satisfies the (weak) stationary hydrodynamic equation

$$\int_{-1}^{1} dq\, \varphi''(q) \hat{h}(\bar{\varrho}(q)) = 0 \ , \quad \text{with } \bar{\varrho}(\pm 1) = \varrho_{\pm} \ ; \tag{3.11}$$

this, together with the statement of tightness in (1), gives the final result. To understand, in intuitive terms, why the equations in (2) give uniqueness, linearize Eq. (3.10) around the solution $\bar{\varrho}$ as

$$\varrho(q) = \bar{\varrho}(q) + \delta\varrho(q) \tag{3.12}$$

to obtain (formally)

$$\partial_q^2 [D(\bar{\varrho}(q)) E^*(\delta\varrho(q) \delta\varrho(p))] + \partial_p^2 [D(\bar{\varrho}(p)) E^*(\delta\varrho(q) \delta\varrho(p))] = 0 \ . \tag{3.13}$$

This linear equation for $E^*(\delta\varrho(q)\delta\varrho(p))$ is the same as that derived for the fluctuation covariance by fluctuating hydrodynamics [Sch], except that there is no delta-function source term on the right-hand side. It has the solution (unique subject to the conditions of vanishing at $q,p=\pm 1$)

$$E^*(\delta\varrho(q)\delta\varrho(p))=0\ , \tag{3.14}$$

which implies, of course, that the density profile $\varrho(q)$ in the measure P^* is deterministic and given by $\bar{\varrho}(q)$. The proof we sketch below makes rigorous this formal argument.

In order to verify the statements (1) and (2) for Theorem 1, we must first show that Proposition 1 is applicable to the steady state measure μ_{SS}^{ε}. We state this as:

Proposition 2. $\mu_{SS}^{\varepsilon}\in S(\varepsilon)$, i.e.

$$\sigma^{\varepsilon}[\mu_{SS}^{\varepsilon}]\leqq C\cdot\varepsilon \quad \textit{for some} \quad C>0\ . \tag{3.15}$$

Proof. To obtain the upper bound (3.15), we observe that $\sigma^{\varepsilon}[\mu]$ may be written as

$$\sigma^{\varepsilon}[\mu]=-\sum_{\eta}(L^*\mu)(\eta)(\log\mu(\eta)+H(\eta))$$
$$+\lambda_+\sum_{\eta}\mu(\eta)j_+(\eta)+\lambda_-\sum_{\eta}\mu(\eta)j_-(\eta) \tag{3.16}$$

(see [ELS]). In this expression, the boundary currents $j_{\pm}$ are defined by

$$j_{\pm}(\eta)=c(\pm M,\eta)(2\eta_{\pm M}-1)\ , \tag{3.17}$$

and L^* is the adjoint with respect to counting measure of the generator defined in (1.2):

$$(L^*\mu)(\eta)=\tfrac{1}{2}\sum_{\substack{x,y\in\Lambda\\|x-y|=1}}[c(x,y;\eta^{xy})\mu(\eta^{xy})-c(x,y;\eta)\mu(\eta)]$$
$$+\sum_{x:x_1=\pm M}[c(x,\eta^x)\mu(\eta^x)-c(x,\eta)\mu(\eta)]\ . \tag{3.18}$$

Since the defining property of the stationary measure is that $L^*\mu_{SS}^{\varepsilon}=0$, it follows from (3.16) that

$$\sigma^{\varepsilon}[\mu_{SS}^{\varepsilon}]=(\lambda_+-\lambda_-)\langle j_1(0)\rangle_{SS}^{\varepsilon}\ , \tag{3.19}$$

if $\langle\cdot\rangle_{SS}^{\varepsilon}$ denotes expectation with respect to μ_{SS}^{ε}. We have employed the consequence of stationarity and conservation of particle number

$$\langle j_+\rangle_{SS}^{\varepsilon}=-\langle j_-\rangle_{SS}^{\varepsilon}=\langle j_1(x)\rangle_{SS}^{\varepsilon}\ , \tag{3.20}$$

for $x\in\Lambda$. If we define the current

$$j_{\Lambda',1}(\eta)=\frac{1}{2(M-R)}\sum_{x=-(M-R)}^{M-R}j_{x,x+1}(\eta) \tag{3.21}$$

averaged over the interior block $[-(M-R),M-R]$, then the same reasoning implies that

$$\langle j_1(0)\rangle_{SS}^{\varepsilon}=\langle j_{\Lambda',1}\rangle_{SS}^{\varepsilon}\ . \tag{3.22}$$

On the other hand, by the gradient condition (1.7),

$$j_{\Lambda',1}(\eta)=\frac{h_{M-R}(\eta)-h_{-(M-R)}(\eta)}{2(M-R)} \ . \tag{3.23}$$

Because $h_x(\eta)$ is a bounded function uniformly in x, η

$$j_{\Lambda',1}(\eta)=0\left(\frac{1}{M}\right) , \tag{3.24}$$

uniformly in η. Thus, the upper bound (3.15) follows by the combination of (3.19), (3.22) and (3.24). □

(1)
The key point for (1) is that $\mathscr{M}_1$ is a compact, metrizable space in the weak topology. The proof of this is an easy modification of the well-known proof that the set of probability measures $\mathscr{P}(X)$ on a compact metric space X is itself a compact, metrizable space in the weak topology (see e.g. [P]). We therefore make only a few remarks. The proof proceeds by imbedding $\mathscr{M}_1$ in the compact product space $[-2,2]^\omega$ via the mapping

$$T(\varrho)=\left[\int_{-1}^{1} dq g_k(q)\varrho(q)|k\in\omega\right] , \tag{3.25}$$

for $(g_k|k\in\omega)$ a sequence of elements dense in the unit ball of $C[-1,1]$. The important point to verify is that $T[\mathscr{M}_1]$ is closed in $[-2,2]^\omega$. The estimate for $g\in C[-1,1]$

$$|\mu_n(g)|\equiv\left|\int_{-1}^{1} dq g(q)\varrho_n(q)\right|\leqq 2\|g\|_p \tag{3.26}$$

gives for any weak limit μ the same estimate, $|\mu(g)|\leqq 2\|g\|_p$ and, hence, absolute continuity with respect to Lebesgue measure by the Riesz theorem. It is easy to verify the bounds $0\leqq\varrho(q)\leqq 1$ a.e. for the density, so that ϱ is the preimage in $\mathscr{M}_1$ of the limit point of $T(\varrho_n)$ in $[-2,2]^\omega$.

Then, since $\mathscr{M}_1$ is a compact, metric space, it follows automatically that the sequence of measures $(P^\varepsilon|\varepsilon>0)$ on $\mathscr{M}_1$ is relatively compact in the weak topology.

(2)
The proof of (2a) depends upon the fact that

$$L\eta_x=(\Delta h)_x(\eta) \tag{3.27}$$

for $|x\pm M|>R$, which is the consequence of the gradient condition. We shall prove the results first for ϕ with compact support in $[-1,1]$, i.e. vanishing in an interval of the boundaries, and then for arbitrary $\phi\in C_0^\infty[-1,1]$ by approximation. We note that for such ϕ we can choose ε sufficiently small that $\phi(\varepsilon x)=0$ for $|x\pm M|\leqq R$, and that

$$\begin{aligned}\varepsilon^{-2}LX^{\varepsilon,l}(\phi)&=\varepsilon\sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-l)}\phi(\varepsilon x)\varepsilon^{-2}(\Delta h)_x(\eta)\\&=\varepsilon\sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-l)}\phi''(\varepsilon x)h_x(\eta)+O(\varepsilon) \ .\end{aligned} \tag{3.28}$$

By stationarity,

$$\mu_{SS}^{\varepsilon}\left[\varepsilon \sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-l)} \phi''(\varepsilon x) h_x(\eta)\right] = O(\varepsilon) \ . \tag{3.29}$$

We now introduce a class of smooth "averaging functions" $\psi_{q,l}$, $q \in [-1, 1-l]$, $l \in (0, l_0)$, defined by

$$\psi_{q,l}(q') = \begin{cases} 1/l & q' \in [q+\delta, q+l-\delta] \\ 0 & q' \in [-1,1] \backslash [q, q+l] \ , \end{cases} \tag{3.30}$$

and which interpolate between these values on the intervals $(q, q+\delta)$ and $(q+l-\delta, q+l)$ as a C^∞ function, $\leqq 1/l$. (A δ-dependence should be indicated for the function $\psi_{q,l}$, which we have omitted; we shall consider a limit in which $l \to 0$, $\delta/l \to 0$.) We note that

$$\left| X^\varepsilon(\psi_{\varepsilon x, l}) - \frac{\varepsilon}{l} \sum_{y=x}^{x+\varepsilon^{-1} l} \eta_y \right| \leqq \frac{2\delta}{l} \ , \tag{3.31}$$

and thus,

$$\begin{aligned} &\mu_{SS}^{\varepsilon}\left(\left|\varepsilon \sum_{x=\varepsilon^{-1}}^{\varepsilon^{-1}(1-l)} \phi''(\varepsilon x)\left(\hat{h}(X^\varepsilon(\psi_{\varepsilon x,l})) - \hat{h}\left(\frac{\varepsilon}{l} \sum_{y=x}^{x+\varepsilon^{-1}l} \eta_y\right)\right)\right|\right) \\ &\leqq \|\hat{h}'\|_\infty \|\phi''\|_1 \cdot \frac{2\delta}{l} \ , \end{aligned} \tag{3.32}$$

so that from (3.29) and (3.32)

$$\begin{aligned} &\left|\varepsilon \sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-l)} \phi''(\varepsilon x) E^\varepsilon[\hat{h}(X^\varepsilon(\psi_{\varepsilon x,l}))]\right| \leqq \text{const}\, \varepsilon + \|\hat{h}'\|_\infty \|\varphi''\|_1 \frac{2\delta}{l} \\ &+ \mu_{SS}^{\varepsilon}\left(\left|\varepsilon \sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-l)} \phi''(\varepsilon x)\left(h_x(\eta) - \hat{h}\left(\frac{\varepsilon}{l} \sum_{y=x}^{x+\varepsilon^{-1}l} \eta_y\right)\right)\right|\right) \ . \end{aligned} \tag{3.33}$$

Thus, by (3.33) and the fundamental Proposition 1 (a),

$$\lim_{\substack{l\to 0 \\ \delta/l \to 0}} \lim_{\varepsilon \to 0} \varepsilon \sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}(1-l)} \phi''(\varepsilon x) E^\varepsilon[\hat{h}(X^\varepsilon(\psi_{\varepsilon x,l}))] = 0 \ , \tag{3.34}$$

or

$$\lim_{\substack{l\to 0 \\ \delta/l \to 0}} \lim_{\varepsilon \to 0} \int_{-1}^{1-l} dq \phi''(q) E^\varepsilon[\hat{h}(X(\psi_{q,l}))] = 0 \ , \tag{3.35}$$

using $|\varepsilon [\![q\varepsilon^{-1}]\!] - q| \leqq \varepsilon$ and the nice properties of ϕ, $\hat{h}$ and $\psi_{q,l}$.

Since $\hat{h}(X(\psi_{q,l}))$ is a bounded, continuous function on $\mathscr{M}_1$, it follows that for any weakly convergent subsequence $(P^{\varepsilon_n} | n \in \omega)$, $P^{\varepsilon_n} \to P^*$, that

$$\lim_{n\to 0} E^{\varepsilon_n}[\hat{h}(X(\psi_{q,l}))] = E^*[\hat{h}(X(\psi_{q,l}))] \tag{3.36}$$

and by dominated convergence

$$\lim_{\substack{l\to 0 \\ \delta/l \to 0}} \int_{-1}^{1-l} dq \phi''(q) E^*[\hat{h}(X(\psi_{q,l}))] = 0 \ . \tag{3.37}$$

Since, by the Lebesgue-Vitali theorem and dominated convergence,

$$\lim_{\substack{l\to 0\\ \delta/l\to 0}} E^*[\hat{h}(X(\psi_{q,l}))]=E^*[\hat{h}(\varrho(q))] \text{ a.e. } q\in[-1,1] \ , \tag{3.38}$$

it follows from (3.37) (again by dominated convergence) that

$$\int_{-1}^{1} dq\phi''(q)E^*[\hat{h}(\varrho(q))]=0 \ . \tag{3.39}$$

Finally, the restriction to ϕ with compact support can be removed easily by approximating elements of $C_0^\infty[-1,1]$ by such ϕ and applying (once more!) dominated convergence.

The proof of (2b) is very similar, but is based on the identity

$$L(\eta_x\eta_y)-(L\eta_x)\eta_y-\eta_x(L\eta_y)=(\delta_{x,y}-\delta_{x+1,y})c(x,x+1;\eta)$$
$$+(\delta_{x,y}-\delta_{x-1,y})c(x,x-1;\eta) \tag{3.40}$$

instead of (3.27), for both $|x\pm M|>R$, $|y\pm M|>R$. From this, the gradient condition, and stationarity, it follows that for φ,ψ of compact support

$$\sum_{x,y} \varphi(\varepsilon x)\psi(\varepsilon y)\mu^\varepsilon_{SS}[(\Delta h)_x(\eta)\eta_y+\eta_x(\Delta h)_y(\eta)]$$
$$=-\sum_x \mu^\varepsilon_{SS}[c(x,x+1;\eta)][\varphi(\varepsilon x)\psi(\varepsilon x)-\varphi(\varepsilon x+\varepsilon)\psi(\varepsilon x)$$
$$-\varphi(\varepsilon x)\psi(\varepsilon x+\varepsilon)+\varphi(\varepsilon x+\varepsilon)\psi(\varepsilon x+\varepsilon)]+O(\varepsilon) \tag{3.41}$$

or,

$$\mu^\varepsilon_{SS}\left[\left(\varepsilon\sum_x \varphi''(\varepsilon x)h_x(\eta)\right)\left(\varepsilon\sum_y \psi(\varepsilon y)\eta_y\right)+\left(\varepsilon\sum_x \varphi(\varepsilon x)\eta_x\right)\left(\varepsilon\sum_y \psi''(\varepsilon y)h_y(\eta)\right)\right]$$
$$=-\varepsilon^2\sum_x \varphi'(\varepsilon x)\psi'(\varepsilon x)\mu^\varepsilon_{SS}(c(x,x+1;\eta))+O(\varepsilon^2) \ . \tag{3.42}$$

Thus,

$$\lim_{\varepsilon\to 0} E^\varepsilon[X(h;\varphi'')X(\psi)+X(\varphi)X(h;\psi'')]=0 \ , \tag{3.43}$$

$\left(\text{with, of course, } X(h;\varphi'')=\varepsilon\sum_{x=-\varepsilon^{-1}+R}^{\varepsilon^{-1}-R}\varphi''(\varepsilon x)h_x(\eta).\right)$ From here we proceed similarly as for (2a), so we sketch the argument in broad strokes.

Using Proposition 1 (a)

$$\lim_{l\to 0}\lim_{\varepsilon\to 0} E^\varepsilon\left|\varepsilon\sum_x \varphi(\varepsilon x)\left(h_x(\eta)-\hat{h}\left(\frac{1}{l}\sum_{y=x}^{x+\varepsilon^{-1}l}\right)\right)\right|=0 \ , \tag{3.44}$$

and the bounds $|X^\varepsilon(\psi)|\leqq \text{const}\,\|\psi\|_1$, $|X^\varepsilon(\varphi)|\leqq\text{const}\,\|\varphi\|_1$, and introducing the averaging functions $\psi_{q,l}$, we obtain from (3.43)

$$\lim_{\substack{l\to 0\\ \delta/l\to 0}}\lim_{\varepsilon\to 0}\left\{\int_{-1}^{1-l} dq\varphi''(q)E^\varepsilon[\hat{h}(X(\psi_{q,l}))X(\psi)]\right.$$
$$\left.+\int_{-1}^{1-l} dp\psi''(p)E^\varepsilon[\hat{h}(X(\psi_{p,l}))X(\varphi)]\right\}=0 \ . \tag{3.45}$$

Then, by dominated convergence

$$\lim_{\substack{l\to 0\\ \delta/l\to 0}} \left\{ \int_{-1}^{1-l} dq\varphi''(q)E^*[\hat{h}(X(\psi_{q,l}))X(\psi)] + \int_{-1}^{1-l} dp\psi''(p)E^*[\hat{h}(X(\psi_{p,l}))X(\phi)] \right\} = 0 \tag{3.46}$$

by considering a subsequence $P^{\varepsilon_n}\to P^*$. Replacing $X(\psi_{q,l})$ by $\frac{1}{l}\int_q^{q+l} dq'\varrho(q')$, employing the Lebesgue-Vitali theorem and dominated convergence gives

$$\int_{-1}^{1} dq\varphi''(q)E^*[\hat{h}(\varrho(q))X(\psi)] + \int_{-1}^{1} dp\psi''(p)E^*[X(\varphi)\hat{h}(\varrho(p))] = 0 \tag{3.47}$$

which is (2b).

We may now give the argument for a deterministic limit $\delta_{\bar{\varrho}}$. As a matter of fact, note that there is a unique solution $\bar{\varrho}$ of the weak stationary hydrodynamic equations

$$\int_{-1}^{1} dq\varphi''(q)\hat{h}(\varrho(q)) = 0 , \quad \varphi\in C_0^\infty[-1,1] , \tag{3.48}$$

with the weak boundary conditions $\lim_{l\to 0}\frac{1}{l}\int_{1-l}^{1} dq\varrho(q) = \varrho_+$, $\lim_{l\to 0}\frac{1}{l}\int_{-1}^{-1+l} dq\varrho(q) = \varrho_-$. This follows here by elementary arguments, since (3.48) implies that, as a distribution on $C_0^\infty[-1,1]$, $\hat{h}(\varrho(q))$ is a linear *function* of q. Also, $\hat{h}(\varrho)$ is smooth for $\varrho\in[0,1]$ and, further

$$\hat{h}'(\varrho) = \frac{1}{2\chi(\varrho)}\langle c(0,e_1)(\eta_0-\eta_{e_1})^2\rangle_\varrho > 0 , \tag{3.49}$$

so that $\hat{h}$ is strictly monotonic. Hence, the unicity follows easily and, in fact, the result that $\bar{\varrho}$ is C^∞.

We consider now the following correlation function $G:[-1,1]^2\to\mathbb{R}$ defined by

$$\begin{aligned} G(q,p) &= E^*[\delta\varrho(q)\delta h(p)] \\ &\equiv E^*[(\varrho(q)-\bar{\varrho}(q))(\hat{h}(\varrho(p))-\hat{h}(\bar{\varrho}(p)))] , \end{aligned} \tag{3.50}$$

as a bounded continuous function. By monotonicity of $\hat{h}$, on the diagonal

$$G(q,q) \geqq 0 , \tag{3.51}$$

and by the P^*-a.s. b.c.'s (2.71),

$$G(q,p) = 0 \quad \text{for} \quad q,p\in\partial[-1,1]^2 . \tag{3.52}$$

Hence, we may expand G as a Fourier series

$$G(q,p) = \sum_{n,m} c_{n,m} e^{2\pi i(nq+mp)} , \tag{3.53}$$

where the convergence is for a.e. (q,p). In fact, we have the following

Lemma 5. *The distributional partial derivatives*

$$\partial_q G\ , \quad \partial_p G \quad \partial^2_{q,p} G \in L^2([-1,1]^2) \tag{3.54}$$

and thus $\sum_{n,m} c_{n,m} e^{2\pi i(nq+mp)}$ *converges absolutely (therefore uniformly) to* $G(q,p)$.

Proof. Consider $\partial_q G$. By the definition of the distributional derivative and the Fubini theorem, it follows that

$$\partial_q G(q,p) = E^*[\delta\varrho'(q)\delta h(p)]\ , \tag{3.55}$$

assuming the regularity of Lemma 4. Then, by Cauchy-Schwartz and the boud (2.68) it follows that

$$\begin{aligned}\int_{-1}^{1} dq \int_{-1}^{1} dp|\partial_q G(q,p)|^2 &\leqq \int_{-1}^{1} dq E^*[(\delta\varrho'(q))^2] \int_{-1}^{1} dp E^*[(\delta h(p))^2] \\ &\leqq \text{const}\ .\end{aligned} \tag{3.56}$$

The proofs for $\partial_p G$, $\partial^2_{q,p} G$ are identical.

The Fourier coefficients associated to $G(q,p)$ are given, we recall, by

$$c_{n,m} = \int_{-1}^{1} \frac{dq}{2} \int_{-1}^{1} \frac{dp}{2} e^{-2\pi i(nq+mp)} G(q,p)\ . \tag{3.57}$$

We now observe that

$$\begin{aligned}\sum_{n,m} |c_{n,m}| &= |c_{0,0}| + \sum_{|n|\geqq 1} \frac{1}{n}\cdot n|c_{n,0}| + \sum_{|m|\geqq 1} \frac{1}{m}\cdot m|c_{0,m}| + \sum_{|n|\geqq 1,|m|\geqq 1} \frac{1}{n\cdot m} n\cdot m|c_{n,m}| \\ &\leqq \frac{1}{4}\|G\|_1 + \sqrt{2\sum_{n\geqq 1}\frac{1}{n^2}}\sqrt{\sum_n n^2|c_{n,0}|^2} \\ &\quad + \sqrt{2\sum_{m\geqq 1}\frac{1}{m^2}}\sqrt{\sum_m m^2|c_{0,m}|^2} \\ &\quad + \left(2\sum_{n\geqq 1}\frac{1}{n^2}\right)\sqrt{\sum_{n,m} n^2 m^2 |c_{n,m}|^2}\ .\end{aligned} \tag{3.58}$$

Then for example,

$$\begin{aligned}\sum_n n^2|c_{n,0}|^2 &= \frac{1}{(2\pi)^2}\sum_n (2\pi n)^2 |c_{n,0}|^2 \\ &= \frac{1}{(2\pi)^2}\int_{-1}^{1}\frac{dq}{2}\left[\int_{-1}^{1}\frac{dp}{2}\partial_q G(q,p)\right]^2 \\ &\leqq \frac{1}{(2\pi)^2}\|\partial_q G\|_2^2 \quad \text{by Cauchy-Schwartz}\ .\end{aligned} \tag{3.59}$$

Likewise,

$$\sum_m m^2|c_{0,m}|^2 \leqq \frac{1}{(2\pi)^2}\|\partial_p G\|_2^2 \tag{3.60}$$

and

$$\sum_{n,m} n^2m^2|c_{n,m}|^2 \leqq \frac{1}{(2\pi)^4} \|\partial^2_{q,p}G\|_2^2 \ , \tag{3.61}$$

so that the result follows.

Because $\bar{\varrho}$ satisfies the stationary hydrodynamic equation and the equations (3.9), (3.10) are valid, the equation

$$\partial_p^2 G(q,p) + \partial_q^2 G(p,q) = 0 \tag{3.62}$$

holds in a distributional sense. Consequently, for all n, m

$$m^2 c_{n,m} + n^2 c_{m,n} = 0 \ ; \tag{3.63}$$

in particular, for $m = -n$,

$$c_{n,-n} + c_{-n,n} = 0 \ . \tag{3.64}$$

Now, by absolute convergence of the Fourier series, we have that

$$\begin{aligned} \int_{-1}^{1} dqG(q,q) &= \sum_{n,m} \int_{-1}^{1} dq e^{2\pi i(n+m)q} \cdot c_{n,m} = \sum_n c_{n,-n} \\ &= \tfrac{1}{2} \sum_n (c_{n,-n} + c_{-n,n}) = 0 \ . \end{aligned} \tag{3.65}$$

On the other hand, for *every* $\varrho \in \mathscr{M}_1$, $q \in [-1, 1]$, by monotonicity of $\hat{h}$,

$$(\varrho(q) - \bar{\varrho}(q))(\hat{h}(\varrho(q)) - \hat{h}(\bar{\varrho}(q)) \geqq 0 \ . \tag{3.66}$$

It therefore follows from

$$\int_{-1}^{1} dqE^* [(\varrho(q) - \bar{\varrho}(q))(\hat{h}(\varrho(q)) - \hat{h}(\bar{\varrho}(q)))] = 0 \ , \tag{3.67}$$

that

$$\varrho(q) = \bar{\varrho}(q) \text{ a.e.} q \ , \qquad P^*\text{-a.s.} \tag{3.68}$$

This concludes the proof of Theorem 1. □

The attentive reader may have noted that considerably more was established than simply the static law of large numbers for the empirical density field. An immediate consequence of the arguments of Proposition 1 is a rather weak form of the local equilibrium property, namely, that the marginal distribution in any microscopic block approaches, as $\varepsilon \to 0$, a convex combination of canonical Gibbs measures. A little more work, in fact, produces from the proofs of Proposition 1 and Theorem 1 a somewhat stronger result, the law of large numbers for an arbitrary extensive field, and two consequences of special physical interest: Fick's law for the current field and the normal transport property. Here, we harvest the work of the previous sections to derive these results.

First, we have the following: let $g_0(\eta)$ be any bounded, local (depending only on variables at sites $x : |x| \leqq R$) function of the configuration, and define an $\mathscr{M}_g$-valued random variable

$$g^\varepsilon(q) \equiv g_{[\![\varepsilon^{-1}q]\!]} \ , \tag{3.69}$$

with $\mathcal{M}_g$ the set of measurable functions $\mathscr{g}$ on $[-1,1]$ such that $\|\mathscr{g}\|_\infty \leqq \sup_\eta |g(\eta)|$. Alternatively, we may consider the random field

$$G^\varepsilon(\psi) \equiv \sum_{x=-\varepsilon^{-1}+R}^{\varepsilon^{-1}-R} \psi(\varepsilon x) g_x(\eta) \ . \tag{3.70}$$

At the risk of some confusion, we denote both the probability measure on $\mathcal{M}_g$ induced by μ_{SS}^ε and the law of the random field $G^\varepsilon(\psi)$ by P_g^ε. We define also the deterministic field

$$G(\psi) = \int_{-1}^{1} dq \psi(q) \hat{g}(\bar{\varrho}(q)) \tag{3.71}$$

with the law P_g (equivalently, the measure $P_g = \delta_{\hat{g} \circ \bar{\varrho}}$ on $\mathcal{M}_g$). Then, we have the result:

Theorem 2 (Convergence of the extensive fields). *P_g is the weak limit of P_g^ε as $\varepsilon \to 0$.*

Proof. For the purposes of the proof, it is convenient to consider P_g^ε to be the joint distribution of the random fields $G^\varepsilon(\psi)$, $X^\varepsilon(\varphi)$. Then, tightness of $(P_g^\varepsilon | \varepsilon > 0)$ follows as before. The result (a) of Proposition 1,

$$\lim_{l \to 0} \lim_{\varepsilon \to 0} \mu_{SS}^\varepsilon \left(\left| \varepsilon \sum_x \psi(\varepsilon x) \left(g_x(\eta) - \hat{g} \left(\frac{\varepsilon}{l} \sum_{y=x}^{x+\varepsilon^{-1} l} \eta_y \right) \right) \right|^2 \right) = 0 \ , \tag{3.72}$$

along with Theorem 1 is then seen to give for any weak limit point P_g^* that

$$P_g^* \left(\left| G(\psi) - \int_{-1}^{1} dq \psi(q) \hat{g}(\bar{\varrho}(q)) \right|^2 \right) = 0 \ , \tag{3.73}$$

i.e.

$$G(\psi) = \int_{-1}^{1} dq \psi(q) \hat{g}(\bar{\varrho}(q)) \ P_g^*\text{-a.s.} \tag{3.74}$$

Thus, the stated result follows. □

From this corollary of the proof of Theorem 1 we may infer the following:

Proposition 3 (Fick's law and normal transport). *Consider the bounded, local function $j_1(\eta) = c(0,1;\eta)(\eta_0 - \eta_1)$, which is the (systematic) current for the configuration η. Then, as $\varepsilon \to 0$,*

$$\varepsilon^{-1} X^\varepsilon(j_1;\psi) \xrightarrow[\text{weakly}]{} -\int_{-1}^{1} dq \psi(q) D(\bar{\varrho}(q)) \partial_q \bar{\varrho}(q) \ ; \tag{3.75}$$

in particular, the current field converges in probability with respect to μ_{SS}^ε to the deterministic limit given by Fick's law. Furthermore, for every $q \in [-1,1]$,

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \langle j_{1,[\![\varepsilon^{-1} q]\!]} \rangle_{SS}^\varepsilon = -D(\bar{\varrho}(q)) \partial_q \bar{\varrho}(q) \ . \tag{3.76}$$

(However, note that both sides of the equality are independent of q!)

Proof. By employing the gradient condition we can write

$$\varepsilon^{-1} X^\varepsilon(j_1 : \psi) = X^\varepsilon(h; \psi') + O(\varepsilon) \ . \tag{3.77}$$

Hence, applying the previous result we have that the law for the random field $\varepsilon^{-1}X^{\varepsilon}(j_1;\psi)$ converges weakly to that for the deterministic field

$$X(h;\psi')=\int_{-1}^{1} dq\psi'(q)\hat{h}(\bar{\varrho}(q))$$
$$=-\int_{-1}^{1} dq\psi(q)\hat{h}'(\bar{\varrho}(q))\partial_q\bar{\varrho}(q)$$
$$=-\int_{-1}^{1} dq\psi(q)D(\bar{\varrho}(q))\partial_q\bar{\varrho}(q)\ . \tag{3.78}$$

This gives (3.75). Now consider any $\psi\in C_0^{\infty}[-1,1]$ such that $\int_{-1}^{1} dq\psi(q)=1$. From the first part of the Proposition and the fact that $\varepsilon^{-1}X^{\varepsilon}(j_1;\psi)=X^{\varepsilon}(h;\psi')+O(\varepsilon)$ is uniformly bounded (in η), it follows that

$$\mu_{SS}^{\varepsilon}(\varepsilon^{-1}X^{\varepsilon}(j_1;\psi))\xrightarrow[\varepsilon\to 0]{}-\int_{-1}^{1} dq\psi(q)D(\bar{\varrho}(q))\partial_q\bar{\varrho}(q)\ . \tag{3.79}$$

However, since both $\mu_{SS}^{\varepsilon}(j_{1,[\![\varepsilon^{-1}q]\!]})$ and $\bar{j}(q)\equiv -D(\bar{\varrho}(q))\partial_q\bar{\varrho}(q)$ are in fact independent of q, we can infer the pointwise statement (3.76). □

4. Discussion

The models we have been investigating in this work satisfy, in the hydrodynamic limit, the expectations from standard nonequilibrium thermodynamics of stationary states. The theorems we have proved for lattice gases are likely to hold also, *mutatis mutandis*, for more realistic but intractable cases, such as the hard sphere fluid. Of course, our methods of proof will not carry over in an obvious way. The dynamical properties such as "ergodicity" or "stochasticity" which it would presumably be necessary to establish *a priori* for deterministic dynamical models in order to derive hydrodynamics, are here incorporated by hand by our adoption of a stochastic dynamics. Even for the generic lattice gas of the type presented in the Introduction, we have not been able to carry through the proofs. On the other hand, our models may help to clarify some of the unusual features of the structure of the non-equilibrium steady state.

As an example of this, we would like to discuss at some length the subject of *long-ranged hydrodynamic correlations*. These are well-known to occur for non-equilibrium steady states in general [Sch, GLMS] and rigorously proved to be present for certain stochastic lattice gas models [Sp]. At first thought, this might cast doubt on a law of large numbers for such a measure, since usually such results depend upon some rapid decay of correlations. In fact, we have found a law of large numbers to hold, and, furhermore, our method of proof was essentially to show that the correlations *vanish* in the hydrodynamic limit! The question arises how this may be reconciled with the feature of long-ranged correlations.

The key point is that the long-range density-density correlation is a weaker effect (higher order in ε) than is seen in the hydrodynamic scaling. To observe the correlations we must consider the fluctuations about the deterministic limit. Introduce, for any bounded, local function $g(\eta)$, the fluctuation field

$$Y^{\varepsilon}(g;\varphi)=\varepsilon^{1/2}\sum_{x=-\varepsilon^{-1}+R}^{\varepsilon^{-1}-R}\varphi(\varepsilon x)[g_x(\eta)-\mu_{SS}^{\varepsilon}(g_x)]\ . \tag{4.1}$$

It follows then directly from the identity (3.40) that

$$\mu_{SS}^{\varepsilon}[Y^{\varepsilon}(h;\varphi'')Y^{\varepsilon}(\psi)]+\mu_{SS}^{\varepsilon}[Y^{\varepsilon}(\phi)Y^{\varepsilon}(h;\psi'')]=-\mu_{SS}^{\varepsilon}[X^{\varepsilon}(c;\phi'\psi')]+O(\varepsilon)\ . \tag{4.2}$$

We have introduced here the special notation

$$Y^{\varepsilon}(\phi)=\varepsilon^{1/2}\sum_{x=-\varepsilon^{-1}}^{\varepsilon^{-1}}\varphi(\varepsilon x)(\eta_x-\mu_{SS}^{\varepsilon}(\eta_x)) \tag{4.3}$$

for the fluctuation field of the conserved density. The convergence of the extensive fields established in Theorem 2 of Sect. 3 implies then that

$$\begin{aligned}\lim_{\varepsilon\to 0}\mu_{SS}^{\varepsilon}[X^{\varepsilon}(c;\varphi'\psi')]&=\int dq\varphi'(q)\psi'(q)\langle c(0,1)\rangle_{\bar{\varrho}(q)}\\&=\int dq\varphi'(q)\psi'(q)(2\chi D)(\bar{\varrho}(q))\ .\end{aligned} \tag{4.4}$$

On the other hand, one expects that as $\varepsilon\to 0$, the fluctuations of the fast variables are just projections onto the fluctuations of the slow, conserved variables (here, the density):

$$Y^{\varepsilon}(g;\varphi)\sim Y^{\varepsilon}(A_g\varphi)\ , \tag{4.5}$$

where A_g is the linear operator

$$(A_g\varphi)(q)\equiv\hat{g}'(\bar{\varrho}(q))\varphi(q)\ . \tag{4.6}$$

(We refer the reader to the discussion in [DPSW].) For the fluctuation covariance C, given by

$$\int dq\int dp\varphi(q)\psi(p)C(q,p)=\lim_{\varepsilon\to 0}\mu_{SS}^{\varepsilon}[Y^{\varepsilon}(\phi)Y^{\varepsilon}(\psi)]\ , \tag{4.7}$$

one derives, from (4.2) and a (presumed) rigorous version of (4.5), the equation

$$\begin{aligned}\int dq\int dp[\varphi''(q)\psi(p)D(\bar{\varrho}(q))+\varphi(q)\psi''(p)D(\bar{\varrho}(p))]C(q,p)\\=-2\int dq\phi'(q)\psi'(q)(\chi D)(\bar{\varrho}(q))\ ,\end{aligned} \tag{4.8}$$

which is a weak version of the equation

$$(AC)(q,p)+(CA^{*})(q,p)=2\partial_q[(\chi D)(\bar{\varrho}(q))\partial_q\delta(q-p)]\ , \tag{4.9}$$

where A is the linearized evolution operator

$$(Af)(q)=\partial_q^2[D(\bar{\varrho}(q))f(q)]\ . \tag{4.10}$$

This is the same equation for the fluctuation covariance as derived by the method of fluctuating hydrodynamics [Sch, Sp1]. It is a form of the fluctuation-dissipation theorem valid for the non-equilibrium steady state (equivalent to the so-called "extended local equilibrium hypothesis"). However, the solution of (4.9) has a behavior quite different from the equilibrium case, where the corresponding covariance

$$C_{\mathrm{eq}}(q,p)=\chi(\varrho)\delta(q-p) \tag{4.11}$$

is delta-correlated. To exhibit this, it is useful to decompose $C(q,p)$ into a local equilibrium part

$$C_L(q,p)=\chi(\bar{\varrho}(q))\delta(q-p) \tag{4.12}$$

and a "mode-coupling" part C_M, as

$$C(q,p)=C_L(q,p)+C_M(q,p)\ . \tag{4.13}$$

Substituting (4.12–13) into (4.9), one easily finds the equation satisfied by C_M as

$$(AC_M)(q,p)+(C_M A^*)(q,p)=-(\chi D)(\bar{\varrho}(q))''\delta(q-p)\ . \tag{4.14}$$

The formal solution of (5.14) is

$$C_M(q,p)=\int_0^\infty dt \int dr e^{At}(q,r)(\chi D)(\bar{\varrho}(r))'' e^{A^*t}(r,p)\ , \tag{4.15}$$

which is hard to evaluate exactly, in general, because of the spatial variation of $(\chi D)(\bar{\varrho}(r))''$. For the case of symmetric, simple exclusion $(\chi D)(\bar{\varrho}(r))''=\frac{1}{2}(\varrho_+-\varrho_-)^2$ and A is the Laplacian Δ on $[-1,1]$ with Dirichlet boundary conditions, leading to

$$C_M(q,p)=\tfrac{1}{4}(\varrho_+-\varrho_-)^2\Delta^{-1}(q,p)\ , \tag{4.16}$$

which behaves, away from the boundary, asymptotically like $\sim|q-p|$. (In dimension d, the behavior is like $\sim|q-p|^{-d+2}$.) The same qualitative behavior is present in all cases.

From this discussion it should be clear that the presence of "long-ranged correlations" is perfectly consistent with a deterministic limit. The latter require that the function $G(q,p)=0$, with G the correlation function defined in (3.49):

$$G(q,p)=E^*[\delta\varrho(q)\delta h(p)]\ . \tag{4.17}$$

However, the long-ranged decay occurs in the fluctuation covariance $C(q,p)$, which probes a different scale.

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