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A Feynman-Kac Formula for the Quantum Heisenberg Ferromagnet. I

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Abstract. The Hamiltonian of the (anisotropic) quantum Heisenberg (anti-) ferromagnet on an arbitrary finite lattice is lifted to a Hamiltonian acting on sections of the bundle obtained by twisting a certain line bundle over the classical spin configuration space (which is a Kähler manifold) with the Dolbeault complex. This procedure is extended from $SU(2)$ to arbitrary compact semi-simple Lie groups and arbitrary irreducible representations. The Bott-Borel-Weil theorem gives a heat kernel representation for the original partition function in an external magnetic field. The $U(1)$-gauged local Hamiltonian is the sum of the free, supersymmetric, twisted Dolbeault Laplace operator (multiplied by the inverse of an arbitrary small mass parameter) plus the lifted Hamiltonian.

The resulting (Euclidean) Lagrangian is nonlocal and describes bosons which do and fermions which do not propagate through the lattice. All fields couple to the external magnetic field. The Lagrangian contains Yukawa and Luttinger type interactions.

1. Introduction: Motivation and Outline of the Approach

The isotropic $2D$ quantum Heisenberg antiferromagnet and the Hubbard model have received renewed attention in attempts to understand high T_c superconductivity. In particular this has been pursued by Anderson and his collaborators (see e.g. [A 1, A 2, ZA]) and by members of the Landau school (see [DPW, Pol, Wi 1, Wi 2]). In these last articles the aim is to exhibit the appearance of what the authors call Pomeranchuk fermions [Pom], which are supposed to be neutral spin 1/2 excitations describing the antiferromagnetic magnons of the theory. Similarly, the fermion solitons appearing in the resonating-valence bond theory (see e.g. [KRS]) are called spinons in [ZA]. In [DPW] and [Wi 1, Wi 2] (cf. also [FS]) the authors try to relate the quantum Heisenberg model via a Feynman-Kac formula to a $U(1)$-gauged CP_1-quantum field theory in $2D+1$ dimensions. The motivation is that the Pauli spin matrices are inappropriate operators when one tries to exhibit critical behaviour, since there are no "Paulions". Only fields appearing in the Lagrangian for the Feynman-Kac formula have a chance of yielding an appropriate dynamical description in the scaling limit.

In this article we rigorously provide such a Feynman-Kac formula for the Heisenberg model on an arbitrary but finite lattice. Our result for the D-dimensional Heisenberg model is that the underlying Euclidean field theory (on a D-dimensional lattice and with continuous time) is indeed a $U(1)$-gauged CP_1-model. Ab initio it contains in addition a neutral spin zero fermion. The free part of the Lagrangian is supersymmetric and contains a small mass parameter which may be chosen freely and which serves as an infrared regularization. The clue is provided by the Bott-Borel-Weil theorem by which the partition function of the Heisenberg model may be written as a supertrace of a heat kernel for a certain Hamiltonian which is an elliptic differential operator. By standard techniques, this heat kernel in turn has a Feynman-Kac type path integral representation.

Our approach has some similarities with heat kernel proofs of index theorems based on the observation that the Hamiltonian involved is supersymmetric (see e.g. [Wt, AG, FW]). In fact, the independence of the index of the Euclidean time there corresponds to the indpendence of the infrared regulator mass here. In contrast, since our total Hamiltonian is not supersymmetric, we obtain a representation for the partition function and not just for an index. Only when the interaction is switched off, an index does appear, namely the dimension of the Hilbert space of the underlying theory. From the probabilistic point of view our approach is related to Bismut's proof of the Lefschetz fixed point formula of Atiyah and Singer [Bi2]. In fact, as a by-product we shall also give a stochastic representation of the character of any finite dimensional irreducible representation of an arbitrary compact connected semisimple Lie group. Note that the Weyl character formula is a special case of the Lefschetz fixed point formula.

We recall that techniques involving the introduction of fermions to the Heisenberg model and its variants have a long history (see e.g. [LM2] for an account of and references to the early history). Its first culmination was the work of Lieb, Mattis, and Schultz [LSM] where the antiferromagnetic $X-Y$ chain was solved. The fundamental construction involved is the non-local Jordan-Wigner transformation. Our approach, however, is completely different and can be described as follows.

Let Λ be a finite lattice in $\mathbb{Z}^D$. The Hamiltonian of the isotropic quantum Heisenberg (anti-)ferromagnet in an external magnetic field is given as

$$H_\Lambda = -J \sum_{k,k' \in \Lambda_{N,N}} \boldsymbol{\sigma}^k \cdot \boldsymbol{\sigma}^{k'} + \sum_{k \in \Lambda} \mathbf{h}(k) \cdot \boldsymbol{\sigma}^k \tag{1.1}$$

and acts in the Hilbert space $\mathscr{H}_\Lambda = \otimes_{k \in \Lambda} \mathbb{C}^2$. The choices $J > 0$ and $J < 0$ correspond to ferromagnetic and antiferromagnetic interactions. Here $\boldsymbol{\sigma}^k = (\sigma_1^k, \sigma_2^k, \sigma_3^k)$ are the standard Pauli matrices $\boldsymbol{\sigma}$ acting on the k-th component in $\mathscr{H}_\Lambda$, and $\Lambda_{N.N}$ in (1.1) indicates that the summation is carried out over nearest neighbour lattice points only. Also $\mathbf{h}(k) \in \mathbb{R}^3$ denotes the magnetic field at the lattice side $k \in \Lambda$. Note that $\mathbf{h} \in \mathbb{R}^3$ defines a Lie-algebra element $i\mathbf{h} \cdot \boldsymbol{\sigma} = i \sum_{\alpha=1,2,3} h^\alpha \sigma_\alpha$ in $su(2)$.

Since $(\boldsymbol{\sigma}^k)^2 = 3$ for all $k \in \Lambda$, one may alternatively consider the Hamiltonian

$$\tilde{H}_\Lambda = \frac{J}{2} \sum_{k,k' \in \Lambda_{N,N}} (\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^{k'})^2 + \sum_{k \in \Lambda} \mathbf{h}(k) \cdot \boldsymbol{\sigma}^k \tag{1.2}$$

which differs from H_Λ by an additive constant $\sim |\Lambda|$.

The aim is to obtain a Feynman-Kac formula for the matrix elements

$$\langle \psi | e^{-tH_\Lambda} | \psi' \rangle, \qquad \psi, \psi' \in \mathscr{H}_\Lambda \tag{1.3}$$

which in particular would lead to a Feynman-Kac representation for the partition function

$$Z_\Lambda(t) = \mathrm{Trace}_{\mathscr{H}_\Lambda} e^{-tH_\Lambda}. \tag{1.4}$$

We propose to approach this aim using the theory of coherent states.

For quantum spin systems such states have been emphasized by various authors (see e.g. [Kl, Pe1, Pe2] and references therein). They have turned out to provide a powerful tool in establishing classical limit theorems for the partition function of quantum spin systems (spin $\to \infty$) (see e.g. [FL, Gi, Lie, Si]) and in calculating the corresponding quantum corrections (cf. [HPS, ST1, ST2]). Coherent states were also crucial in showing that the Wong equations are the classical limit (as $\hbar \to 0$) of the Schrödinger equation for a quantum mechanical particle carrying isospin and moving in an external Yang-Mills potential [HPS]. In the present context their usefulness has also been recognized in e.g. [DPW, Wi1, Wi2, FS].

Our approach may already be explained in the particular case of one lattice point, $|\Lambda| = 1$. There a coherent state is a complex vector with two components and labelled by a point $z \in S^2$ (the unit sphere, which is the configuration space of a classical spin); it may be chosen to be of the form

$$|z\rangle = \begin{pmatrix} e^{i\varphi/2} \cos(\vartheta/2) \\ e^{-i\varphi/2} \sin(\vartheta/2) \end{pmatrix}, \tag{1.5}$$

where z is parametrized by its polar angles (ϑ, φ). Note that the parametrization is well defined and smooth for $0 \leqq \varphi < 2\pi$, $0 < \vartheta < \pi$. Thus in this context, one would like to have a Feynman-Kac formula for

$$K_t^{\mathbf{h}}(z, z') = \langle z | \exp\{-t\mathbf{h} \cdot \boldsymbol{\sigma}\} | z' \rangle \tag{1.6}$$

with $\mathbf{h} \in \mathbb{R}^3$ and $t > 0$, as an integral over the space of all continuous paths $z(s)$ on S^2 starting at z and ending at z' in time t. Of course these matrix elements may be calculated explicitly, but we are interested in a formula that extends to the general case, i.e. arbitrary Λ with $|\Lambda| < \infty$. Now, as it stands and contrary to attempts in the literature, it is not possible to find such a Feynman-Kac formula for (1.6). The reason is that the coherent states form an overcomplete set of states in $\mathbb{C}^2$ such that in particular $\langle z | z' \rangle \neq 0$ a.e. and there is no mechanism to force $z(s)$ and $z(s')$ to be close when s and s' are close. This is reflected in the fact that the $K_t^{\mathbf{h}}(z, z')$ fail to form kernels of a semigroup on $L^2(S^2)$ since obviously

$$K_{t=0}^{\mathbf{h}}(z, z') = \langle z | z' \rangle. \tag{1.7}$$

On the other hand, due to the semigroup behaviour of $\exp\{-t\mathbf{h}\cdot\boldsymbol{\sigma}\}$, they almost satisfy the semigroup property since

$$2\int_{S^2} K_t^{\mathbf{h}}(z,z'')K_{t'}^{\mathbf{h}}(z'',z')dz''=K_{t+t'}^{\mathbf{h}}(z,z'), \tag{1.8}$$

where dz'' is the normalized canonical measure on S^2.

As a preliminary ansatz for a remedy one may try to regularize (1.6) by considering instead the quantity

$$K_t^{\mathbf{h},m}(z,z')=e^{-\frac{t}{2m}\Delta}(z,z')K_t^{\mathbf{h}}(z,z'). \tag{1.9}$$

Here $m>0$ is a mass parameter and $e^{-t\Delta}(z,z')$ is the kernel of the contraction semigroup of the Laplace-Beltrami Operator $\Delta\geqq 0$ on S^2. Note that this kernel in (1.9) is the transition probability for a Brownian motion on S^2 whose diffusion coefficient is $(2m)^{-1}$. Now for $t>0$,

$$|e^{-t\Delta}(z,z')-1|\leqq e^{-ct} \tag{1.10}$$

for some constant $c>0$. Hence as $m\to 0$ the kernel $K_t^{\mathbf{h},m}(z,z')$ converges to $K_t^{\mathbf{h}}(z,z')$ uniformly on $S^2\times S^2$ and t away from zero, reflecting the fact that diffusion increases with decreasing m.

On the other hand, for $m>0$ fixed and $t\to 0$ the kernel $e^{-\frac{t}{2m}\Delta}(z,z')$ approaches the Dirac δ-function $\delta_z(z')$ at z, thus forcing z and z' to be close for all small times t. Hence the presence of $e^{-\frac{t}{2m}\Delta}(z,z')$ counteracts the overlap property of $|z\rangle$ and $|z'\rangle$ mentioned above.

As it stands, the ansatz (1.9) has two drawbacks. First, the $K_t^{\mathbf{h},m}(z,z')$ still do not form the kernels of a semigroup. Secondly, the states (1.5) are not globally defined and the choice $e^{i\phi(z)}|z\rangle$ for any real valued $\phi(z)$ are as well possible. In fact, only the associated one-dimensional orthogonal projection operators

$$Q(z)=\begin{pmatrix} \cos^2\frac{\vartheta}{2} & \frac{1}{2}e^{i\varphi}\sin\vartheta \\ \frac{1}{2}e^{-i\varphi}\sin\vartheta & \sin^2\frac{\vartheta}{2}\end{pmatrix} \tag{1.11}$$

have a globally well defined invariant meaning. In a natural way they define a complex hermitian line bundle $\mathscr{L}$ over S^2, associated to the principal $U(1)$-bundle

$$\begin{array}{c} U(1)\to S^3\cong SU(2)\\ \downarrow\\ S^2\end{array} \tag{1.12}$$

(the Hopf fibration) via the one-dimensional self-representation of $U(1)$.

Both of the obstructions above may now be overcome by working directly in the Hilbert space $L^2(\mathscr{L})$ of square integrable sections in $\mathscr{L}$. First, the self-representation of $SU(2)$ on $\mathbb{C}^2$ may be "lifted" to a unitary representation π_0 of $SU(2)$ on $L^2(\mathscr{L})$. Secondly there is a Laplace operator $\bar{\square}\geqq 0$ (see below) commuting with every $\pi_0(g)$ for $g\in SU(2)$ on $L^2(\mathscr{L})$ such that as a consequence of the Borel-

Weil theorem

$$\lim_{t\to\infty} \mathrm{Trace}_{L^2(\mathscr{L})}\{e^{-t\bar{\Box}}\pi_0(g)\} = \mathrm{Trace}_{\mathbb{C}^2} g\,, \qquad g\in SU(2)\,. \tag{1.13}$$

We may even go further. For this we observe that S^2 is a compact Kähler manifold and that $\mathscr{L}$ is a holomorphic line bundle. Thus $\mathscr{L}$ may be twisted with the Dolbeault complex. On the resulting hermitian vector bundle $\bar{\Lambda}(\mathscr{L})$ there is a Dolbeault operator, also denoted by $\bar{\partial}$. The Laplacian $\bar{\Box}$ is now defined to be the twisted Dolbeault-Laplace operator $(\bar{\partial}+\bar{\partial}^*)^2$ which by construction is a supersymmetric Hamiltonian. Furthermore, on $L^2(\bar{\Lambda}(\mathscr{L}))$ there is a unitary representation π of $SU(2)$ extending π_0 on $L^2(\mathscr{L})\subset L^2(\bar{\Lambda}(\mathscr{L}))$, commuting with $\bar{\partial}$ and $\bar{\partial}^*$ and preserving the degree. If ε is $+1$ on even forms and -1 on odd forms, then as a consequence of the Bott-Borel-Weil theorem,

$$\mathrm{Trace}_{L^2(\bar{\Lambda}(\mathscr{L}))}\{\varepsilon e^{-\frac{t}{m}\bar{\Box}}\pi(g)\} = \mathrm{Trace}_{\mathbb{C}^2} g\,, \tag{1.14}$$

valid for all $t>0$ and $m>0$. By analytic continuation, relation (1.14) still remains valid if g is replaced by $\exp\{-t\mathbf{h}\cdot\boldsymbol{\sigma}\}$ and $\pi(g)$ is replaced by the formal expression $\exp\{itd\pi(i\mathbf{h}\cdot\boldsymbol{\sigma})\}$. More precisely, the Hamiltonian in $L^2(\bar{\Lambda}(\mathscr{L}))$ becomes

$$H = \frac{1}{m}\bar{\Box} - id\pi(i\mathbf{h}\cdot\boldsymbol{\sigma})\,. \tag{1.15}$$

This is an elliptic operator bounded below, and for the resulting left-hand side of (1.14) it is now straightforward to write down a Feynman-Kac type integral representation. Due to the Grassmann structure of $\bar{\Lambda}(\mathscr{L})$, a scalar neutral fermi field appears. For later purposes we therefore refer to the subspace $L^2(\mathscr{L})$ of $L^2(\bar{\Lambda}(\mathscr{L}))$ (and to the corresponding situation for a general lattice Λ) as the purely bosonic sector.

This approach now readily extends to arbitrary finite Λ. In fact, this is achieved by "lifting" the Hamiltonian (1.1) or (1.2) to an operator acting on sections of the bundle $\bar{\Lambda}(\mathscr{L}_\Lambda)$ over the Kähler manifold $S^2_\Lambda = \times_{k\in\Lambda} S^2$ via the substitution $i\sigma^k_\alpha \to d\pi(i\sigma^k_\alpha)$. Here the line bundle $\mathscr{L}_\Lambda$ over S^2_Λ is the product of the line bundles obtained by pulling $\mathscr{L}$ over the component $S^2_k \cong S^2$ back to S^2_Λ. This lifted interaction Hamiltonian then becomes a second order differential operator. By definition, the free supersymmetric Hamiltonian is just the sum of the lattice site Hamiltonians $(1/m)\bar{\Box}$. The resulting free supersymmetric Lagrangian then contains one fermionic degree of freedom for each lattice site. It is not related to the Lagrangian considered by Zumino [Zu]. Actually, there is local fermion number conservation preventing fermions from propagation across the lattice. The total Lagrangian, however, describes bosonic propagation. Also it is nonlocal and contains Yukawa and Luttinger type interactions. Its power series coefficients in J are all local, and couple nearest neighbours, next nearest neighbours, ..., etc. Due to the $U(1)$-gauge from (1.12) the Lagrangian also contains a Wess-Zumino term of the form already considered e.g. in [FS].

Our approach allows the following four types of generalizations. First, the coupling may be chosen to be anisotropic in spin space. Secondly, it is possible to choose an arbitrary finite lattice Λ and the coupling need not be only between nearest neighbours. Thus the Hamiltonian (1.1) may be replaced by

$$H_{\Lambda} = -J \sum_{\substack{k,k' \in \Lambda \\ k \neq k'}} c^{\alpha\beta}(k,k') \boldsymbol{\sigma}_{\alpha}^{k} \boldsymbol{\sigma}_{\beta}^{k'} + \sum_{k \in \Lambda} \mathbf{h}(k) \cdot \boldsymbol{\sigma}^{k} . \tag{1.16}$$

Here $c^{\alpha\beta}(k,k') = c^{\beta\alpha}(k',k)$ for any $k, k' \in \Lambda$ may be arbitrary real symmetric 3×3 matrices. Particular choices give the Ising or the $X-Y$ model. Thirdly, instead of the self-representation of $SU(2)$ (spin $= 1/2$) one may as well consider an arbitrary irreducible representation of $SU(2)$. And finally, following Simon [Si], the group $SU(2)$ itself may be replaced by an arbitrary compact semisimple Lie group.

As a consequence, the classical spin configuration space S^2 for one lattice point will be replaced by another homogeneous space (i.e. a coadjoint orbit) which is still Kähler but now of higher dimension. Correspondingly the number of fermions per lattice site will also change, being equal to the complex dimension of this manifold.

Within our set-up, all these generalizations will come at no extra charge, since presently we are not concerned with the more intricate physical questions such as for instance the existence of the thermodynamic limit or the scale limit and the related questions about possible critical temperatures and the nature of the phase transitions. From the start therefore our presentation will be given in this general context.

The article is organized as follows. In Sect. 2 we review facts from the theory of group representations and the structure of Kähler manifolds needed to establish (1.13) and (1.14), and to calculate the Lagrangian for the case $|\Lambda| = 1$. The material is basically well known but will be presented in such a way that the derivation of analogous relations in the interacting case (i.e. $|\Lambda| \neq 1$) in Sect. 3 will then be straightforward.

In Sect. 4 we determine the resulting Lagrangian, first for the purely bosonic sector and then for the full theory including fermions. Also here we start with the case $|\Lambda| = 1$.

Our investigations are continued in a second paper with Sects. 5–7. Section 5 contains a stochastic derivation of a Feynman-Kac formula for the heat kernel of the Hamiltonian (1.15) as well as a stochastic representation of the character (1.14). The methods employed are inspired by techniques used by Bismut [Bi 1]. In Sect. 6 we extend the discussion to arbitrary $|\Lambda| < \infty$. Finally, in Sect. 7 classical limit theorems in the purely bosonic sector are obtained by letting the representation of the group tend to infinity in a way well known from the theory of quantum spin systems.

2. Preliminaries: Representation Theory and Kähler Geometry

For easy reference and to fix the notation, in this section we will summarize some facts from the theory of representations of compact Lie groups and from Kähler geometry needed in the remaining part of the paper.

2.1. Notation

Let G be compact connected semisimple Lie group with Lie algebra $\mathbf{g}$. We fix a maximal torus T in G; its Lie algebra will be denoted by $\mathbf{t}$. We use the notation $\mathbf{g}_{\mathbb{C}}$ and $\mathbf{t}_{\mathbb{C}}$ for the complexifications of the Lie algebras $\mathbf{g}$ and $\mathbf{t}$, respectively. Note that $\mathbf{t}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathbf{g}_{\mathbb{C}}$. Furthermore, there exists a complex semisimple Lie group $G_{\mathbb{C}}$ with Lie algebra $\mathbf{g}_{\mathbb{C}}$ such that G is a maximal compact subgroup of $G_{\mathbb{C}}$ (cf. [Che, p. 200]). We denote by $T_{\mathbb{C}}$ the complex subgroup of $G_{\mathbb{C}}$ corresponding to $\mathbf{t}_{\mathbb{C}}$, and by $\Phi = \Phi(\mathbf{g}_{\mathbb{C}}, \mathbf{t}_{\mathbb{C}})$ the set of nonzero roots of $\mathbf{g}_{\mathbb{C}}$ with respect to $\mathbf{t}_{\mathbb{C}}$. For a given $\alpha \in \Phi$, the associated root space will be denoted by $\mathbf{g}_{\mathbb{C}}^{\alpha}$. Since $\mathbf{g}$ is a compact real form of $\mathbf{g}_{\mathbb{C}}$, all roots assume real values on $\mathbf{t}_{\mathbb{R}} = i\mathbf{t} = \sqrt{-1}\,\mathbf{t}$. We will regard Φ as a subset of $\mathbf{t}_{\mathbb{R}}^*$, the dual space of $\mathbf{t}_{\mathbb{R}}$. The Cartan-Killing form

$$B(Y, Y') = \mathrm{Trace}\{\mathrm{ad}(Y) \circ \mathrm{ad}(Y')\}$$

with $Y, Y' \in \mathbf{g}_{\mathbb{C}}$ restricts to a positive definite bilinear form on $\mathbf{t}_{\mathbb{R}}$, and by duality determines an inner product $(\cdot, \cdot)$ on $\mathbf{t}_{\mathbb{R}}^*$. The hyperplanes $\Sigma_\alpha = \{\mu \in \mathbf{t}_{\mathbb{R}}^* | (\mu, \alpha) = 0\}$ with $\alpha \in \Phi$ divide $\mathbf{t}_{\mathbb{R}}^*$ into a finite number of closed convex cones, the so-called Weyl chambers. The group generated by the reflections about the hyperplanes Σ_α is the Weyl group W. Elements of $\mathbf{t}_{\mathbb{R}}^*$ are called singular if they lie on a hyperplane Σ_α for some $\alpha \in \Phi$, and nonsingular otherwise.

A system of positive roots for $(\mathbf{g}_{\mathbb{C}}, \mathbf{t}_{\mathbb{C}})$ is a subset $\Delta^+ \subset \Phi$ of the form

$$\Delta^+ = \{\alpha \in \Phi | (\lambda, \alpha) > 0\}$$

for some particular nonsingular $\lambda \in \mathbf{t}_{\mathbb{R}}^*$. Equivalently, such a set Δ^+ can be described as the set of all elements of Φ that are positive with respect to some suitably chosen order on $\mathbf{g}_{\mathbb{R}}^*$. To each system of positive roots Δ^+ there corresponds a distinguished Weyl chamber

$$C = \{\mu \in \mathbf{t}_{\mathbb{R}}^* | (\alpha, \mu) \geqq 0 \text{ for every } \alpha \in \Delta^+\}.$$

C is called the positive Weyl chamber with respect to Δ^+. From now on, we will fix such a system Δ^+ of positive roots. As usual, we set

$$\varrho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

2.2. Irreducible Representations

The finite dimensional representations of G are in one-to-one correspondence with the holomorphic representations of $G_{\mathbb{C}}$. This correspondence is determined by restricting a given holomorphic representation of $G_{\mathbb{C}}$ to G. An element $\lambda \in \mathbf{t}_{\mathbb{R}}^*$ is called a weight if it is the differential of a character of T. The weights form a lattice L in $\mathbf{t}_{\mathbb{R}}^*$, and L is contained in the lattice

$$\left\{\lambda \in \mathbf{t}_{\mathbb{R}}^* \,\middle|\, 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z} \text{ for every } \alpha \in \Phi\right\}.$$

Let C be the positive Weyl chamber with respect to the given system of positive roots Δ^+. The finite dimensional irreducible representations of G are naturally parametrized by $\lambda \in L \cap C$. Given $\lambda \in L \cap C$, we shall denote the corresponding irreducible representation of G by π_λ and the representation space by $\mathscr{H}_\lambda$. Then λ is the highest weight of π_λ with respect to the given ordering on L. Moreover, in $\mathscr{H}_\lambda$ there exists a (up to a complex scalar) unique λ-weight vector ψ_λ characterized by the following properties: (i) $d\pi_\lambda(Y)\psi_\lambda = \lambda(Y)\psi_\lambda$ for all $Y \in \mathbf{t}_{\mathbb{C}}$, (ii) $d\pi_\lambda(Y)\psi_\lambda = 0$ for every $Y \in \mathbf{g}^\alpha_{\mathbb{C}}$ and $\alpha \in \Delta^+$.

Given $\lambda \in L \cap C$, we have $\lambda = i\tilde{\lambda}$ with $\tilde{\lambda} \in \mathbf{t}^*$. We extend $\tilde{\lambda}$ to an element of $\mathbf{g}^*$ by setting it equal to zero on the orthogonal complement of $\mathbf{t}$ with respect to the Cartan-Killing form. Therefore we may regard λ as a linear form on $\mathbf{g}$ with purely imaginary values given by

$$\lambda(Y) = \langle \psi_\lambda | d\pi_\lambda(Y)\psi_\lambda \rangle \tag{2.1}$$

for $Y \in \mathbf{g}$, i.e. λ is an element of $\mathbf{g}^*_{\mathbb{R}}$, the real dual of $\mathbf{g}_{\mathbb{R}} = i\mathbf{g}$.

2.3. *The Borel-Weil Theorem*

The Borel-Weil theorem provides a geometric realization of the finite dimensional irreducible representations of G. Consider the quotient manifold

$$M = G/T.$$

M carries various G-invariant complex structures depending on the choice of a system of positive roots Δ^+. To describe the complex structure of M associated to our fixed system of positive roots, we introduce the nilpotent subalgebra $\mathbf{n}$ of $\mathbf{g}_{\mathbb{C}}$ by

$$\mathbf{n} = \sum_{\alpha \in \Delta^+} \mathbf{g}^{-\alpha}.$$

Note that $\mathbf{n}$ is $\mathrm{Ad}\,T$-invariant and satisfies

$$\mathbf{g}_{\mathbb{C}} = \mathbf{t}_{\mathbb{C}} \oplus \bar{\mathbf{n}} \oplus \mathbf{n}.$$

Furthermore, by $\mathbf{b} = \mathbf{t}_{\mathbb{C}} \oplus \mathbf{n}$ a Borel subalgebra of $\mathbf{g}_{\mathbb{C}}$ is defined; let $B \subset G_{\mathbb{C}}$ be its corresponding complex analytic subgroup. Then $G_{\mathbb{C}}/B$ is a compact complex manifold and the invariant hermitian metric of $G_{\mathbb{C}}/B$ is a Kähler metric. Since the real span of $\mathbf{g}$ and $\mathbf{b}$ is all of $\mathbf{g}_{\mathbb{C}}$, the G-orbit of $eB \in G_{\mathbb{C}}/B$ is open. On the other hand, this orbit is closed due to the compactness of G. The isotropy group of eB is $B \cap G = T$. Hence we get a canonical diffeomorphism

$$G/T \cong G_{\mathbb{C}}/B.$$

This is one of the G-invariant complex structures on G/T. Recall that a complex structure on a manifold M defines a splitting of the complexified tangent space

$$T_{z,\mathbb{C}}M = T_z^{(1,0)}M \oplus T_z^{(0,1)}M$$

at every point $z \in M$. Here $T_z^{(1,0)}M$ is the holomorphic tangent space at z and $T_z^{(0,1)}M$ the antiholomorphic tangent space at z. In our case, the holomorphic tangent space at $eT \in G/T$ can be identified with $\bar{\mathbf{n}}$ and the antiholomorphic tangent space at eT with $\mathbf{n}$. Since $-B(\bar{Y}, Y')$ is an $\mathrm{Ad}\,T$-invariant inner product on $\bar{\mathbf{n}} \oplus \mathbf{n}$ (which is the complexified tangent space of M at eT), we obtain by translation a G-invariant hermitian metric on M. This metric is known to be Kähler [W].

Next we turn to the discussion of homogeneous holomorphic line bundles over M, i.e. holomorphic vector bundles of fibre dimension one admitting a lift of the action of $G_{\mathbb{C}}$ on $M=G_{\mathbb{C}}/B$. Let $\mathscr{L}$ be such a vector bundle. The action of the isotropy group B on the fibre of $\mathscr{L}$ over eB determines a holomorphic character $\chi: B\to\mathbb{C}^*$. Via this character, $\mathscr{L}$ is associated to the holomorphic principal bundle $B\to G_{\mathbb{C}}\to M$. Conversely, given a holomorphic character $\chi: B\to\mathbb{C}^*$, we can construct a vector bundle $\mathscr{L}=G_{\mathbb{C}}\times_{\chi}\mathbb{C}$ over M. The points of $\mathscr{L}$ are equivalence classes of pairs (g,w) under $(gb,w)\sim(g,\chi(b)^{-1}w)$. Denote the equivalence class by $[g,w]$. The action of $G_{\mathbb{C}}$ on $\mathscr{L}$ is given by $g_1[g,w]=[g_1g,w]$. Thus $\mathscr{L}$ is a homogeneous holomorphic line bundle. As a C^∞ vector bundle, $\mathscr{L}$ is associated to the principal bundle $T\to G\to M$ via the restriction of χ to $T=B\cap G$. We may identify the space $C^\infty(\mathscr{L})$ of C^∞ sections of $\mathscr{L}$ with the space of C^∞ functions $\phi: G\to\mathbb{C}$ satisfying $\phi(g_1g_2)=\chi(g_2)^{-1}\phi(g_1)$ for all $g_2\in T$ and $g_1\in G$.

Let $\lambda\in \mathrm{L}$ and denote by e^λ the corresponding character of T. We extend λ to a linear functional on $\mathbf{b}=\mathbf{t}_{\mathbb{C}}\oplus\mathbf{n}$ by putting it equal to zero on $\mathbf{n}$. This infinitesimal character may be lifted to a holomorphic character $\chi_\lambda: B\to\mathbb{C}^*$ because the fundamental groups of B and T are the same. The resulting holomorphic extension of e^λ is the only possible one since $\mathbf{n}$ must act trivially on any irreducible $\mathbf{b}$-module. Let $\mathscr{L}^\lambda$ denote the associated homogeneous holomorphic line bundle over M. Using the fact that $e^\lambda=\chi_{\lambda|T}$ is unitary, by translation is determines a G-invariant hermitian metric in $\mathscr{L}^\lambda$. This hermitian metric is unique up to multiplication by a constant.

Now consider the antiholomorphic cotangent bundle $T^{*(0,1)}M$ of M. This is clearly a homogeneous vector bundle, i.e. the left action of G on M lifts to an action of G on $T^{*(0,1)}M$, the lift being given by $(dL_{g^{-1}})^*$, where $L_g: M\to M$ denotes left translation by $g\in G$. This isotropy representation of T is the coadjoint representation

$$\mathrm{Ad}^*_{\mathbf{n}}: T\to GL(\mathbf{n}^*).$$

Via this representation, $T^{*(0,1)}M$ is associated to the principal bundle $T\to G\to M$. Furthermore, a G-invariant hermitian metric in $T^{*(0,1)}M$ is obtained by translating the Ad T-invariant inner product on $\mathbf{n}^*$ induced by the negative of the Cartan-Killing form. For $q=0,1,\dots,n=\dim_{\mathbb{C}}M$ we set

$$\bar{\Lambda}^q(\mathscr{L}^\lambda)=\mathscr{L}^\lambda\otimes\Lambda^qT^{*(0,1)}M \tag{2.2}$$

and

$$\bar{\Lambda}(\mathscr{L}^\lambda)=\bigoplus_{q=0}^{n}\bar{\Lambda}^q(\mathscr{L}^\lambda). \tag{2.3}$$

The space of C^∞ sections of $\bar{\Lambda}^q(\mathscr{L}^\lambda)$ is usually denoted by $\Lambda^{0,q}(M,\mathscr{L}^\lambda)$ which by definition is the space of $(0,q)$-forms on M with coefficients in $\mathscr{L}^\lambda$.

Since $\bar{\Lambda}^q(\mathscr{L}^\lambda)$ is a homogeneous vector bundle, we get a natural action of G in $\Lambda^{0,q}(M,\mathscr{L}^\lambda)$. If $\omega\in\Lambda^{0,q}(M,\mathscr{L}^\lambda)$ and $g\in G$, then this action is determined by setting

$$(g\omega)(z)=g\omega(L_{g^{-1}}z)$$

for all $z\in M$. The hermitian metric on M together with the G-invariant metric in $\mathscr{L}^\lambda$ give an inner product for the sections of $\bar{\Lambda}^q(\mathscr{L}^\lambda)$. On the resulting Hilbert space $L^2(\bar{\Lambda}^q(\mathscr{L}^\lambda))$ a unitary representation π^λ_q is obtained by the same construction as above. Let $\mathrm{Ind}_T^G(e^\lambda \otimes \Lambda^q \mathrm{Ad}_{\mathfrak{n}}^*)$ be the representation induced by the representation $e^\lambda \otimes \Lambda^q \mathrm{Ad}_{\mathfrak{n}}^* : T \to GL(\Lambda^q \mathfrak{n}^*)$. By definition, the Hilbert space of this representation is just the space of L^2 functions $\varphi : G \to \Lambda^q \mathfrak{n}^*$ satisfying $\varphi(g_1 g_2) = \exp(-\lambda(\log(g_2)))$ $\mathrm{Ad}_{\mathfrak{n}}^*(g_2^{-1})\varphi(g_1)$ for all $g_1 \in G$ and $g_2 \in T$. It is easy to see that the Hilbert spaces are naturally isomorphic and, with respect to this identification,

$$\pi_q^\lambda = \mathrm{Ind}_T^G(e^\lambda \otimes \Lambda^q \mathrm{Ad}_{\mathfrak{n}}^*). \tag{2.4}$$

Furthermore, $\Lambda^{0,q}(M, \mathscr{L}^\lambda) \subset L^2(\bar{\Lambda}^q(\mathscr{L}^\lambda))$ is the space of C^∞ vectors of π_q^λ.

Next we recall the definition of the $\bar{\partial}$ operator

$$\bar{\partial}_q : \Lambda^{0,q}(M, \mathscr{L}^\lambda) \to \Lambda^{0,q+1}(M, \mathscr{L}^\lambda).$$

Given $\omega \in \Lambda^{0,q}(M, \mathscr{L}^\lambda)$ and a coordinate neighbourhood $V \subset M$, we can write $\omega_{|V} = s \otimes \omega_0$, where s is a holomorphic section of $\mathscr{L}^\lambda$ over V and ω_0 a $(0,q)$-form on V. Then $\bar{\partial}_q \omega = s \otimes \bar{\partial}_q \omega_0$ and $\bar{\partial}_q \omega_0$ is the usual $\bar{\partial}$ operator applied to ω_0. Clearly, $\bar{\partial}_q \omega$ is independent of the choice of the local holomorphic section s. Moreover, $\bar{\partial}_q$ satisfies $\bar{\partial}_{q+1} \circ \bar{\partial}_q = 0$ and we arrive at the following elliptic complex, the so-called Dolbeault complex

$$\cdots \xrightarrow{\bar{\partial}_{q-1}} \Lambda^{0,q}(M, \mathscr{L}^\lambda) \xrightarrow{\bar{\partial}_q} \Lambda^{0,q+1}(M, \mathscr{L}^\lambda) \xrightarrow{\bar{\partial}_{q+1}} \cdots .$$

Let $H^q(M, \mathscr{L}^\lambda)$ be the q-th cohomology group of this complex, i.e. $H^q(M, \mathscr{L}^\lambda) = \mathrm{kernel}\,(\bar{\partial}_q)/\mathrm{range}\,(\bar{\partial}_{q-1})$. Since the left translation by $g \in G$ is a holomorphic map of M, the operator $\bar{\partial}$ commutes with the action of G on $\Lambda^{0,q}(M, \mathscr{L}^\lambda)$. Therefore, π_q^λ induces a representation of G in $H^q(M, \mathscr{L}^\lambda)$.

This representation can be described in another way as follows. Let

$$\bar{\partial}_q^* : \Lambda^{0,q}(M, \mathscr{L}^\lambda) \to \Lambda^{0,q-1}(M, \mathscr{L}^\lambda)$$

be the formal adjoint of $\bar{\partial}_{q-1}$, and set

$$\bar{\square}_q = \bar{\partial}_{q-1} \bar{\partial}_q^* + \bar{\partial}_{q+1}^* \bar{\partial}_q .$$

Then $\bar{\square}_q$ is an elliptic second order G-invariant differential operator acting in $\Lambda^{0,q}(M, \mathscr{L}^\lambda)$ and its kernel is the space $\mathscr{H}^{0,q}(M, \mathscr{L}^\lambda)$ of harmonic $(0,q)$-forms with coefficients in $\mathscr{L}^\lambda$. Equivalently, $\mathscr{H}^{0,q}(M, \mathscr{L}^\lambda)$ is the space of $\mathscr{L}^\lambda$-valued smooth $(0,q)$-forms ω obeying $\bar{\partial}\omega = 0$ and $\bar{\partial}^*\omega = 0$. The G-invariance of $\bar{\square}_q$ implies the invariance of $\mathscr{H}^{0,q}(M, \mathscr{L}^\lambda)$ under π_q^λ, thus leading to a representation of G on $\mathscr{H}^{0,q}(M, \mathscr{L}^\lambda)$ also denoted by π_q^λ. The well-known Hodge theorem gives an isomorphism

$$\mathscr{H}^{0,q}(M, \mathscr{L}^\lambda) \cong H^q(M, \mathscr{L}^\lambda)$$

which is compatible with the action of G on both sides. Now we are ready to state the Borel-Weil theorem [Se].

Theorem (Borel-Weil). *Let C be the positive Weyl chamber and assume that $\lambda \in L \cap C$. Then*

(i) $H^q(M, \mathscr{L}^\lambda) = 0$ *for* $q > 0$.

(ii) *The representation π_0^λ of G in $H^0(M, \mathscr{L}^\lambda)$ is equivalent to that irreducible representation π_λ of G in $\mathscr{H}_\lambda$ which has λ as its highest weight.*

Note that $H^0(M, \mathcal{L}^\lambda)$ is the space of holomorphic sections of the line bundle $\mathcal{L}^\lambda \to M$.

This theorem has been generalized by Bott [Bo]. In place of B one considers now any closed subgroup $P \subset G_{\mathbb{C}}$ containing B. Any such P is called a parabolic subgroup of $G_{\mathbb{C}}$. The homogeneous space $M = G_{\mathbb{C}}/P$ is again a compact complex manifold. Let $U = P \cap G$; then U is a subgroup of G containing T. Moreover there exists a canonical diffeomorphism $G_{\mathbb{C}}/P \simeq G/U$ and M carries a canonical G-invariant Kähler metric. Let $\mathfrak{u}$ be the Lie algebra of U and $\Psi = \Psi(\mathfrak{u}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ the corresponding root system. Again we regard Ψ as a subset of $\mathfrak{t}^*_{\mathbb{R}}$. Choosing a system of positive roots Ψ^+ with $\Psi^+ \subset \Delta^+$, we take a unitary representation $\sigma_\lambda : U \to GL(V)$ such that λ is its highest weight with respect to Ψ^+. As above, σ_λ determines a homogeneous holomorphic vector bundle $\mathcal{E}^\lambda \to M$ equipped with a G-invariant hermitian metric. Let $\Lambda^{0,q}(M, \mathcal{E}^\lambda)$ be the space of smooth $\mathcal{E}^\lambda$-valued $(0,q)$-forms, i.e. the space of C^∞ sections of $\mathcal{E}^\lambda \otimes \Lambda^q T^{*(0,1)}M$. Then the q^{th} cohomology group $H^q(M, \mathcal{E}^\lambda)$ of the Dolbeault complex

$$\cdots \xrightarrow{\bar\partial_{q-1}} \Lambda^{0,q}(M, \mathcal{E}^\lambda) \xrightarrow{\bar\partial_q} \Lambda^{0,q+1}(M, \mathcal{E}^\lambda) \xrightarrow{\bar\partial_{q+1}} \cdots$$

admits a canonical G-action which we denote by π^λ_q. The following generalization of the Borel-Weil theorem is due to Bott [Bo] (cf. also [Wa] for a comprehensive discussion).

Theorem. *If $\lambda + \varrho$ is singular, i.e. $(\lambda + \varrho, \alpha) = 0$ for some $\alpha \in \Phi$, then $H^q(M, \mathcal{E}^\lambda) = 0$ for all q. Otherwise, for nonsingular $\lambda + \varrho$ let $w \in W$ be that element which carries $\lambda + \varrho$ into the positive Weyl chamber, and let k be the number of $\alpha \in \Delta^+$ such that $w(\alpha) < 0$. Then $H^q(M, \mathcal{E}^\lambda) = 0$ for $q \neq k$ and the representation π^λ_k of G on $H^k(M, \mathcal{E}^\lambda)$ is irreducible with highest weight $w(\lambda + \varrho) - \varrho$.*

For our purposes we need the Bott-Borel-Weil theorem only for the particular case of σ_λ being a character of U.

Finally, we restate the Bott-Borel-Weil theorem in a heat kernel formulation. To this end, let $\mathcal{L}^\lambda \to G_{\mathbb{C}}/P = M$ be a homogeneous holomorphic line bundle associated with the holomorphic character $\chi_\lambda : P \to \mathbb{C}^*$. By $\bar\Box$ we denote the elliptic differential operator

$$\bar\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$$

on $C^\infty(\bar\Lambda(\mathcal{L}^\lambda)) = \bigoplus_{q=0}^{n} \Lambda^{0,q}(M, \mathcal{L}^\lambda)$ as well as its closure in $L^2(\bar\Lambda(\mathcal{L}^\lambda))$; the resulting heat operator $\exp(-t\bar\Box)$ is of trace class for all $t > 0$. Since $\bar\Box$ commutes with the induced representation π^λ of G in $L^2(\bar\Lambda(\mathcal{L}^\lambda))$, it commutes also with $d\pi^\lambda(Y)$ for all $Y \in \mathfrak{g}$. Now, for each $m > 0$ the operator sum

$$\frac{1}{m}\bar\Box - id\pi^\lambda(Y)$$

defines an elliptic, selfadjoint operator bounded from below; the semigroup associated to this operator is of trace class. Furthermore, we introduce the linear operator ε in $L^2(\bar\Lambda(\mathcal{L}^\lambda))$ which is equal to $+1$ on even forms and equal to -1 on odd forms.

Theorem 2.1. *For $\lambda \in L \cap C$ let $(\pi_\lambda, \mathscr{H}_\lambda)$ be the irreducible representation with highest weight λ. Then for all Lie algebra elements $Y \in \mathbf{g}$:*

(i) *For all $t > 0$*

$$\lim_{m \to 0^+} \mathrm{Trace}_{L^2(\mathscr{L}^\lambda)}\left\{\exp\left\{-t\left(\frac{1}{m}\bar{\square} - id\pi^\lambda(Y)\right)\right\}\right\} = \mathrm{Trace}_{\mathscr{H}_\lambda}\{\exp(itd\pi_\lambda(Y))\}\,. \quad (2.5)$$

(ii) *For all $t > 0$ and $m > 0$*

$$\mathrm{Trace}_{L^2(\bar{\Lambda}(\mathscr{L}^\lambda))}\left\{\varepsilon \exp\left\{-t\left(\frac{1}{m}\bar{\square} - id\pi^\lambda(Y)\right)\right\}\right\} = \mathrm{Trace}_{\mathscr{H}_\lambda}\{\exp(itd\pi_\lambda(Y))\}\,. \quad (2.6)$$

Proof. First we recall the well-known fact that for any selfadjoint operator $A \geqq 0$ whose semigroup $\exp(-tA)$ for $t > 0$ is of trace class, the strong limit $s-\lim_{t \to \infty} \exp(-tA)$ gives just the orthogonal projection onto the kernel of A. Choosing $A = \bar{\square}$, the relation (2.5) becomes a direct consequence of the Bott-Borel-Weil theorem.

To prove statement (ii), we replace the factor i on both sides of (2.6) by a complex parameter w. Then

$$\mathrm{Trace}_{\mathscr{H}_\lambda}\{\exp\{twd\pi_\lambda(Y)\}\} \quad (2.7)$$

is analytic in w. Furthermore, for $w \in \mathbb{C}$

$$\frac{1}{m}\bar{\square} - wd\pi^\lambda(Y)$$

is a holomorphic family of operators of type A (see [Ka, p. 375]). Thus

$$\mathrm{Trace}_{L^2(\bar{\Lambda}(\mathscr{L}^\lambda))}\left\{\varepsilon \exp\left\{-t\left(\frac{1}{m}\bar{\square} - wd\pi^\lambda(Y)\right)\right\}\right\} \quad (2.8)$$

does exist and defines an analytic function of w. Consequently, we need only to show that (2.7) and (2.8) are equal for w real. This, however, is an easy consequence of supersymmetry and the Bott-Borel-Weil theorem. In fact, let $0 = \mu_0 < \mu_1 < \ldots < \mu_j < \ldots$ be the eigenvalues of $\bar{\square}$ and l_j the corresponding eigenspaces. Then $l_j = l_j^+ \oplus l_j^-$, where $l_j^\pm$ are the subspaces of even $(+)$ or odd $(-)$ forms; these subspaces are invariant under π^λ, so π^λ can be restricted to finite dimensional representations $\pi_j^\pm$ of G on $l_j^\pm$. Employing the Bott-Borel-Weil theorem, the expression (2.8) becomes

$$\mathrm{Trace}_{\mathscr{H}_\lambda}\{\pi_\lambda(e^{twY})\} + \sum_{j>0} e^{-m^{-1}t\mu_j}\{\mathrm{Trace}_{l_j^+}\{\pi_j^+(e^{twY})\} - \mathrm{Trace}_{l_j^-}\{\pi_j^-(e^{twY})\}\}\,. \quad (2.9)$$

But for $j > 0$, the operator

$$U_j = \frac{1}{\sqrt{\mu_j}}(\bar{\partial} + \bar{\partial}^*)_{|l_j^+}$$

provides a unitary equivalence of the representations π_j^+ and π_j^-, implying statement (ii). An alternative argument leading to (ii) and not based on the analytic continuation of (2.6) will be given in Sect. 3.

2.4. Coadjoint Orbits

As before, a given $\lambda \in L \cap C$ will be considered as an element of $\mathbf{g}_{\mathbb{R}}^*$. Let M_λ be the coadjoint orbit in $\mathbf{g}_{\mathbb{R}}^*$ through λ. Obviously M_λ is diffeomorphic to the left coset space G/U_λ, where $U_\lambda = \{g \in G | \mathrm{Ad}^*(g)\lambda = \lambda\} \supset T$ denotes the stabilizer of λ. Equivalently, with π_λ being the irreducible representation in $\mathscr{H}_\lambda$ corresponding to λ, the stabilizer U_λ can be characterized as the subgroup of all those $g \in G$ leaving the one-dimensional subspace of $\mathscr{H}_\lambda$ spanned by ψ_λ invariant. In addition, we have

$$\dim_{\mathbb{R}} U_\lambda = \mathrm{rank}\, G + 2 \times \#\{\alpha \in \Delta^+ | (\lambda, \alpha) = 0\}. \tag{2.10}$$

Therefore, in the generic case, i.e. when λ lies in the interior of the positive Weyl chamber C, the equality $U_\lambda = T$ holds. Note that under the diffeomorphism $M_\lambda \simeq G/U_\lambda$ the coadjoint action of G on M_λ corresponds to the left translation on the homogeneous space G/U_λ.

On M_λ a normalized measure $d\mu_{M_\lambda}$ is defined as the push forward of the normalized Haar measure μ on G,

$$\mu_{M_\lambda}(W) = \mu(\{g \in G | \mathrm{Ad}^*(g)\lambda \in W\}).$$

This measure can be equivalently characterized in terms of the involved Riemannian structure. Since $\mathbf{g}_{\mathbb{R}}^*$ carries the Euclidean metric induced by the Cartan-Killing form on $\mathbf{g}_{\mathbb{R}}$ and $M_\lambda \subset \mathbf{g}_{\mathbb{R}}^*$ is a compact subset, M_λ becomes a compact Riemannian submanifold of $\mathbf{g}_{\mathbb{R}}^*$. Note that this metric on M_λ induced by the Euclidean metric on $\mathbf{g}_{\mathbb{R}}^*$ coincides (up to a scale) with the G-invariant Kähler metric on G/U_λ considered above. In fact, define P_λ to be the parabolic subgroup of $G_{\mathbb{C}}$ leaving the one dimensional space spanned by ψ_λ invariant. Then we have $P_\lambda \cap G = U_\lambda$ and by the above discussion $G_{\mathbb{C}}/P_\lambda \cong G/U_\lambda \cong M_\lambda$.

The holomorphic line bundle $\mathscr{L}^\lambda \to G/U_\lambda$ may also be described in these terms. Namely, let Q_λ^0 be the orthogonal projection in $\mathscr{H}_\lambda$ onto the one dimensional subspace spanned by ψ_λ. For any $g \in G$ consider the one dimensional orthogonal projection

$$Q_\lambda(g) = \pi_\lambda(g) Q_\lambda^0 \pi_\lambda(g)^{-1} \tag{2.11}$$

with range spanned by the vector $\pi_\lambda(g)\psi_\lambda$. If we set $z = \mathrm{Ad}^*(g)\lambda \in M_\lambda$, then $Q_\lambda(g)$ depends on z only and, by abuse of notation, we write $Q_\lambda(z) = Q_\lambda(g)$ such that $Q_\lambda(\lambda) = Q_\lambda^0$. The transformation property

$$\pi_\lambda(g') Q_\lambda(z) \pi_\lambda(g')^{-1} = Q_\lambda(\mathrm{Ad}^*(g')z) \tag{2.12}$$

is obvious. With $d_\lambda = \dim \mathscr{H}_\lambda$ the completeness relation

$$d_\lambda \int_{M_\lambda} Q_\lambda(z) d\mu_{M_\lambda}(z) = \mathrm{id}_{\mathscr{H}_\lambda} \tag{2.13}$$

is a consequence of Schur's lemma; in particular, it implies the relation (1.8) in Sect. 1. Now, with the help of the $Q_\lambda(z)$, a subbundle of the trivial product bundle $\mathscr{H}^\lambda = M_\lambda \times \mathscr{H}_\lambda$ is constructed by setting the fibre over each z equal to $Q_\lambda(z)\mathscr{H}_\lambda = \mathrm{range}\, Q_\lambda(z)$. The resulting bundle is isomorphic to the line bundle $\mathscr{L}^\lambda \to G/U_\lambda$, and the G-invariant hermitian metric in $\mathscr{L}^\lambda$ is the one induced from $\mathscr{H}^\lambda$. Henceforth $\mathscr{L}^\lambda$ will therefore be viewed as a bundle over the coadjoint orbit M_λ.

Finally, we discuss the induced representations. Using the projections $Q_\lambda(z)$, a linear map $Q_\lambda: L^2(\mathscr{H}^\lambda) \to L^2(\mathscr{L}^\lambda)$ is defined by $(Q_\lambda \varphi)(z) = Q_\lambda(z)\varphi(z)$. This map satisfies $Q_\lambda \circ j_\lambda = \mathrm{id}$, where $j_\lambda: L^2(\mathscr{L}^\lambda) \to L^2(\mathscr{H}^\lambda)$ denotes the canonical embedding. Furthermore, we define a unitary representation $\tilde{\pi}_0^\lambda$ of G in $L^2(\mathscr{H}^\lambda)$ by

$$(\tilde{\pi}_0^\lambda(g)\varphi)(z) = \pi_\lambda(g)\varphi(\mathrm{Ad}^* g^{-1} z). \tag{2.14}$$

Then the induced representation π_0^λ of G in $L^2(\mathscr{L}^\lambda)$ is given by

$$\pi_0^\lambda(g) = Q_\lambda \circ \tilde{\pi}_0^\lambda(g) \circ j_\lambda. \tag{2.15}$$

The induced representation π^λ of G in $L^2(\bar{\Lambda}(\mathscr{L}^\lambda))$ can be described in a similar way. Namely, with respect to the isomorphism $L^2(\bar{\Lambda}(\mathscr{L}^\lambda)) \cong L^2(\mathscr{L}^\lambda) \otimes L^2(\Lambda^* T^{*(0,1)} M_\lambda)$ we have

$$\pi^\lambda = \pi_0^\lambda \otimes \sigma. \tag{2.16}$$

Here σ stands for the induced representation of G in $L^2(\Lambda^* T^{*(0,1)} M_\lambda)$. More precisely, if as usual we denote by $d\psi^*: T^*M \to T^*M$ the map induced by a diffeomorphism $\psi: M \to M$, then for all $\omega \in L^2(\Lambda^* T^{*(0,1)} M_\lambda)$ and $z \in M_\lambda$ the representation σ is given as

$$(\sigma(g)\omega)(z) = \Lambda^*((d\mathrm{Ad}^* g^{-1})^*)\omega(\mathrm{Ad}^* g^{-1} z).$$

Note that $\mathrm{Ad}^* g$ defines a holomorphic map such that $\Lambda^*((d\mathrm{Ad}^* g^{-1})^*)$ is well defined on $\Lambda^* T^{*(0,1)} M_\lambda$. Let K_Y be the Killing vector field associated with the Lie algebra element $Y \in \mathbf{g}$, i.e.

$$K_Y(z) = \frac{d}{dt} \mathrm{Ad}^* e^{-tY} z_{|t=0}, \tag{2.17}$$

and let L_{K_Y} be the Lie derivative with respect to K_Y. The definition of σ implies

$$d\sigma(Y)\omega = L_{K_Y}\omega$$

for all $Y \in \mathbf{g}$. Hence, by virtue of (2.16), we obtain

$$d\pi^\lambda(Y) = d\pi_0^\lambda(Y) \otimes \mathrm{id} + \mathrm{id} \otimes L_{K_Y}. \tag{2.18}$$

To compute $d\pi_0^\lambda(Y)$ we have to apply the relations (2.1), (2.14), and (2.15). With $z = \mathrm{Ad}^*(g)\lambda \in M_\lambda$ and $\varphi \in C^\infty(\mathscr{L}^\lambda)$ it follows

$$\begin{aligned}(d\pi_0^\lambda(Y)\varphi)(z) &= \lambda(\mathrm{Ad}\, g^{-1} Y)\varphi(z) + (Q_\lambda \circ K_Y(j_\lambda \varphi))(z) \\ &= Y(z)\varphi(z) + \nabla_{0 K_Y}^\lambda \varphi(z).\end{aligned} \tag{2.19}$$

Here the purely imaginary quantity $Y(z)$ is defined by $Y(z) = z(Y)$ where we view $z \in M_\lambda$ as an element of $\mathbf{g}_{\mathbb{R}}^*$. We have also used the fact that in terms of the trivial connection d of $\mathscr{H}^\lambda$ the hermitian connection ∇_0^λ in $\mathscr{L}^\lambda$ can be expressed as

$$\nabla_0^\lambda = Q_\lambda \circ d \circ j_\lambda. \tag{2.20}$$

2.5. Connections and Kähler Geometry

Recall that for any holomorphic hermitian vector bundle $E \to M$ over a complex manifold M there exists a unique connection $\nabla: C^\infty(E) \to C^\infty(E \otimes T_{\mathbb{C}}^* M)$ satisfying

(i) $\nabla'' = \bar{\partial}$, where $\nabla = \nabla' + \nabla''$ is the decomposition of ∇ into type, i.e. $\nabla'' = p \circ \nabla$ with p being the projection onto the space of E-valued $(0,1)$-forms.

(ii) $d(s,s') = (\nabla s, s') + (s, \nabla s')$ for all $s, s' \in C^\infty(E)$.

This ∇ is called the hermitian connection of E (cf. Chap. IX, Sect. 10 of [KN]).

In particular, if M carries a hermitian metric, we may consider the hermitian connection in the holomorphic tangent bundle $T^{(1,0)}M$ of M. We extend the covariant differentiation to $T_{\mathbb{C}}M$ as follows

$$\nabla_{X_1 + iX_2}(Z_1 + iZ_2) = \nabla_{X_1}(Z_1) - \nabla_{X_2}(Z_2) + i\nabla_{X_1}(Z_2) + i\nabla_{X_2}(Z_1),$$

where X_1, X_2, Z_1, Z_2 are vector fields on M. This generalized connection induces a corresponding connection in all tensor bundles. If the hermitian metric on M is Kähler, then the hermitian connection is torsion free.

In the case of $M = M_\lambda$ being the coadjoint orbit through λ and $\mathscr{L}^\lambda \to M_\lambda$ the holomorphic line bundle discussed above, we denote by ∇^λ the connection in $\bar{\Lambda}^q(\mathscr{L}^\lambda) = \mathscr{L}^\lambda \otimes \Lambda^q T^{*(0,1)} M_\lambda$ which is obtained from the hermitian connection in $\mathscr{L}^\lambda$ and the generalized connection in $\Lambda^q T^{*(0,1)} M_\lambda$ just defined. This connection ∇^λ can be described more explicitly in local holomorphic coordinates z^a on M_λ, where $1 \leqq a \leqq \dim_{\mathbb{C}} M_\lambda$. Namely, in any tangent space $T_z M_\lambda$ the inner product defined by the hermitian metric g on M_λ admits an unique extension to a symmetric bilinear form in the complexified tangent space $T_{z\,\mathbb{C}} M_\lambda$. Set

$$g_{a\bar{b}} = g\left(\frac{\partial}{\partial z^a}, \frac{\partial}{\partial \bar{z}^b}\right)$$

and define $g_{\bar{a}b}$, g_{ab}, and $g_{\bar{a},\bar{b}}$ similarly. Then $g_{ab} = g_{\bar{a}\bar{b}} = 0$ and the metric on M_λ becomes

$$ds^2 = 2 \sum_{a,b} g_{a\bar{b}} dz^a d\bar{z}^b .$$

If g is a Kähler metric, the Christoffel symbols are given by (see e.g. [KN])

$$\Gamma^c_{ab} = \sum_d \frac{\partial g_{b\bar{d}}}{\partial z^a} g^{c\bar{d}} = \Gamma^c_{ba}, \qquad \Gamma^{\bar{c}}_{\bar{a}\bar{b}} = \overline{\Gamma^c_{ab}}, \tag{2.21}$$

and all the other terms are zero.

We take s to be a local nonvanishing holomorphic section of $\mathscr{L}^\lambda$ over the given coordinate neighborhood, and set

$$l_a = \frac{\partial}{\partial z^a} \log(\|s(z)\|^2), \tag{2.22}$$

where $\|\cdot\|$ is the norm in the fibre of $\mathscr{L}^\lambda$ over z determined by the metric of $\mathscr{L}^\lambda$. Furthermore, let

$$\varphi = s \otimes \left(\sum_{b_1 < \ldots < b_q} \varphi_{\bar{b}_1 \ldots \bar{b}_q} d\bar{z}^{b_1} \wedge \ldots \wedge d\bar{z}^{b_q} \right)$$

be a smooth $(0,q)$-form with values in $\mathscr{L}^\lambda$ defined on the given coordinate neighborhood. The covariant derivative of φ in $\bar{\Lambda}^q(\mathscr{L}^\lambda)$ is then given by the following formulas

$$\begin{aligned}(\nabla^\lambda_{\partial/\partial z^a}\varphi)_{\bar b_1\ldots\bar b_q} &= \frac{\partial}{\partial z^a}\varphi_{\bar b_1\ldots\bar b_q} - l_a\varphi_{\bar b_1\ldots\bar b_q},\\ (\nabla^\lambda_{\partial/\partial\bar z^a}\varphi)_{\bar b_1\ldots\bar b_q} &= \frac{\partial}{\partial\bar z^a}\varphi_{\bar b_1\ldots\bar b_q} - \sum_{r,b}\Gamma^{\bar b}_{\bar a\bar b_r}\varphi_{\bar b_1\ldots\bar b_{r-1}\bar b\bar b_{r+1}\ldots\bar b_q}.\end{aligned} \tag{2.23}$$

Also, by virtue of (2.19) and (2.20), the relation (2.18) can be rewritten as

$$(d\pi^\lambda(Y)\varphi)(z) = Y(z)\varphi(z) + (\nabla^\lambda_{0K_Y}\otimes\mathrm{id})\varphi(z) + (\mathrm{id}\otimes L_{K_Y})\varphi(z) \tag{2.24}$$

for $z\in M_\lambda$ and $Y\in\mathbf{g}$.

Our next goal is to express the right-hand side of (2.24) in terms of the hermitian connection ∇^λ. It suffices to calculate the Lie derivative L_{K_Y} on $T^{(0,1)}M_\lambda$, the dual of $T^{*(0,1)}M_\lambda$. Since the hermitian connection ∇ on $T^{(0,1)}M_\lambda$ is torsion free, we have for $Y\in\mathbf{g}$

$$L_{K_Y}\left(\frac{\partial}{\partial\bar z^b}\right) - \nabla_{K_Y}\left(\frac{\partial}{\partial\bar z^b}\right) = \left[K_Y, \frac{\partial}{\partial\bar z^b}\right] - \nabla_{K_Y}\left(\frac{\partial}{\partial\bar z^b}\right) = -\nabla_{\partial/\partial\bar z^b}(K_Y). \tag{2.25}$$

Moreover, in the local coordiante expression for K_Y,

$$K_Y = \sum_c\left(K^c_Y\frac{\partial}{\partial z_c} + K^{\bar c}_Y\frac{\partial}{\partial\bar z_c}\right) \tag{2.26}$$

the coefficients K^c_Y and $K^{\bar c}_Y$ are holomorphic and antiholomorphic, respectively. In fact, since $\mathrm{Ad}^*(e^{tY})$ acts on M_λ as a one parameter family of holomorphic transformations, $K_Y f$ is holomorphic for any local holomorphic function f on M_λ. Thus

$$\nabla_{\partial/\partial\bar z^b}(K_Y) = \sum_a\left(\frac{\partial K^{\bar a}_Y}{\partial\bar z^b} + \sum_c\Gamma^{\bar a}_{\bar b\bar c}K^{\bar c}_Y\right)\frac{\partial}{\partial\bar z^a}. \tag{2.27}$$

We define the "Fermi field" ψ^{*a} to be the (local) bundle endomorphism of $\Lambda^*T^{*(0,1)}M_\lambda$ given by exterior multiplication with $d\bar z^a$; its adjoint with respect to the hermitian metric on $\Lambda^*T^{*(0,1)}M_\lambda$ defines ψ^a, and ψ_a is then defined as

$$\psi_a = \sum_b g_{b\bar a}\psi^b. \tag{2.28}$$

The ψ give rise to a Clifford algebra with the canonical anticommutation relations

$$\begin{aligned}\{\psi^{*a},\psi_b\} &= \delta^a_b,\\ \{\psi^{*a},\psi^{*b}\} &= \{\psi_a,\psi_b\} = 0.\end{aligned} \tag{2.29}$$

Let the vector bundle endomorphism E_Y be locally given by

$$\begin{aligned}E_Y &= \sum_{a,b}\left(\frac{\partial K^{\bar b}_Y}{\partial\bar z^a} + \sum_c\Gamma^{\bar b}_{\bar a\bar c}K^{\bar c}_Y\right)(\mathrm{id}\otimes\psi^{*a}\psi_b)\\ &= \sum_{a,b}(\nabla_{\partial/\partial\bar z^a}K_Y)^{\bar b}(\mathrm{id}\otimes\psi^{*a}\psi_b).\end{aligned} \tag{2.30}$$

Now, the wanted formula for the induced representation follows easily from (2.24) and the observations above

$$(d\pi^{\lambda}(Y)\varphi)(z)=Y(z)\varphi(z)+\nabla^{\lambda}_{K_Y}\varphi(z)+(E_Y\varphi)(z). \tag{2.31}$$

Finally, we want to recall a Weitzenböck type formula for the operator $\bar{\square}$ in the form given by Patodi [Pa]. It reads:

$$\bar{\square}=\bar{\Delta}-D\tilde{K}+DS. \tag{2.32}$$

Here $\bar{\Delta}$ denotes the following Laplace operator given locally as

$$\bar{\Delta}=-\sum_{a,b} g^{a\bar{b}}\nabla^{\lambda}_{a}\nabla^{\lambda}_{\bar{b}}, \tag{2.33}$$

where we use the abbreviations

$$\nabla^{\lambda}_{a}=\nabla_{\partial/\partial z^a}, \qquad \nabla^{\lambda}_{\bar{a}}=\nabla_{\partial/\partial \bar{z}^a}.$$

The other two terms appearing in (2.32) are strict vector bundle endomorphisms; $D\tilde{K}$ is defined in terms of the Ricci curvature tensor $\tilde{K}$ of M_{λ}, and DS is defined in terms of the curvature form S of the hermitian connection of the line bundle $\mathscr{L}^{\lambda}$.

More precisely, consider the following components of the curvature tensor

$$K^{\bar{d}}_{\bar{c}a\bar{b}}=\frac{\partial\Gamma^{\bar{d}}_{\bar{b}\bar{c}}}{\partial z_a}. \tag{2.34}$$

The symmetry relation

$$K^{\bar{d}}_{\bar{c}a\bar{b}}=K^{\bar{d}}_{\bar{b}a\bar{c}} \tag{2.35}$$

is special for Kähler manifolds. With

$$K^{\bar{a}\bar{c}}_{\bar{b}\bar{d}}=\sum_{e} g^{e\bar{c}}K^{\bar{a}}_{\bar{b}e\bar{d}}, \tag{2.36}$$

the Ricci curvature tensor of M_{λ} is defined by

$$\tilde{K}^{\bar{a}}_{\bar{c}}=\sum_{b} K^{\bar{a}\bar{b}}_{\bar{b}\bar{c}}. \tag{2.37}$$

Similarly, the curvature form S is given by

$$S^{\bar{a}}_{\bar{b}}=\sum_{c} g^{c\bar{a}}S_{c\bar{b}}, \tag{2.38}$$

where

$$S_{a\bar{b}}=-\frac{\partial l_a}{\partial \bar{z}^b}=-\frac{\partial^2}{\partial z^a\partial\bar{z}^b}\log(\|s(z)\|^2) \tag{2.39}$$

and l_a was introduced in (2.22). Thus, with these notations the endomorphisms $D\tilde{K}$ and DS take the form

$$\begin{aligned} D\tilde{K}&=\sum_{a,b}\tilde{K}^{\bar{a}}_{\bar{b}}\mathrm{id}\otimes\psi^{*b}\psi_a,\\ DS&=\sum_{a,b} S^{\bar{a}}_{\bar{b}}\mathrm{id}\otimes\psi^{*b}\psi_a. \end{aligned} \tag{2.40}$$

Formula (2.32) allows us to relate $\bar{\Box}$ to the Bochner Laplacian $-(\nabla^\lambda)^2$ for the hermitian connection ∇^λ. Namely, rewriting its definition

$$\begin{aligned} -(\nabla^\lambda)^2 &= -g^{a\bar{b}}(\nabla_a^\lambda \nabla_{\bar{b}}^\lambda + \nabla_{\bar{b}}^\lambda \nabla_a^\lambda) \\ &= 2\bar{\Delta} - g^{a\bar{b}}(\nabla_{\bar{b}}^\lambda \nabla_a^\lambda - \nabla_a^\lambda \nabla_{\bar{b}}^\lambda) \\ &= 2\bar{\Delta} - g^{a\bar{b}} S_{a\bar{b}} - D\tilde{K} \end{aligned} \tag{2.41}$$

and setting $\operatorname{tr} S = g^{a\bar{b}} S_{a\bar{b}}$, the relation

$$\bar{\Box} = -\tfrac{1}{2}(\nabla^\lambda)^2 - \tfrac{1}{2} D\tilde{K} + DS + \tfrac{1}{2}\operatorname{tr} S \tag{2.42}$$

is a simple consequence of (2.32) and (2.41).

Notice that in the special case of the trivial line bundle where $2\bar{\Box} = \Delta$ (see e.g. [We]), by (2.35) relation (2.42) reduces to the usual Weitzenböck formula on Kähler manifolds. Furthermore, since the Dirac operator on Kähler manifolds is $\sqrt{2}(\bar{\partial} + \bar{\partial}^*)$, the relation (2.42) is akin to the Lichnerowicz formula for the spinor Laplacian [Lic, Hi]. For a later purpose, we remark that according to (2.23) the hermitian connection ∇ in $\Lambda^* T^{*(0,1)} M_\lambda$ may be written in local coordinates as

$$\begin{aligned} \nabla_{\bar{a}} &= \frac{\partial}{\partial \bar{z}^a} + C_{\bar{a}}, \\ \nabla_a &= \frac{\partial}{\partial z^a}, \end{aligned} \tag{2.43}$$

where

$$C_{\bar{a}} = -\sum_{b,c} \Gamma_{\bar{a}\bar{b}}^{\bar{c}} \psi^{*b} \psi_c. \tag{2.44}$$

This motivates the definition of the local one form

$$C = \sum_a C_{\bar{a}} d\bar{z}^a \tag{2.45}$$

taking values in the space of (local) vector bundle endomorphisms of $\Lambda^* T^{*(0,1)} M_\lambda$. Sometimes it will be convenient to work with the real coordiantes $x^j = \Re e z^j$, $x^{j+n} = \Im m z^j$, where $1 \leqq j \leqq n = \dim_{\mathbb{C}} M_\lambda$. Then with

$$C = \sum_{j=1}^{2n} C_j dx^j \tag{2.46}$$

we have

$$\nabla_{\partial/\partial x^j} = \frac{\partial}{\partial x^j} + C_j \tag{2.47}$$

on $\Lambda^* T^{*(0,1)} M_\lambda$. This yields in particular

$$\nabla_{\partial/\partial x^j}^\lambda = \nabla_{0\,\partial/\partial x^j}^\lambda \otimes \mathrm{id} + \mathrm{id} \otimes \left(\frac{\partial}{\partial x^j} + C_j\right) \tag{2.48}$$

on $\bar{\Lambda}(\mathscr{L}^\lambda)$.

3. The Generalized Quantum Heisenberg Ferromagnet

Following Simon [Si] we generalize the quantum Heisenberg ferromagnet to an arbitrary G and an arbitrary representation π_λ. Then we establish a supertrace representation for the partition function analogous to the one for the one lattice point theory.

Let Λ be a finite lattice as described in the introduction. We set $G_\Lambda = \times_{k\in\Lambda} G$ and write $g=\{g(k)\}_{k\in\Lambda}\in G_\Lambda$ for its elements; the group G will be identified with the diagonal subgroup of G_Λ. The Lie algebra of G_Λ is then $\mathbf{g}_\Lambda = \bigoplus_{k\in\Lambda} \mathbf{g}$. The irreducible representation of G in $\mathscr{H}_\lambda$ defines an irreducible representation $\pi_{\lambda,\Lambda}(g) = \bigotimes_{k\in\Lambda} \pi_\lambda(g(k))$ of G_Λ in the finite dimensional Hilbert space $\mathscr{H}_{\lambda,\Lambda} = \bigotimes_{k\in\Lambda} \mathscr{H}_\lambda$. For any $Y\in\mathbf{g}$ and $k\in\Lambda$ we let Y^k be the element of $\mathbf{g}_\Lambda$ that agrees with Y in the k^{th} place and is zero otherwise. Let Y_α $(1\leqq\alpha\leqq\dim(\mathbf{g}))$ be a basis of $\mathbf{g}$ and let $\mathbf{c}=\{c^{\alpha\beta}(k,k')\}_{1\leqq\alpha,\beta\leqq\dim G; k,k'\in\Lambda}$ be any set of real numbers such that $c^{\alpha\beta}(k,k')=c^{\alpha\beta}(k',k)$, and for any fixed k,k' with $k\neq k'$ the matrix $c^{\alpha\beta}(k,k')$ is symmetric in α and β. Also let $\mathbf{h}\in\mathbf{g}_\Lambda$ be arbitrary and J real. We define the following selfadjoint operator in $\mathscr{H}_{\lambda,\Lambda}$:

$$H_{\lambda,\Lambda} = H_{\lambda,\Lambda}(J,\mathbf{c},\mathbf{h}) = -J\sum_{\alpha,\beta}\sum_{\substack{k,k'\in\Lambda\\ k\neq k'}} d\pi_{\lambda,\Lambda}(Y_\alpha^k)\,d\pi_{\lambda,\Lambda}(Y_\beta^{k'})\,c^{\alpha\beta}(k,k') - i\,d\pi_{\lambda,\Lambda}(\mathbf{h}) \tag{3.1}$$

to be the generalized quantum Heisenberg model in the external magnetic field $\mathbf{h}$. The choice $-c^{\alpha\beta}(k,k')=b^{\alpha\beta}$ (= inverse of $B(Y_\alpha, Y_\beta)$ with B being the Killing form) if k,k' are nearest neighbours, and zero otherwise, leads to an (in "spin" space) isotropic Heisenberg ferromagnet ($J\geqq 0$) or Heisenberg antiferromagnet ($J\leqq 0$). Then $H_{\lambda,\Lambda}$ for the case $\mathbf{h}=0$ is globally gauge invariant, i.e. commutes with $\pi_{\lambda,\Lambda}(g)$ for $g\in G$. Specializing to $G=SU(2)$ and its self-representation we recover the usual quantum Heisenberg (anti-)ferromagnet. Also for other appropriate choices of $\mathbf{c}$ we obtain (generalizations of) the $X-Y$ or the Ising model.

One is interested in the partition function

$$\operatorname{Trace}_{\mathscr{H}_{\lambda,\Lambda}} e^{-tH_{\lambda,\Lambda}}. \tag{3.2}$$

We extend our one lattice point discussion in the following way. First we introduce the Kähler manifold

$$M_{\lambda,\Lambda} = \times_{k\in\Lambda} M_{\lambda,k}, \tag{3.3}$$

where each $M_{\lambda,k}$ is a copy of M_λ. Points in $M_{\lambda,\Lambda}$ are written as $\mathbf{z}=\{z(k)\}_{k\in\Lambda}$ with $z(k)\in M_\lambda$. Then

$$d\mu_{M_{\lambda,\Lambda}}(\mathbf{z}) = \prod_{k\in\Lambda} d\mu_{M_\lambda}(z(k)) \tag{3.4}$$

defines a probability measure on $M_{\lambda,\Lambda}$.

Next we introduce the holomorphic and hermitian line bundle $\mathscr{L}_\Lambda^\lambda = \bigotimes_{k\in\Lambda} \mathscr{L}_k^\lambda$ over $M_{\lambda,\Lambda}$, where $\mathscr{L}_k^\lambda$ is the pull back of $\mathscr{L}^\lambda$ via the holomorphic projection $\pi_k: M_{\lambda,\Lambda}\to M_{\lambda,k}$. Via this pullback the connection ∇_0^λ on $\mathscr{L}^\lambda$ determines for all

$k\in\Lambda$ a connection $\nabla^\lambda_{0,k}$ on $\mathscr{L}^\lambda_k$ and hence a connection $\nabla^\lambda_{0,\Lambda}$ on the tensor product $\mathscr{L}^\lambda_\Lambda$. A holomorphic and hermitian vector bundle over $M_{\lambda,\Lambda}$ is defined by $\bar{\Lambda}(\mathscr{L}^\lambda_\Lambda) = \mathscr{L}^\lambda_\Lambda \otimes \Lambda^* T^{*(0,1)}_{\mathbb{C}} M_{\lambda,\Lambda}$. Obviously $\mathscr{L}^\lambda_\Lambda$ may be regarded as a subbundle of $\bar{\Lambda}(\mathscr{L}^\lambda_\Lambda)$. By ∇^λ_Λ we denote the hermitian connection on $\bar{\Lambda}(\mathscr{L}^\lambda_\Lambda)$ obtained from $\nabla^\lambda_{0,\Lambda}$ on $\mathscr{L}^\lambda_\Lambda$ and the canonical hermitian (torsionfree) connection ∇_Λ on $\Lambda^* T^{*(0,1)}_{\mathbb{C}} M_{\lambda,\Lambda}$. We set

$$\bar{\Box}_\Lambda = \sum_{k\in\Lambda} \bar{\Box}_k \tag{3.5}$$

mapping $C^\infty(\bar{\Lambda}(\mathscr{L}^\lambda_\Lambda))$ into itself and leaving $C^\infty(\mathscr{L}^\lambda_\Lambda)$ invariant. Here $\bar{\Box}_k$ is $\bar{\Box}$ acting on the k^{th} variable $z(k)$ of $\mathbf{z}$. The operator (3.5) is densely defined and has a unique selfadjoint extension in $L^2(\bar{\Lambda}(\mathscr{L}^\lambda_\Lambda))$ which is denoted by the same symbol.

Finally, in analogy to the construction of $\pi_{\lambda,\Lambda}$ the unitary representation π^λ of G on $L^2(\bar{\Lambda}(\mathscr{L}^\lambda))$ gives a unitary representation $\pi^\lambda_\Lambda(g) = \bigotimes_{k\in\Lambda} \pi^\lambda(g(k))$ of G_Λ on $L^2(\bar{\Lambda}(\mathscr{L}^\lambda_\Lambda)) \cong \bigotimes_{k\in\Lambda} L^2(\bar{\Lambda}(\mathscr{L}^\lambda))$. Its restriction to $L^2(\mathscr{L}^\lambda_\Lambda)$ is just $\pi^\lambda_{0,\Lambda}(g) = \bigotimes_{k\in\Lambda} \pi^\lambda_0(g(k))$.

It will again be crucial that $\pi^\lambda_\Lambda(g)$ for all $g\in G_\Lambda$ commutes with $\bar{\partial}_\Lambda = \sum_{k\in\Lambda} \bar{\partial}_k$ and hence also with its adjoint $\bar{\partial}^*_\Lambda$.

Now a second order differential operator acting on $C^\infty(\bar{\Lambda}(\mathscr{L}^\lambda_\Lambda))$ and leaving $C^\infty(\mathscr{L}^\lambda_\Lambda)$ invariant is defined by

$$H^\lambda_\Lambda = H^\lambda_\Lambda(J,\mathbf{c},\mathbf{h}) = -J\sum_{\alpha,\beta}\sum_{\substack{k,k'\in\Lambda\\ k\neq k'}} d\pi^\lambda_\Lambda(Y^k_\alpha)\, d\pi^\lambda_\Lambda(Y^{k'}_\beta)\, c^{\alpha\beta}(k,k') - i\, d\pi^\lambda_\Lambda(\mathbf{h}), \tag{3.6}$$

where $\mathbf{c}$ and $\mathbf{h}$ are as in (3.1). In other words, H^λ_Λ is obtained from $H_{\lambda,\Lambda}$ by the replacement $\pi_{\lambda,\Lambda} \to \pi^\lambda_\Lambda$. As was the case for the operator (3.5), the closure of (3.6) determines uniquely a selfadjoint operator in $L^2(\bar{\Lambda}(\mathscr{L}^\lambda_\Lambda))$ which we continue to denote by H^λ_Λ.

Again, letting ε be 1 on even forms and -1 on odd forms, the next theorem is a consequence of the Bott-Borel-Weil theorem.

Theorem 3.1. *For small $m|J|$ the operator $\frac{1}{m}\bar{\Box}_\Lambda + H^\lambda_\Lambda$ is elliptic. The following representation holds for the partition function* (3.2)

$$\lim_{m\to 0^+} \mathrm{Trace}_{L^2(\mathscr{L}^\lambda_\Lambda)}\, e^{-t\{\frac{1}{m}\bar{\Box}_\Lambda + H^\lambda_\Lambda\}} = \mathrm{Trace}_{\mathscr{H}_{\lambda,\Lambda}}\, e^{-tH_{\lambda,\Lambda}}. \tag{3.7}$$

Moreover for all sufficiently small $m>0$,

$$\mathrm{Trace}_{L^2(\bar{\Lambda}(\mathscr{L}^\lambda_\Lambda))}\{\varepsilon e^{-t\{\frac{1}{m}\bar{\Box}_\Lambda + H^\lambda_\Lambda\}}\} = \mathrm{Trace}_{\mathscr{H}_{\lambda,\Lambda}}\, e^{-tH_{\lambda,\Lambda}}. \tag{3.8}$$

Using Lemma 3.2 below, the arguments leading to (3.7) and (3.8) are the same as those employed in Sect. 2. In fact, in the present context one simply has to work with the supercharge operator

$$\mathscr{D}_\Lambda = \sum_{k\in\Lambda} \mathscr{D}_k = \sum_{k\in\Lambda} (\bar{\partial}_k + \bar{\partial}^*_k) \tag{3.9}$$

which anticommutes with ε, commutes with each $\pi^\lambda_\Lambda(g)$ for $g\in G_\Lambda$ and satisfies $\mathscr{D}^2_\Lambda = \bar{\Box}_\Lambda$.

Lemma 3.2. *The operator*

$$\frac{1}{m}\bar{\Box}_\Lambda + H_\Lambda^\lambda \tag{3.10}$$

acting in $L^2(\bar{\Lambda}(\mathscr{L}_\Lambda^\lambda))$ is selfadjoint and has pure point spectrum. For a given λ and **c** *this operator is elliptic for all* **h** *and the resulting semigroup of trace class, provided $m|J|$ is sufficiently small.*

Proof. We first prove selfadjointness and the spectrum property. Let $l_\mu \subseteq C^\infty(\bar{\Lambda}(\mathscr{L}_\Lambda^\lambda))$ be the eigenspace of $\bar{\Box}_\Lambda$ for the eigenvalue $\mu \geqq 0$. Each space l_μ is finite dimensional and is left invariant by the selfadjoint operator H_Λ^λ with $\mathscr{D}(H_\Lambda^\lambda) \supseteq C^\infty(\bar{\Lambda}(\mathscr{L}_\Lambda^\lambda))$. Hence the claim follows. To establish the remaining properties, we will bound the Hamiltonian (3.10) below by a sum of one lattice point Hamiltonians. In fact, since the Cartan-Killing form on **g** is negative definite, there is $c>0$ depending on $c^{\alpha\beta}(k,k')$ such that the operator inequality

$$\begin{aligned}&\pm 2 \sum_{\alpha\beta} d\pi_\Lambda^\lambda(Y_\alpha^k)\, d\pi_\Lambda^\lambda(Y_\beta^{k'})\, c^{\alpha\beta}(k,k')\\ &\leqq c \sum_{\alpha\beta} d\pi_\Lambda^\lambda(Y_\alpha^k)\, d\pi_\Lambda^\lambda(Y_\beta^k)\, b^{\alpha\beta} + c \sum_{\alpha\beta} d\pi_\Lambda^\lambda(Y_\alpha^{k'})\, d\pi_\Lambda^\lambda(Y_\beta^{k'})\, b^{\alpha\beta}\end{aligned} \tag{3.11}$$

holds for all $k \neq k'$, $k, k' \in \Lambda$. Note that the $d\pi_\Lambda^\lambda(Y_\alpha^k)$ are antiselfadjoint operators. Define $N = N(\Lambda) \geqq 1$ to be the maximum of the number of k''s any lattice point $k \in \Lambda$ can be coupled to, i.e.

$$N(\Lambda) = \sup_{k\in\Lambda} \#\{k' \mid \exists \alpha, \beta \text{ with } c^{\alpha\beta}(k,k') \neq 0\}. \tag{3.12}$$

Then by (3.11) we have

$$\frac{1}{m}\bar{\Box}_\Lambda + H_\Lambda^\lambda \geqq \sum_{k\in\Lambda} H_k^\lambda, \tag{3.13}$$

where each summand is a one lattice point operator [i.e. acts only on the k^{th} component in $L^2(\bar{\Lambda}(\mathscr{L}_\Lambda^\lambda)]$). More precisely, in $L^2(\bar{\Lambda}(\mathscr{L}^\lambda))$ these operators take the form

$$H_k^\lambda = \frac{1}{m}\bar{\Box} - \frac{|J|}{2} Nc \sum_{\alpha,\beta} d\pi^\lambda(Y_\alpha)\, d\pi^\lambda(Y_\beta)\, b^{\alpha\beta} - i\, d\pi^\lambda(\mathbf{h}(k)). \tag{3.14}$$

Therefore it suffices to show that for all small $m|J|$ each of these one lattice point operators is elliptic, bounded below and that the resulting semigroup is of trace class. This, however, follows easily from the fact that $\bar{\Box} \geqq 0$ is a second order elliptic operator, and $d\pi^\lambda(\mathbf{h}(k))$ and $d\pi^\lambda(Y_\alpha)$ are first order differential operators and that M_λ is compact, thus concluding the proof of Lemma 3.2.

In the case of an isotropic Heisenberg ferromagnet (i.e. $J \geqq 0$ and $c^{\alpha\beta}(k,k') = -b^{\alpha\beta} c(k,k')$ for suitable nonnegative $c(k,k')$) we may drop the restriction that $m|J|$ be small in the following way. Consider the Hamiltonian

$$\begin{aligned}\tilde{H}_{\lambda,\Lambda} &= \tilde{H}_{\lambda,\Lambda}(J, \mathbf{c}, \mathbf{h})\\ &= \frac{J}{2} \sum_{\alpha,\beta} \sum_{k,k'\in\Lambda} d\pi_{\lambda,\Lambda}(Y_\alpha^k - Y_\alpha^{k'})\, d\pi_{\lambda,\Lambda}(Y_\beta^k - Y_\beta^{k'})\, b^{\alpha\beta} c(k,k') - i\, d\pi_{\lambda,\Lambda}(\mathbf{h}).\end{aligned} \tag{3.15}$$

We define $\tilde{H}_\Lambda^\lambda$ by replacing $\pi_{\lambda,\Lambda}$ by π_Λ^λ. Then the observation that the first term of $\tilde{H}_\Lambda^\lambda$ resulting from the corresponding one in (3.15) is a nonnegative operator yields the following theorem.

Theorem 3.3. *For all $J \geqq 0$, $t > 0$, $m > 0$,*

$$\mathrm{Trace}_{L^2(\bar{\Lambda}(\mathscr{L}_\Lambda^\lambda))}\{\varepsilon e^{-t\{\frac{1}{m}\bar{\square}_\Lambda + \tilde{H}_\Lambda^\lambda\}}\} = \mathrm{Trace}_{\mathscr{H}_{\lambda,\Lambda}} e^{-t\tilde{H}_{\lambda,\Lambda}}. \tag{3.16}$$

Also for all $J \geqq 0$, $t > 0$

$$\lim_{m \to 0^+} \mathrm{Trace}_{L^2(\mathscr{L}_\Lambda^\lambda)} e^{-t\{\frac{1}{m}\bar{\square}_\Lambda + \tilde{H}_\Lambda^\lambda\}} = \mathrm{Trace}_{\mathscr{H}_{\lambda,\Lambda}} e^{-t\tilde{H}_{\lambda,\Lambda}}. \tag{3.17}$$

We remark that relations (3.16) and (3.17) continue to hold if $b^{\alpha\beta}$ is replaced by any negative semidefinite matrix.

It would be an interesting attempt to try to evaluate (3.16) in the limit $m \to \infty$. Note that in the corresponding one lattice point situation (see (2.6)) the limit $m \to \infty$ gives the Weyl character formula. When combined with the zero temperature limit ($t \to 0$), this could shed some new light on the structure of the ground state of the Heisenberg quantum ferromagnet [LM1].

4. Lagrangians

In this section we calculate the Lagrangians entering the (formal) Feynman-Kac formula for the partition function of the generalized Heisenberg quantum ferromagnet. As a preparation we first determine the Lagrangian for the one lattice point theory. It will turn out to be convenient to work in real coordinates (see Sect. 2).

Our first goal is to find the Lagrangian for the Feynman-Kac formula of

$$\mathrm{Trace}_{L^2(\mathscr{L}^\lambda)} e^{-t\{\frac{1}{m}\bar{\square} - i d\pi^\lambda(h)\}}, \tag{4.1}$$

where $h \in \mathbf{g}$. We recall that on $L^2(\mathscr{L}^\lambda) \subseteq L^2(\bar{\Lambda}(\mathscr{L}^\lambda))$ by (2.42)

$$\bar{\square}_{|L^2(\mathscr{L}^\lambda)} = -\tfrac{1}{2}(\nabla_0^\lambda)^2 + \tfrac{1}{2}\,\mathrm{tr}\, S. \tag{4.2}$$

To exhibit the fact that ∇_0^λ is a hermitian connection in $\mathscr{L}^\lambda$, we choose the following local gauge on $\mathscr{L}^\lambda$: Given a local holomorphic nonvanishing section s over $V \subset M_\lambda$ as discussed in Sect. 2, we set

$$s_0(z) = \|s(z)\|^{-1} s(z) \tag{4.3}$$

such that s_0 is fibrewise normalized. Then any other section φ may be written locally as $\varphi = s_0 f$, where f is a C^∞ function on $V \subset M_\lambda$. With the help of s_0 a real valued local 1-form is defined by

$$A(z) = -i\langle s_0(z) | ds_0(z)\rangle = \sum_j A_j(z)\, dx^j, \tag{4.4}$$

where the scalar product is taken in $\mathscr{H}_\lambda$. Now a short calculation shows that

$$-\frac{1}{2m}(\nabla_0^\lambda)^2 - i\, d\pi_0^\lambda(h) + \frac{1}{2}\,\mathrm{tr}\, S$$

applied to φ corresponds to

$$-\frac{1}{2m}(\nabla+iA)(d+iA)-ih+\frac{1}{i}K_h+(A,K_h)+\frac{1}{2}\operatorname{tr}S \tag{4.5}$$

applied to f. Here the first term in (4.5) is of course

$$-\frac{1}{2m\sqrt{g}}(\partial_j+iA_j)\sqrt{g}g^{jk}(\partial_k+iA_k), \tag{4.6}$$

and (A,K_h) denotes the canonical pairing of the 1-form A with the Killing vector field K_h. As in relations (2.19) and (2.24), in the second term of (4.5) we regard $h\in\mathbf{g}$ as a C^∞ function on M_λ.

Classically, (4.5) corresponds to the Hamiltonian

$$\begin{aligned} H^{\mathrm{cl}}(p+A,z)&=\frac{1}{2m}(p+A(z))^2-ih(z)+(p+A(z),K_h(z))+\frac{1}{2}\operatorname{tr}S(z)\\ &=H^{0,\mathrm{cl}}(p+A,z)+\frac{1}{2}\operatorname{tr}S(z)\end{aligned} \tag{4.7}$$

for $p\in T_z^*M_\lambda$. Again p^2 stands for the length squared of p with respect to the metric on M_λ. Applying a Legrende transformation, (4.7) leads to the following classical Lagrange function:

$$L^{\mathrm{cl}}(\dot z,z)=\frac{m}{2}\dot z^2+\frac{m}{2}(K_h^*,K_h)(z)+ih(z)-(A(z)+mK_h^*(z),\dot z)-\frac{1}{2}\operatorname{tr}S(z). \tag{4.8}$$

Here $\dot z\in T_zM_\lambda$ and K_h^* is the 1-form dual to K_h with respect to the metric on M_λ. Furthermore, after the substitution

$$\dot z\to -i\dot z \tag{4.9}$$

in L^{cl} describing the Wick rotation, we obtain the Euclidean Lagrange function

$$L(\dot z,z)=-\frac{m}{2}\dot z^2+\frac{m}{2}(K_h^*,K_h)(z)+ih(z)+i(A(z)+mK_h^*(z),\dot z)-\frac{1}{2}\operatorname{tr}S(z). \tag{4.10}$$

This gives the Feynman-Kac integral representation of (4.1),

$$\int_{z(0)=z(t)}\exp\left\{\int_0^t L(\dot z(s),z(s))ds\right\}\prod_{0\leq s<t}d\mu_{M_\lambda}(z(s)) \tag{4.11}$$

on the space of "all loops $z(\cdot)$ on M_λ".

The Lagrange function needed for a Feynman-Kac formula for

$$\operatorname{Trace}_{L^2(\bar A(\mathscr{L}^\lambda))}\{\varepsilon e^{-t\{\frac{1}{m}\bar\Box-id\pi^\lambda(\mathrm{h})\}}\} \tag{4.12}$$

is now obtained with help of the following three observations.

First we have to employ the Weitzenböck type formula (2.32). Secondly, the free Lagrange function for fermions is well known to be given formally by

$$L_0(\psi^*(s),\psi(s))=\sum_a\psi^{*a}(s)\dot\psi_a(s). \tag{4.13}$$

Here $\psi^*(s)$ and $\psi(s)$ are time dependent Fermi fields with

$$\dot{\psi}_a(s) = \frac{d}{ds}\,\psi_a(s)$$

(see e.g. [OK] for a detailed discussion). Since we are taking a supertrace in (4.12), we have to impose periodic boundary conditions for the fermionic field.

Thirdly, in a local gauge the connection ∇^λ on $\bar{A}(\mathscr{L}^\lambda)$ is obtained from the connection ∇_0^λ on $\mathscr{L}^\lambda \subset \bar{A}(\mathscr{L}^\lambda)$ by the simple replacement

$$iA \to iA + C, \tag{4.14}$$

(cf. (2.43)–(2.48)). In addition, the relation (2.31) leads to the substitution rule

$$h \to h + E_h. \tag{4.15}$$

Combining the preceding observations, we arrive at the following Euclidean Lagrange function:

$$\begin{aligned} L(\psi^*, \psi, \dot{z}, z) = &-\frac{m}{2}\,\dot{z}^2 + L_0(\psi^*, \psi) + \frac{m}{2}\,(K_h^*, K_h)(z) \\ &+ (iA(z) + C(\psi^*, \psi, z) + imK_h^*(z), \dot{z}) + ih(z) - \frac{1}{2}\,\mathrm{tr}\,S(z) \\ &+ \frac{1}{2}\,D\tilde{K}(\psi^*, \psi, z) - DS(\psi^*, \psi, z) + iE_h(\psi^*, \psi, z)\,. \end{aligned} \tag{4.16}$$

Note that in (4.16) the term $(C(\psi^*, \psi, z), \dot{z})$ represents a typical Yukawa coupling, while $E_h(\psi^*, \psi, z)$ describes a coupling of the Fermi fields to the external magnetic field h. We also recall that $D\tilde{K}$ and DS are quadratic in the Fermi fields.

Consequently, (4.12) is given as a Feynman-Kac formula of the form

$$\int\limits_{\substack{z(0)=z(t)\\ \psi^*(0)=\psi^*(t)\\ \psi(0)=\psi(t)}} \mathscr{P}\exp\left\{\int_0^t L(\psi^*, \psi, \dot{z}, z)\,ds\right\} \prod_{0 \leq s < t} (d\mu_{M_\lambda}(z(s))\,\Pi_a\, d\psi^{*a}(s)\, d\psi_a(s))\,, \tag{4.17}$$

where the fermionic integrations are carried out in the sense of Berezin [Bez] and $\mathscr{P}$ denotes path ordering. Note that the term $i(A(z), \dot{z})$ in the Lagrangian leads to the familiar gauge contribution $i\int_0^t A(z(s))ds$ in (4.17); in the particular case of $G = SU(2)$ this becomes i spin × (Area enclosed by the curve $z(\cdot)$ on S^2). Expressed in terms of the projections $Q_\lambda(z)$ of Sect. 2, it is easily recognized as an integrated Berry phase [Ber] and may be interpreted as a Wess-Zumino term (see the discussion in e.g. [FS]).

Since the fermions appear only quadratically, the fermion integration may be carried out formally in the usual way. This results in an effective Lagrangian for the purely "bosonic mode" $\dot{z}, z$. Moreover, we remark that the terms in the Lagrangian containing h destroy the supersymmetry. In fact, for $h = 0$, as a consequence of the Bott-Borel-Weil theorem, (4.17) is indeed a supersymmetric index equal to the dimension d_λ of $\mathscr{H}_\lambda$. A precise discussion of (4.17) will be given in Sect. 5.

We turn now to the case with interaction. Here, some more notation is required. Recall that by $\mathbf{z}=\{z(k)\}_{k\in\Lambda}$ we denote an arbitrary point in $M_{\lambda,\Lambda}=\times_{k\in\Lambda}M_\lambda$. Let $\mu=(m,k)$, $\nu=(n,k'),\dots$ with $1\leqq m,n\leqq\dim_{\mathbb{R}}M_\lambda$ and $k,k'\in\Lambda$; let $x^m(k)$ be local real coordinates for $z(k)\in V_k$, where V_k may vary with k (cf. Sect. 2). We define $x^\mu=x^m(k)$ to be real coordinates for $\mathbf{z}$ in $V_\Lambda=\times_{k\in\Lambda}V_k\subset M_{\lambda,\Lambda}$.

In analogy to the one lattice point theory the operator

$$\left(\frac{1}{m}\bar{\Box}_\Lambda+\mathrm{H}_\Lambda^\lambda\right)_{|C^\infty(\mathscr{L}^\lambda_{\Lambda|V_\Lambda})} \tag{4.18}$$

corresponds to the following operator on $C^\infty(V_\Lambda)$:

$$\begin{aligned}&-\frac{1}{2m}(\nabla_\Lambda+i\mathbf{A})(d+i\mathbf{A})\\&-J\sum_{k,k'\in\Lambda}\sum_{\alpha\beta}(Y^k_\alpha+i(\mathbf{A},K_{Y^k_\alpha})+K_{Y^k_\alpha})(Y^{k'}_\alpha+i(\mathbf{A},K_{Y^{k'}_\alpha})+K_{Y^{k'}_\alpha})c^{\alpha\beta}(k,k')\\&-i\mathbf{h}+\frac{1}{i}K_{\mathbf{h}}+\frac{1}{2}\operatorname{tr}\mathbf{S}_\Lambda.\end{aligned} \tag{4.19}$$

Here

$$\mathbf{A}(\mathbf{z})=\sum_\mu A_\mu(\mathbf{z})dx^\mu$$

is the local real 1-form corresponding to the local (product) gauge over V_Λ and $(\mathbf{A},K_{\mathrm{Y}})$ is again the canonical pairing of the 1-form $\mathbf{A}$ with the Killing vector field K_{Y}, where $\mathbf{Y}\in\mathbf{g}_\Lambda$. Also we set

$$\operatorname{tr}\mathbf{S}_\Lambda(\mathbf{z})=\sum_{k\in\Lambda}\operatorname{tr}S(z(k))$$

and ∇_Λ denotes the Levi-Civita connection on $M_{\lambda,\Lambda}$.

The operator (4.18) corresponds to the following classical Hamiltonian on $T^*M_{\lambda,\Lambda}$:

$$\begin{aligned}H^{\mathrm{cl}}(\mathbf{p}+\mathbf{A},\mathbf{z})=&\frac{1}{2m}(\mathbf{p}+\mathbf{A}(\mathbf{z}))^2\\&+J\sum_{\substack{k,k'\\k\neq k'}}\sum_{\alpha\beta}(-iY^k_\alpha(\mathbf{z})+(\mathbf{p}+\mathbf{A}(\mathbf{z}),K_{Y^k_\alpha}(\mathbf{z})))(-iY^{k'}_\beta(\mathbf{z})\\&+(\mathbf{p}+\mathbf{A}(\mathbf{z}),K_{Y^{k'}_\beta}(\mathbf{z})))c^{\alpha\beta}(k,k')\\&-i\mathbf{h}(z)+(\mathbf{p}+\mathbf{A}(\mathbf{z}),K_{\mathbf{h}}(z))+\frac{1}{2}\operatorname{tr}\mathbf{S}_\Lambda(\mathbf{z})=H^{0,\mathrm{cl}}(\mathbf{p}+\mathbf{A},\mathbf{z})+\frac{1}{2}\operatorname{tr}\mathbf{S}_\Lambda(\mathbf{z}),\end{aligned} \tag{4.20}$$

where $\mathbf{p}\in T^*_{\mathbf{z}}M_{\lambda,\Lambda}$ and $\mathbf{p}^2$ is the norm of $\mathbf{p}$ with respect to the metric tensor $g_\Lambda(\mathbf{z})=\bigotimes_{k\in\Lambda}g$ on $M_{\lambda,\Lambda}$ at $\mathbf{z}$.

To determine the resulting classical Lagrangian via a Legendre transformation we have to calculate

$$\dot{\mathbf{z}}=\frac{\partial H^{\mathrm{cl}}}{\partial\mathbf{p}}. \tag{4.21}$$

Let $\mathbf{p} \to \mathbf{p}_*$ be the canonical isomorphism from $T_{\mathbf{z}}^* M_{\lambda,\Lambda}$ onto $T_{\mathbf{z}} M_{\lambda,\Lambda}$ induced by the metric g_Λ. Then (4.20) gives

$$\dot{\mathbf{z}} = \frac{1}{m}(\mathbf{p}_* + \mathbf{A}_*(\mathbf{z})) + \frac{1}{m} D(\mathbf{z})(\mathbf{p}_* + \mathbf{A}_*(\mathbf{z})) + \dot{\mathbf{z}}_0 + K_{\mathbf{h}}(\mathbf{z}) \tag{4.22}$$

with $\dot{\mathbf{z}}_0 \in T_{\mathbf{z}} M_{\lambda,\Lambda}$ being given by

$$\dot{\mathbf{z}}_0 = -iJ \sum_{\substack{k,k' \\ k \neq k'}} \sum_{\alpha\beta} (Y_\beta^{k'}(\mathbf{z}) K_{Y_\alpha^k}(\mathbf{z}) + Y_\alpha^k(\mathbf{z}) K_{Y_\beta^{k'}}(\mathbf{z})) c^{\alpha\beta}(k,k'), \tag{4.23}$$

and where the linear transformation

$$D(\mathbf{z}) : T_{\mathbf{z}} M_{\lambda,\Lambda} \to T_{\mathbf{z}} M_{\lambda,\Lambda}$$

is defined as

$$D(\mathbf{z})\mathbf{p}_* = m \sum_{\substack{k,k' \\ k \neq k'}} \sum_{\alpha\beta} ((\mathbf{p}, K_{Y_\beta^{k'}}(\mathbf{z})) K_{Y_\alpha^k}(\mathbf{z}) + (\mathbf{p}, K_{Y_\alpha^k}(\mathbf{z})) K_{Y_\beta^{k'}}(\mathbf{z})) c^{\alpha\beta}(k,k'). \tag{4.24}$$

As in Sect. 3 we will assume $m|J|$ to be sufficiently small such that $(1 + D(z))$ is invertible on $T_{\mathbf{z}} M_{\lambda,\Lambda}$ for all $\mathbf{z} \in M_{\lambda,\Lambda}$.

Then, after some straightforward calculations the following classical Lagrangian results:

$$L^{\mathrm{cl}}(\dot{\mathbf{z}}, \mathbf{z}) = \frac{m}{2}((1 + D(\mathbf{z}))^{-1}(\dot{\mathbf{z}} - \dot{\mathbf{z}}_0 - K_{\mathbf{h}}(\mathbf{z})), \dot{\mathbf{z}} - \dot{\mathbf{z}}_0 - K_{\mathbf{h}}(\mathbf{z})) - (\mathbf{A}(\mathbf{z}), \dot{\mathbf{z}}) + i\mathbf{h}(\mathbf{z})$$

$$-\frac{1}{2} \operatorname{tr} \mathbf{S}_\Lambda(z) + J \sum_{\substack{k,k' \\ k \neq k'}} \sum_{\alpha\beta} Y_\alpha^k(\mathbf{z}) Y_\beta^{k'}(\mathbf{z}) c^{\alpha\beta}(k,k'). \tag{4.25}$$

To reveal the structure of this L^{cl} more clearly, we introduce a new metric $\hat{g}$ on $M_{\lambda,\Lambda}$ by

$$\hat{g}(\mathbf{z}) = g_\Lambda(\mathbf{z})(1 + D(\mathbf{z}))^{-1}. \tag{4.26}$$

Then

$$(\dot{\mathbf{z}}, \dot{\mathbf{z}})^\Lambda = ((1 + D(\mathbf{z}))^{-1} \dot{\mathbf{z}}, \dot{\mathbf{z}}) \tag{4.27}$$

is just the length squared of $\dot{\mathbf{z}} \in T_{\mathbf{z}} M_{\lambda,\Lambda}$ with respect to this new metric. Note that $\hat{g}$ is invariantly defined. In other words, if in a given coordinatization we write

$$(D(z)\mathbf{p}_*)^\mu = \sum_\nu D^{\mu\nu}(z) p_\nu, \tag{4.28}$$

then

$$\hat{g}^{\mu\nu}(z) = g_\Lambda^{\mu\nu}(z) + D^{\mu\nu}(z). \tag{4.29}$$

Repeating the one lattice point discussion we now derive the Lagrange function including fermions in the following way. Let

$$L_0(\psi^*, \psi) = \sum_{k \in \Lambda} \sum_a \psi^{*a}(k) \dot{\psi}_a(k) \tag{4.30}$$

be the free fermionic Lagrangian where $\psi^*(k)$ and $\psi(k)$ are the Fermi fields for the lattice side $k \in \Lambda$ as introduced in Sect. 3. A local 1-form $\mathbf{C}$ on $M_{\lambda,\Lambda}$, bilinear in ψ^*

and ψ, is defined by

$$\mathbf{C}(\mathbf{z})=\sum_{k\in\Lambda} C(z(k))=\sum_{\mu=(m,k)} C_m(\psi^*(k),\psi(k),z(k))dx^\mu, \tag{4.31}$$

where the 1-form $C(z(k))$ of (2.43) is now supposed to act on the k^{th} variable. With a similar convention

$$\begin{aligned} D\tilde{\mathbf{K}}(\mathbf{z})&=\sum_{k\in\Lambda} D\tilde{K}(z(k)),\\ D\mathbf{S}(\mathbf{z})&=\sum_{k\in\Lambda} DS(z(k)),\\ \mathbf{E}_{\mathbf{h}}(\mathbf{z})&=\sum_{k\in\Lambda} E_{h(k)}(z(k)). \end{aligned} \tag{4.32}$$

From Sect. 3 it is clear that all these terms are quadratic in the fermionic fields. Furthermore, in analogy to the prescription (4.15) we apply the rule

$$Y^k_\alpha(\mathbf{z})=Y_\alpha(z(k))\to Y_\alpha(z(k))+E_{Y_\alpha}(z(k))=Y^k_\alpha(\mathbf{z})+\mathbf{E}_{Y^k_\alpha}(z). \tag{4.33}$$

In particular this implies the substitution rule

$$\dot{\mathbf{z}}_0\to\dot{\mathbf{w}}_0 \tag{4.34}$$

with

$$\begin{aligned} \dot{\mathbf{w}}_0=-iJ\sum_{\substack{k,k'\\k\neq k'}}\sum_{\alpha\beta}\{&(Y^{k'}_\beta(\mathbf{z})+\mathbf{E}_{Y^{k'}_\beta}(\mathbf{z}))K_{Y^k_\alpha}(\mathbf{z})\\ &+(Y^k_\alpha(\mathbf{z})+\mathbf{E}_{Y^k_\alpha}(\mathbf{z}))K_{Y^{k'}_\beta}(\mathbf{z})\}c^{\alpha\beta}(k,k'). \end{aligned} \tag{4.35}$$

These relations combined with the substitution $i\mathbf{A}\to i\mathbf{A}+\mathbf{C}$ finally lead to the Euclidean Lagrangian

$$\begin{aligned} L(\psi^*,\psi,\dot{\mathbf{z}},\mathbf{z})=&-\frac{m}{2}(\dot{\mathbf{z}},\dot{\mathbf{z}})^\Lambda+L_0(\psi^*,\psi)+im(\dot{\mathbf{w}}_0+K_{\mathbf{h}},\dot{\mathbf{z}})^\Lambda-\frac{1}{2}\operatorname{tr}\mathbf{S}_\Lambda(\mathbf{z})\\ &+\frac{m}{2}(\dot{\mathbf{w}}_0+K_{\mathbf{h}},\dot{\mathbf{w}}_0+K_{\mathbf{h}})^\Lambda+(i\mathbf{A}+\mathbf{C},\dot{\mathbf{z}})+i\mathbf{h}(\mathbf{z})+zi\mathbf{E}_{\mathbf{h}}(\mathbf{z})\\ &+\frac{1}{2}D\tilde{\mathbf{K}}(\mathbf{z})-D\mathbf{S}(\mathbf{z})\\ &+J\sum_{\substack{k,k'\\k\neq k'}}\sum_{\alpha\beta}(Y^k_\alpha(\mathbf{z})+\mathbf{E}_{Y^k_\alpha}(\mathbf{z}))(Y^{k'}_\beta(\mathbf{z})+\mathbf{E}_{Y^{k'}_\beta}(\mathbf{z}))c^{\alpha\beta}(k,k'). \end{aligned} \tag{4.36}$$

The following remarks are in order. First, the interaction described in terms of this Lagrangian is nonlocal, even if the interaction in the original Hamiltonian is local, say if $c^{\alpha\beta}(k,k')=0$ unless k and k' are nearest neighbours. This is an effect of the metric $\hat{g}$ being such that in general $\hat{g}_{\mu\nu}$ does not vanish for arbitrary μ, ν, i.e. the coupling is not only between nearest neighbours. If one expands $\hat{g}_{\mu\nu}$ in a power series in J, then the first order terms will couple nearest neighbours, the second order terms will couple next nearest neighbours, and so on. However, in terms of the

Hamiltonian

$$\frac{1}{m}\bar{\square}+H_\Lambda^\lambda \tag{4.37}$$

we have a local theory in the sense that only nearest neighbours are coupled. The reason for this somewhat unfamiliar picture is that the corresponding classical Hamiltonian couples nearest neighbour momenta. The situation here is to a certain extent opposite to the one in the Ising model when viewed as a Euclidean lattice field theory. There the Euclidean Lagrangian, viz. the negative of the ordinary Ising Hamiltonian, is local whereas the resulting quantum mechanical Hamiltonian, defined as (minus) the logarithm of the transfer matrix, turns out to be nonlocal. In combination with relation (4.37) it would be interesting to see whether the term $i(A,\dot{z})$ in (4.35) or the nonlocality of the Lagrangian is responsible for the known violation of spatial reflection positivity for the quantum Heisenberg model [Sp].

To enhance our understanding of what is happening in this context, let us consider the first term in (4.36). The classical equations of motion for the Lagrangian $-(m/2)(\dot{\mathbf{z}},\dot{\mathbf{z}})^A$ are of course just the geodesic equations with respect to the metric $\hat{g}$. When $J \neq 0$, its Christoffel symbols are such that the equation for the acceleration $\ddot{z}(k)$ involes the velocities $\dot{z}(k')$ for all other $k' \in \Lambda$. Thus the equations of motion are nonlocal. This contrasts with the equations of motion for the Lagrangian $-(m/2)(\dot{\mathbf{z}},\dot{\mathbf{z}})$ in the metric g_Λ. There each $z(k)$ performs a geodesic motion on M_λ independent of the other $z(k')$.

In Sect. 6 we shall give a rigorous proof of a Feynman-Kac formula for the Hamiltonian (4.37) by constructing a horizontal stochastic motion on the bundle $\bar{\Lambda}(\mathscr{L}_\Lambda^\lambda)$ whose projection on $M_{\lambda,\Lambda}$ is a Brownian motion with infinitesimal generator given essentially (i.e. up to a drift term) by the Laplace-Beltrami operator for the metric $\hat{g}$.

Note also that we have local fermion number conservation for the Lagrangian (4.36); fermions do not propagate through the lattice. This is of course a consequence of the fact that the second order differential operator (4.37) on $C^\infty(\bar{\Lambda}(\mathscr{L}_\Lambda^\lambda))$ preserves the local order of the form [where the "local order" of a form at $k\in\Lambda$ is given by the "number of $d\bar{z}^a(k)$ involved"]. Nevertheless, (4.36) contains 4-fermion interaction terms of the Luttinger type $\psi^*(k)\psi(k)\psi^*(k')\psi(k')$ with $k\neq k'$, $k, k'\in\Lambda$ which are also nonlocal. This follows from the fact that the $\mathbf{E}$ and $\dot{\mathbf{w}}_0$ include terms quadratic in the Fermi fields. Besides these interactions we have again Yukawa type couplings as well as couplings of both the bosonic and fermionic modes to the magnetic field $\mathbf{h}$.

The Lagrangian (4.36) gives the Feynman-Kac formula

$$\begin{aligned}&\mathrm{Trace}_{L^2(\bar{\Lambda}(\mathscr{L}_\Lambda^\lambda))}\left\{\varepsilon\exp\left\{-t\left(\frac{1}{m}\bar{\square}+H_\Lambda^\lambda\right)\right\}\right\}\\&=\int_{\mathrm{PBC}}\mathscr{P}\exp\left\{\int_0^t L(\psi^*,\psi,\dot{\mathbf{z}},\mathbf{z})(s)\,ds\right\}\prod_{0\leqq s<t}(d\mu_{M_{\lambda,\Lambda}}(\mathbf{z}(s))\Pi_{a,k}\psi^{*a}(k)(s)\psi_a(k)(s)),\end{aligned} \tag{4.38}$$

where PBC stands for periodic boundary conditions in time: $\mathbf{z}(0)=\mathbf{z}(t)$, $\psi^*(0)=\psi^*(t)$, and $\psi(0)=\psi(t)$. Combining this with (3.8), a Feynman-Kac formula for the partition function of the generalized Heisenberg model results.

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