Hydrodynamic Limit for a System with Finite Range Interactions

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Abstract. We study a system of interacting diffusions. The variables present the amount of charge at various sites of a periodic multidimensional lattice. The equilibrium states of the diffusion are canonical Gibbs measures of a given finite range interaction. Under an appropriate scaling of lattice spacing and time, we derive the hydrodynamic limit for the evolution of the macroscopic charge density.

1. Introduction

The derivation of the hydrodynamic equation for infinite particle systems with conservation law has been the subject of active research. One such model is the *Ginzburg–Landau* model [11]. The hydrodynamic equation for this model is obtained in [3] and [4]. In this model charges are located at the various sites of a periodic multidimensional lattice. The flow of these charges from one site to another is governed by a suitable diffusion law. After an appropriate space and time scaling, the microscopic charge density converges to a deterministic limit which is characterized as the solution of a nonlinear parabolic equation.

The passage to the hydrodynamic limit for the Ginzburg–Landau model under certain conditions was studied in [4]. We describe these conditions.

For any positive integer N, let S_N denote the periodic lattice $\{j: j = 0, 1, \ldots, N\}$ with 0 and N identified, and let S_N^d denote the product of d copies of S_N. For each site a in S_N^d, there is a random variable $x_a = x_a(t)$ which is the amount of charge at site a. The family of x_a undergo a diffusion with generator

$$\mathscr{L}_N^0 = \frac{N^2}{2}\left[\sum\left(\frac{\partial}{\partial x_a} - \frac{\partial}{\partial x_b}\right)^2 - \sum(\phi'(x_a) - \phi'(x_b))\left(\frac{\partial}{\partial x_a} - \frac{\partial}{\partial x_b}\right) \right], \qquad (1.1)$$

where both sums are over the adjacent sites a and b in S_N^d. The generator $\mathscr{L}_N^0$ is formally symmetric with respect to the product measure $\rho_N(d\underline{x})$ defined as

$$\rho_N(d\underline{x}) = \prod_{a \in S_N^d} e^{-\phi(x_a)} dx_a, \qquad (1.2)$$

where $\underline{x}$ denotes the vector $(x_a; a \in S_N^d)$. The function $\phi: \mathbf{R} \to \mathbf{R}$ is a continuously differentiable function with the following properties.

$$\int e^{-\phi(x)} dx = 1, \tag{1.3}$$

$$\int e^{\lambda x - \phi(x)} dx < \infty, \tag{1.4}$$

$$\int e^{\lambda |\phi'(x)| - \phi(x)} dx < \infty \tag{1.5}$$

for all λ in $\mathbf{R}$.

The diffusion generated by $\mathscr{L}_N^0$ starts from an initial distribution which has a density f_N^0 with respect to the measure ρ_N and f_N^0 satisfies the *entropy bound*

$$\frac{1}{N^d} \int f_N^0 \log f_N^0 d\rho_N \leqq C \tag{1.6}$$

uniformly in N and for some finite constant C. Then the evolution corresponding to the diffusion will give us a density f_N^t which satisfies the *forward equation*,

$$\frac{\partial f_N^t}{\partial t} = \mathscr{L}_N^0 f_N^t. \tag{1.7}$$

Associated with the charge configuration $\underline{x} \in \mathbf{R}^{Nd}$ and $t \geqq 0$, we define the signed measure

$$\mu_N(t) = \frac{1}{N^d} \sum_{a \in S_N^d} x_a(t) \delta_{a/N}. \tag{1.8}$$

Let S be the unit circle or the interval $0 \leqq \theta \leqq 1$ with 0 and 1 identified, and let S^d be the product of d copies of S. Then $\mu_N(t) = \mu_N(t, d\theta)$ is a measure on the d-dimensional torus S^d.

We assume that for some nice function $m_0(\theta)$ and any positive δ

$$\lim_{N \to \infty} \int_{E_{N,\delta}} f_N^0 d\rho_N = 0, \tag{1.9}$$

where

$$E_{N,\delta} = \left\{ \underline{x} : \left| \frac{1}{N^d} \sum J(a/N) x_a - \int J(\theta) m_0(\theta) d\theta \right| \geqq \delta \right\}.$$

Here $d\theta$ denotes the normalized Haar measure on S^d. It is shown in [4] that there exists a function $m(t, \theta)$ such that

$$\lim_{N \to \infty} \int J(\theta) \mu_N(t, d\theta) = \int J(\theta) m(t, \theta) d\theta, \tag{1.10}$$

and this function satisfies a nonlinear parabolic equation which will be described below (see (1.13)).

We generalize the above result by adding an *interaction* term to the function $-\phi(x)$ in the measure ρ_N. This means that we replace the product measure ρ_N with a Gibbs measure which has an interaction of finite range. The important aspect of our work is that we do not exclude the occurrence of phase transition in the space of Gibbs measures.

Let F be a function which depends on a fixed finite number of coordinates. We assume that F is bounded, continuously differentiable and has bounded first derivatives. The *interaction energy* corresponding to the *interaction field* F is defined by

$$\mathscr{H}_N(\underline{x}) = \sum_{a \in S_N^d} F(\tau_a \underline{x}),$$

where τ_a denotes the shift operator in the space S_N^d. We now replace the generator $\mathscr{L}_N^0$ with

$$\mathscr{L}_N = \frac{N^2}{2} \left[\sum \left(\frac{\partial}{\partial x_a} - \frac{\partial}{\partial x_b} \right)^2 \right.$$
$$\left. - \sum \left(\phi'(x_a) - \phi'(x_b) + \frac{\partial \mathscr{H}_N}{\partial x_b}(\underline{x}) - \frac{\partial \mathscr{H}_N}{\partial x_a}(\underline{x}) \right) \left(\frac{\partial}{\partial x_a} - \frac{\partial}{\partial x_b} \right) \right]. \qquad (1.11)$$

This generator is reversible with respect to the Gibbs measure v_N defined as

$$v_N(d\underline{x}) = \frac{1}{Z_N} \exp \left(\mathscr{H}_N(\underline{x}) \right) \rho_N(d\underline{x}), \qquad (1.12)$$

where Z_N is the normalizing constant.

We now assume that the diffusion generated by L_N starts from an initial distribution which has a density f_N^0 with respect to the measure v_N and f_N^0 satisfies the entropy bound

$$\frac{1}{N^d} \int f_N^0 \log f_N^0 \, dv_N \leqq C \qquad (1.6')$$

uniformly in N and for some finite constant C. Then the evolution corresponding to the diffusion will give us a density f_N^t which satisfies

$$\frac{\partial f_N^t}{\partial t} = \mathscr{L}_N f_N^t. \qquad (1.7')$$

We also assume that for some nice function $m_0(\theta)$ and any positive δ,

$$\lim_{N \to \infty} \int_{E_{N,\delta}^0} f_N^0 dv_N = 0, \qquad (1.9')$$

where

$$E_{N,\delta}^0 = \left\{ \underline{x} : \left| \frac{1}{N^d} \sum J(a/N) x_a - \int J(\theta) m_0(\theta) d\theta \right| \geqq \delta \right\}.$$

We define h as the *convex conjugate* of the *specific free energy* φ defined by

$$\varphi(\lambda) = \lim_{N \to \infty} \frac{1}{N^d} \log \int \exp \left(\lambda \sum_a x_a \right) v_N(d\underline{x}).$$

We now state our main result.

Theorem 1.1. *If the initial distribution of charges satisfies (1.6') and (1.9'), then for*

all time t, every smooth J and each $\delta > 0$,

$$\lim_{N \to \infty} \int_{E^t_{N,\delta}} f^t_N dv_N = 0,$$

where

$$E^t_{N,\delta} = \left\{ \underline{x} : \left| \frac{1}{N^d} \sum J(a/N) x_a - \int J(\theta) m(t,\theta) d\theta \right| \geq \delta \right\},$$

and $m(t, \theta)$ satisfies weakly the following nonlinear parabolic equation:

$$\frac{\partial m}{\partial t} = \tfrac{1}{2} \Delta h'(m(t,\theta)), \quad m(0,\theta) = m_0(\theta). \tag{1.13}$$

When $F = 0$, then h' is strictly increasing. It is well known in statistical mechanics that for some nonzero F, the function h' is not strictly increasing, which makes the evolution Eq. (1.13) degenerate. This corresponds to a *phase transition* in physics. Nonetheless we can still derive the hydrodynamic equation for the charge density m. We will also verify properties such as uniqueness of solutions, maximum principle and the asymptotic behavior of the solution $m(t, \theta)$ as time goes to infinity.

In the presence of phase transition, our result implies that the different phases do not segregate macroscopically at the hydrodynamic time scale. This is consistent with the Lifshitz–Slyozov theory. According to this theory, it takes times of the order N^3 for forming a macroscopic *droplet*. Therefore segregation cannot be seen on the hydrodynamic time scale.

Let J be a fixed smooth function and consider the stochastic process

$$y_N(t) = \int J(\theta) \mu_N(t, d\theta) = \frac{1}{N^d} \sum J(a/N) x_a(t).$$

It is not hard to see that

$$dy_N(t) = dM_N(t) + A_N(t)dt,$$

where $M_N(t)$ is a continuous martingale which goes to zero as N goes to infinity, and

$$A_N(t) \approx \frac{1}{2N^d} \sum \Delta J(a/N) G_0(\tau_a \underline{x}(t)),$$

$$G_0(\underline{x}) = \phi'(x_0) - \frac{\partial \mathcal{H}_N}{\partial x_0}(\underline{x}). \tag{1.14}$$

The main step in the proof of the hydrodynamic behavior for our example (like [4]) is that

$$\lim_{N \to \infty} \int_0^t A_N(s)ds = \frac{1}{2} \int_0^t \int \Delta J(\theta) h'(m(\theta,s))ds\,d\theta, \tag{1.15}$$

where m is the solution of (1.13).

When $F = 0$, then for any bounded continuous local function G (i.e. depending on a fixed finite number of coordinates) and for any smooth test function J, it is

shown in [4] that

$$\lim_{N \to \infty} \frac{1}{N^d} \sum_a J(a/N) \int_0^t G(\tau_a \underline{x}(s)) ds = \int_0^t \int J(\theta) B_G(m(s, \theta)) ds \, d\theta, \qquad (1.16)$$

where

$$B_G(y) = \int G(\underline{x}) \prod_a \frac{1}{M(\lambda)} e^{\lambda x_a - \phi(x_a)} dx_a,$$

$$M(\lambda) = \int e^{\lambda x - \phi(x)} dx$$

and

$$\lambda = h'(y).$$

In general, if $F \neq 0$ and if phase transition occurs, we do not expect to have (1.16) for any G. We can however prove (1.16) for $G = G_0$ and obtain (1.15). This is because of a very special form of G_0 which makes it possible to avoid any problem coming from phase transitions.

We expect that the distribution of the charge configuration $\underline{x}(t)$ for $t > 0$, looks somewhat like

$$c_N \exp\left(\sum_a \lambda_a x_a \right) dv_N,$$

where $\lambda_a \approx h'(m(t, a/N))$. As in [4], we use the entropy production as a technical tool to prove the validity of such a picture. This is the content of Sect. 3. Before this, we need to establish a *local ergodic theorem*. This is done in Sect. 2. In this section, we basically use the techniques of large deviations. Finally in Sect. 5 we derive the hydrodynamic equation and study some of its properties.

2. Large Deviations for Conditioned Gibbs Measures

The object of this section is a large deviations result (and its corollaries) for the conditioned Gibbs measures. First we start with some preliminary considerations related to theory of the large deviations. Let X be a Polish space with Borel field $\mathscr{B}$. A function $I: X \to [0, \infty]$ is called a *rate function* if it is lower semicontinuous and the set $\{x | I(x) \leq \ell\}$ is compact for all finite ℓ. A family P_n of probability measures on X is said to have the large deviation property with rate I if there exists a sequence a_n which tends to ∞ and

$$\limsup_{n \to \infty} \frac{1}{a_n} \log \mu_n(C) \leq - \inf_{x \in C} I(x) \qquad (2.1)$$

for all closed subsets C of X, and

$$\liminf_{n \to \infty} \frac{1}{a_n} \log \mu_n(O) \geq - \inf_{x \in O} I(x) \qquad (2.2)$$

for all open subsets O of X.

Our first lemma is a useful form of the so-called *Contraction Principle*.

Lemma 2.1. *Let* $\Phi: X_0 \to Y$, *where* X_0 *is a subset of* X *and* Y *is another Polish space. Suppose that for each positive number* ℓ *there exists a closed subset* C_ℓ *of* X *such that* Φ *restricted to* C_ℓ *is continuous and moreover*

$$\limsup_{\ell \to \infty} \limsup_{n \to \infty} \frac{1}{a_n} \log P_n(X - C_\ell) = -\infty.$$

Then the family $P_n \circ \Phi^{-1}$ *has the large deviation property with the rate* $\hat{I}(x) = \inf_{\Phi(x)=y} I(x)$.

Proof. Let $\hat{C}$ be any closed subset of Y. Then

$$P_n(\Phi \in \hat{C}) \leq P_n(\{\Phi \in \hat{C}\} \cap C_\ell) + P_n(X - C_\ell).$$

The set $C = \{\Phi \in \hat{C}\} \cap C_\ell$ is closed and

$$\limsup_{n \to \infty} \frac{1}{a_n} \log P_n(\{\Phi \in \hat{C}\} \cap C_\ell) \leq -\inf_{x \in C} I(x) \leq -\inf_{y \in \hat{C}} \hat{I}(y).$$

So, we have the upper bound estimate because $P_n(X - C_\ell)$ is superexponentially small when $l \to \infty$.

For the lower bound estimate, let $\hat{O}$ be an open subset of Y,

$$P_n(\Phi \in \hat{O}) \geq P_n(\{\Phi \in \hat{O}\} \cup (X - C_\ell)) - P_n(X - C_\ell).$$

The set $O = \{\Phi \in \hat{O}\} \cup (X - C_\ell)$ is open and

$$\liminf_{n \to \infty} \frac{1}{a_n} \log P_n(\{\Phi \in \hat{O}\} \cup (X - C_\ell)) \geq -\inf_{x \in O} I(x) \geq -\inf_{y \in \hat{O}} \hat{I}(y).$$

Moreover $P_n(X - C_\ell)$ is superexponentially small as $l \to \infty$.

It is not hard to check that $\hat{I}$ is a rate function. We omit the proof of it. $\square$

We recall the following result which can be found in [2].

Lemma 2.2. *Let* W_n *be a sequence of* $\mathbf{R}^d$-*valued random variables on a probability space* $(\Omega_n, \mathcal{F}_n, Q_n)$. *If*

$$c(t) = \lim_{n \to \infty} \frac{1}{a_n} \log E^{Q_n} e^{t \cdot W_n}$$

exists for all t *in* $\mathbf{R}^d$, *then the family*

$$P_n(dx) = Q_n\left(\frac{W_n}{a_n} \in dx\right)$$

has the upper bound large deviation property with the rate $I(t)$ *that is the convex conjugate of* $c(t)$.

Lemma 2.3. *Suppose* $\mathcal{A}$ *is a collection of closed subsets of* X *such that for any closed subset* C *of* X, *there exists a decreasing sequence* A_m *in* $\mathcal{A}$ *satisfying* $\bigcap_m A_m = C$. *If (2.1) holds for all* A *in* $\mathcal{A}$, *then it holds for all closed sets* C.

Proof. Suppose that (2.1) holds for all A in $\mathcal{A}$. Let C be a closed subset of X and

$C = \bigcap_m A_m$ for some decreasing subsequence A_m of $\mathscr{A}$. Then

$$\limsup_{n \to \infty} \frac{1}{a_n} \log P_n(C) \leq \inf_m \limsup_{n \to \infty} \log P_n(A_m) \leq -\sup_m \inf_{x \in A_m} I(x).$$

We will show

$$\inf_{x \in C} I(x) \leq \sup_m \inf_{x \in A_m} I(x).$$

Suppose that $\sup_m \inf_{x \in A_m} I(x)$ is finite. Set $K_l = \{x \mid I(x) \leq l\}$, where

$$l = \sup_m \inf_{x \in A_m} I(x) + 1.$$

Then $\inf_{x \in A_m} I(x) = \inf_{x \in A_m \cap K_l} I(x)$ and therefore there exists $x_m \in A_m \cap K_l$ such that

$$\inf_{x \in A_m} I(x) = I(x_m).$$

We may choose a subsequence of x_m which converges to some $y \in \bigcap_m A_m = C$. Therefore

$$\sup_m \inf_{x \in A_m} I(x) \geq \liminf_{m \to \infty} I(x_m) \geq I(y) \geq \inf_{x \in C} I(x). \quad \square$$

Before we state the main result of this section, let us recall the definition and some properties of Gibbs measures.

We consider the configuration space $X = \mathbf{R}^{\mathbf{Z}^d}$ which is the set of $\underline{x}: \mathbf{Z}^d \to \mathbf{R}$ or $\underline{x} = (x_a; a \in \mathbf{Z}^d)$. The set X is endowed with the product topology which makes X a Polish space. For a, let τ_a be the translation operator on X defined by $(\tau_a \underline{x})(b) = \underline{x}(a + b)$. $\mathscr{M}(X)$ denotes the space of probability measures and $\mathscr{M}_\tau(X)$ denotes the space of translation invariant probability measures on X. Both of these spaces are endowed with the topology of weak convergence.

For each $n \in \mathbf{Z}^+$, we define

$$T_n = \{a \in \mathbf{Z}^d \mid 0 \leq a_j \leq n \quad \text{for all} \quad j, \quad \text{where} \quad a = (a_j : 1 \leq j \leq d)\}.$$

For each $x \in X$ and $n \in \mathbf{Z}^+$, we define $\underline{x}^n$ by

$$\underline{x}^n(a) = \underline{x}(a) \, a \in T_n,$$

$$\underline{x}^n(a + (n + 1)\mathbf{e}^j) = \underline{x}^n(a); \, a \in \mathbf{Z}^d,$$

where $\mathbf{e}_k^j = \delta_{j,k}$. X_n denotes the set of all $\underline{x}^n$. The empirical process is defined as

$$R_{n,\underline{x}} = |T_n|^{-1} \sum_{a \in T_n} \delta_{\tau_a \underline{x}^n},$$

where $|T_n|$ is the number of sites in T_n. It is easy to see that $R_{n,\underline{x}} \in \mathscr{M}_\tau(X)$.

Let $\phi : \mathbf{R} \to \mathbf{R}$ be a continuously differentiable function satisfying

$$\int e^{-\phi(x)} dx = 1, \tag{2.3}$$

$$\int e^{\lambda x - \phi(x)} dx < \infty, \tag{2.4}$$

$$\int e^{\lambda |\phi'(x)| - \phi(x)} dx < \infty \tag{2.5}$$

for all real λ. Equations (2.4) and (2.5) are equivalent to saying that there exists a function $w(x)$ which is symmetric and convex on $\mathbf{R}$ with

$$\lim_{x \to \infty} \frac{|x|}{w(x)} = 0, \quad w(x) \geq |x| \quad \text{for all } x \text{ and}$$

$$\int e^{w(x) - \phi(x)} dx < \infty, \quad \int e^{w(\phi'(x)) - \phi(x)} dx < \infty. \tag{2.6}$$

We consider the following classes of local functions:

$C_{\text{loc}}^0 = \{G: X \to \mathbf{R}$ such that G is bounded continuous and depends on a fixed finite number of coordinates$\}$,

$C_{\text{loc}}^1 = \{G \in C_{\text{loc}}^0 : G$ is continuously differentiable with bounded first derivatives$\}$,

$C_{\text{loc}}^\infty = \{G \in C_{\text{loc}}^0 : G$ is smooth with bounded derivatives$\}$.

Let $\underline{x} \in X$ and T be a subset of $\mathbb{Z}^d$. Then $\underline{x}_T$ denotes the restriction of $\underline{x}$ to T. That is $\underline{x}_T : T \to \mathbf{R}$ and $\underline{x}_T(a) = \underline{x}(a)$ for $a \in T$. If $\underline{z}$ is another configuration, then $\underline{x}_T \vee \underline{z}$ denotes the configuration which agrees with $\underline{x}$ on T and with $\underline{z}$ on $T^c = \mathbb{Z}^d - T$. In other words $\underline{x}_T \vee \underline{z}$ is the configuration $\underline{x}$ on T with the boundary condition $\underline{z}$.

Let F be a function in C_{loc}^0 which depends only on the coordinates x_a for a in a finite set $\Lambda \subset \mathbb{Z}^d$. We define the *interaction energy* by

$$\mathscr{H}_{F,T}(\underline{x}) = \sum F(\tau_a \underline{x}),$$

where the sum is over indices a such that $a + \Lambda \subseteq T$. Given a boundary condition $\underline{z}$ in X, we define

$$\mathscr{H}_{F,T,\underline{z}}(\underline{x}) = \mathscr{H}_{F,T}(\underline{x}) + \sum F((\tau_a \underline{x})_T \vee \underline{z}),$$

where the sum is over a such that

$$(a + \Lambda) \cap T \neq \varnothing \quad \text{and} \quad (a + \Lambda) \cap T^c \neq \varnothing.$$

We also define the interaction energy for a configuration with periodic boundary conditions. For this, one considers the space of periodic configurations X_n and defines

$$\hat{\mathscr{H}}_{F,n}(\underline{x}) = \sum_{a \in S_n} F(\tau_a \underline{x}^n),$$

where S_n is obtained by identifying the opposite faces of T_n.

We take the probability measure $\rho(dx) = e^{-\phi(x)} dx$ and for each subset T of $\mathbb{Z}^d$, the measure ρ^T is the product measure of ρ which is defined on $\mathbf{R}^T$. Now we define the finite volume Gibbs measure $\nu_{F,T}$ associated with F and ρ by

$$\nu_{F,T}(d\underline{x}) = \frac{1}{Z_{F,T}} \exp(\mathscr{H}_{F,T}(\underline{x})) \rho^T(d\underline{x}), \tag{2.7}$$

where

$$Z_{F,T} = \int \exp(\mathscr{H}_{F,T}(\underline{x})) \rho^T(d\underline{x}). \tag{2.8}$$

In the same way we define $\nu_{F,T,\underline{z}}$, $Z_{F,T,\underline{z}}$, $\hat{\nu}_{F,n}$, and $\hat{Z}_{F,n}$ by replacing $\mathscr{H}_{F,T}$ in (2.7) and (2.8) with $\mathscr{H}_{F,T}$ and $\hat{\mathscr{H}}_{F,n}$. We also consider the finite volume canonical Gibbs

measures $v^y_{F,T}$ and $v^y_{F,T,\underline{z}}$ which are obtained by conditioning the measures $v_{F,T}$ and $v_{F,T,\underline{z}}$ respectively, with respect to $m_T(\underline{x}) = y$, where

$$m_T(\underline{x}) = \frac{1}{|T|} \sum_{a \in T} x_a. \tag{2.9}$$

We define $\hat{v}^y_{F,n}$ in the same way by choosing $T = S^d_n$ in (2.9).

Note that a calculation like (2.15) below ensures the existence of a version of the conditioned measure $v^y_{F,T}$ which is continuous in y.

The family of the translation invariant (infinite volume) Gibbs measures and canonical Gibbs measures are defined by

$$\mathscr{G}_F = \{\mu \in \mathscr{M}_\tau \,|\, \mu(\cdot \,|\, x_a = z_a \text{ for } a \notin T) = v_{F,T,\underline{z}} \text{ for all } \underline{z} \text{ and finite } T \subset \mathbb{Z}^d\},$$

$$\mathscr{G}^c_F = \{\mu \in \mathscr{M}_\tau \,|\, \mu(\cdot \,|\, x_a = z_a \text{ for } a \notin T \text{ and } m_T(\cdot) = y)$$
$$= v^y_{F,T,\underline{z}} \text{ for all } \underline{z} \text{ and finite } T \subset \mathbb{Z}^d\}.$$

We denote the distribution of $R_{n,\underline{x}}$ with respect to v_{F,T_n}, by $P_{F,n}$ or simply P_n when there is no danger of confusion. That is

$$P_n(A) = v_{F,T_n}(R_{n,\underline{x}} \in A)$$

for A a Borel subset of $\mathscr{M}_\tau$. The probability measures $P^y_{F,n,\underline{z}}$ and $P_{n,\underline{z}}$ are defined in the same way. It is known [9] that the family P_n has the large deviation property with the rate function I_F. In order to define $I_F(\mu)$ for $\mu \in \mathscr{M}_\tau$, we first define the *pressure* of an interaction F by

$$\Psi(F) = \lim_{n \to \infty} \frac{1}{|T_n|} \log Z_{F,T_n,\underline{z}}. \tag{2.10}$$

Note that this limit is uniform in $\underline{z}$, and $\Psi(F)$ is independent of $\underline{z}$.

The *specific entropy* of a measure μ in $\mathscr{M}_\tau$ with respect to ρ is defined as

$$H_\rho(\mu) = \lim_{n \to \infty} \frac{1}{|T_n|} \int \left(\log \frac{d\mu^{T_n}}{d\rho^{T_n}} \right) d\mu^{T_n},$$

where the measure μ^{T_n} is the projection of μ on the space X_{T_n}. The rate function I_F is defined by

$$I_F(\mu) = H_\rho(\mu) - \int F d\mu + \Psi(F), \tag{2.11}$$

or

$$I_F(\mu) = \lim_{n \to \infty} \frac{1}{|T_n|} \int f_n \log f_n \, dv_{F,T_n}, \tag{2.12}$$

where f_n is the density of the measure μ^{T_n} with respect to the Gibbs measure v_{F,T_n}.

If α and β are two probability measures such that

$$d\alpha = f \, d\beta$$

for some density $f \in L^1(d\beta)$, then

$$\int g \, d\alpha \leq \log \int e^g \, d\beta + \int \log f \, d\alpha$$

for any $g \in L^1(d\alpha)$. We will refer to this inequality as the *basic entropy estimate*.

The large deviation property of the sequence $R_{n,\underline{x}}$ with respect to the Gibbs measure ν_{F,T_n} also implies the large deviation property of the distributions of the sequence $m_{T_n}(\underline{x})$. For this we use the *Contraction Principle*.

Lemma 2.4. *Let $\underline{z}^n$ be an arbitrary sequence in X and $F \in C_{loc}^0$. Then the family $\alpha_n(dy) = \nu_{F,T_n,\underline{z}_n}(m_{T_n} \in dy)$ has the large deviation property with the rate $\hat{I}_F(y) = \inf_{\Phi(\mu)=y} I_F(\mu)$, where $\Phi(\mu) = \int x_0 \, d\mu$ for those measures μ such that $\int |x_0| \, d\mu$ is finite.*

Proof. Set $C_l = \{\mu \mid \int w(x_0)\mu(d\underline{x}) \leqq l\}$. Note that the set C_l is closed in X and Φ restricted to C_l is continuous. Now we can apply Lemma 2.1. For this we need to check $P_n(X - C_l)$ is superexponentially small as $l \to \infty$. For notational convenience we denote $\nu_{F,T_n,\underline{z}}$ by ν_n. Using the Tchebyshev inequality,

$$P_n(X - C_l) = \nu_n \left\{ \underline{x} : \frac{1}{|T_n|} \sum_{a \in T_n} w(x_a) > l \right\} \leqq e^{-l|T_n|} E^{\nu_n} \exp\left(\sum_{a \in T_n} w(x_a) \right).$$

On the other hand

$$\limsup_{n \to \infty} \frac{1}{|T_n|} \log E^{\nu_n} \exp\left(\sum_{a \in T_n} w(x_a) \right)$$

$$= \limsup_{n \to \infty} \frac{1}{|T_n|} \log \int \exp\left(\sum_{a \in T_n} (w(x_a) - \phi(x_a)) \right) \exp\left(\sum_{a \in T_n} F(\tau_a \underline{x}) \right) dx - \Psi(F)$$

$$\leqq \|F\|_\infty + \log \int e^{w(x) - \phi(x)} d(x) - \Psi(F) < \infty.$$

Thus

$$\limsup_{l \to \infty} \limsup_{n \to \infty} \frac{1}{|T_n|} \log P_n(X - C_l) = -\infty,$$

and this completes the proof. □

Once we establish the large deviation property of the sequence $m_{T_n}(\underline{x})$ with respect to Gibbs measures, it is not hard to identify the rate function as the convex conjugate h of the *free energy* φ defined by

$$\varphi(\lambda) = \varphi(F, \lambda) = \limsup_{n \to \infty} \frac{1}{|T_n|} \log \int \exp\left(\lambda \sum_{a \in T_n} x_a \right) \nu_{F,T_n,\underline{z}_n}(d\underline{x}). \tag{2.13}$$

Lemma 2.5.

$$\hat{I}_F(y) := h(y) = \sup_y (\lambda y - \varphi(\lambda)). \tag{2.14}$$

Proof. It is enough to show that

$$\varphi(\lambda) = \sup_y (\lambda y - \hat{I}_F(y)).$$

We only verify

$$\varphi(\lambda) \leqq \sup_y (\lambda y - \hat{I}_F(y)).$$

We omit the easy proof of

$$\varphi(\lambda) \geq \sup_{y} (\lambda y - \hat{I}_F(y))$$

which follows from the lower bound large deviation estimate.

Let C_l and v_n be as in Lemma 2.4. Since the function Φ (defined by $\Phi(\mu) = \int x_0 \, d\mu$) restricted to the set C_l is continuous, by Varadhan's theorem ([13]) we have

$$\limsup_{n \to \infty} \frac{1}{|T_n|} \log \int_{K_l} \exp\left(\lambda \sum_{a \in T_n} x_a\right) dv_n \leq \sup_{y} (\lambda y - \hat{I}_F(y)),$$

where $K_l = \left\{ \underline{x} \Big| \sum_{a \in T_n} w(x_a) \leq l \, |T_n| \right\}$.

Therefore we only need to show

$$\limsup_{l \to \infty} \limsup_{n \to \infty} \frac{1}{|T_n|} \log \int_{K_l^c} \exp\left(\lambda \sum_{a \in T_n} x_a\right) v_n(d\underline{x}) = -\infty$$

or

$$\limsup_{l \to \infty} \limsup_{n \to \infty} \frac{1}{|T_n|} \log \int_{K_l^c} \exp\left(\sum_{a \in T_n} w(x_a)\right) v_n(d\underline{x}) = -\infty.$$

But this is an easy consequence of the Tchebyshev inequality and (2.6). $\square$

Corollary 2.6. *The family μ_n has the large deviation property with rate h.*

Now we state the main result of this section.

Theorem 2.7. *Let $\{\underline{z}_n\}$ be a sequence of the boundary conditions and $\{y_n\}$ be a sequence of real numbers such that $\lim_{n \to \infty} y_n = y$. Then the family $\{P_{F,T_n,\underline{z}_n}^{y_n}\}$ has the upper bound large deviation property with rate*

$$I_F^y(\mu) = \begin{cases} I_F(\mu) - h(y), & \text{if } \int x_0 \, d\mu = y; \\ \infty, & \text{otherwise.} \end{cases}$$

The proof of this Theorem is carried out in several steps.
Let $G \in C_{\text{loc}}^1$. Set

$$\hat{G}_n(y) = \frac{1}{|T_n|} \log \int \exp(\mathscr{H}_{G,T_n}(\underline{x})) v_{F,T_n,\underline{z}_n}^y(d\underline{x}).$$

Let $g_n(F, \underline{z}; y)$ denote the density of m_{T_n} with respect to $v_{F,T_n,\underline{z}}$. Then

Lemma 2.8.

$$\hat{G}_n(y) = \frac{1}{|T_n|} \log g_n(F + G, \underline{z}_n; y) - \frac{1}{|T_n|} \log g_n(F, \underline{z}_n; y)$$

$$+ \frac{1}{|T_n|} \log Z_{F+G,T_n,\underline{z}_n} - \frac{1}{|T_n|} \log Z_{F,T_n,\underline{z}_n}. \tag{2.15}$$

Proof. Let J be a test function,

$$\int J(y)g_n(F,\underline{z}_n;y)\exp\left(|T_n|\hat{G}_n(y)\right)dy$$

$$= \int J(m_{T_n}(\underline{x}))\exp\left(\mathscr{H}_{G,T_n,\underline{z}_n}(\underline{x}) + \mathscr{H}_{F,T_n,\underline{z}_n}(\underline{x})\right)\frac{1}{Z_F}d\rho^{T_n};$$

$$= \frac{Z_{F+G}}{Z_F}\int J(m_{T_n})\exp\left(\mathscr{H}_{F+G,T_n,\underline{z}_n}\right)\frac{1}{Z_{F+G}}d\rho^{T_n};$$

$$= \frac{Z_{F+G}}{Z_F}\int J(y)g_n(F+G,\underline{z}_n;y)dy.$$

Here to simplify the notations we have dropped the indices $\underline{z}$ and T_n from Z. □

The main ingredient to prove Theorem 2.7 is the following Lemma:

Lemma 2.9. *The family $\hat{G}_n$ is equicontinuous on every compact subset of* **R**.

Proof: Step 1. Because of Lemma 2.8, it is enough to show that the sequence

$$\frac{1}{|T_n|}\frac{g_n'(y)}{g_n(y)}$$

is uniformly bounded over any compact subset of **R**. Here, for ease in notation we have used g_n for $g_n(G,\underline{z}_n;y)$.

Let J be a test function. Then we have

$$\frac{1}{|T_n|}\int J(y)g_n'(y)dy = -\frac{1}{|T_n|}\int J'(y)g_n(y)dy$$

$$= -\frac{1}{|T_n|}\int J'\left(\frac{\sum\limits_{a\in T_n}x_a}{|T_n|}\right)\exp\left(\mathscr{H}_{G,T_n}(\underline{x}) - \sum\limits_{a\in T_n}\phi(x_a)\right)\frac{1}{Z_{G,T_n,\underline{z}_n}}d\underline{x}$$

$$= \int J\left(\frac{\sum\limits_{a\in T_n}x_a}{|T_n|}\right)\left\{\frac{1}{|T_n|}\sum\limits_{a\in T_n}\frac{\partial\mathscr{H}}{\partial x_a}(\underline{x}) - \frac{1}{|T_n|}\sum\limits_{a\in T_n}\phi'(x_a)\right\}dv_{G,T_n,\underline{z}_n}$$

$$= \int J(y)g_n(y)\left\{\int\frac{1}{|T_n|}\sum\limits_{a\in T_n}\left(\frac{\partial\mathscr{H}}{\partial x_a}(\underline{x}) - \phi'(x_a)\right)dv_{G,T_n,\underline{z}_n}^y\right\}dy.$$

Therefore

$$\frac{1}{|T_n|}\frac{g_n'(y)}{g_n(y)} = \int\frac{1}{|T_n|}\sum\limits_{a\in T_n}\left(\frac{\partial\mathscr{H}}{\partial x_a} - \phi'(x_a)\right)dv_{G,T_n,\underline{z}_n}^y. \qquad (2.16)$$

So

$$\frac{1}{|T_n|}\left|\frac{g_n'(y)}{g_n(y)}\right| \leq c_0 + \int\frac{1}{|T_n|}\sum\limits_{a\in T_n}|\phi'(x_a)|dv_{G,T_n,\underline{z}_n}^y, \qquad (2.17)$$

where c_0 is a constant depending only on G.

Step 2. Let

$$K_n(y) = \int \frac{1}{|T_n|} \sum_{a \in T_n} |\phi'(x_a)| \, dv^y_{G, T_n, \dot{z}_n}(d\underline{x}).$$

It is easy to see that

$$v^y_{G, T_n, \dot{z}_n}(d\underline{x}) = \frac{1}{\tilde{Z}_y} \exp \left(\mathscr{H}_{G, T_n, \dot{z}_n}(\underline{x}) \right) v^y_n(d\underline{x}).$$

Here $v^y_n(d\underline{x})$ is $v^y_{G, T_n, \dot{z}_n}(d\underline{x})$, where we set $G = 0$, and $\tilde{Z}_y$ is the obvious normalizing constant. Now we use the basic entropy estimate to deduce

$$K_n(y) = \int \left(\frac{1}{|T_n|} \sum_{a \in T_n} |\phi'(x_a)| \right) \exp \left(\mathscr{H}_{G, T_n, \dot{z}_n}(\underline{x}) \right) \frac{1}{\tilde{Z}_y} v^y_n(d\underline{x})$$

$$\leq \frac{1}{|T_n|} \log \int \exp \left(\sum_{a \in T_n} |\phi'(x_a)| \right) v^y_n(d\underline{x})$$

$$+ \frac{1}{|T_n|} \int \log \frac{\exp \left(\mathscr{H}_{G, T_n, \dot{z}_n}(\underline{x}) \right)}{\tilde{Z}_y} v^y_{G, T_n, \dot{z}_n}(d\underline{x})$$

$$= \mathscr{I}_1 + \mathscr{I}_2.$$

On the other hand

$$|\mathscr{H}_{G, T_n, \dot{z}_n}| \leq c_1 |T_n|,$$

where c_1 is a constant depending only on G. Therefore

$$\tilde{Z}_y = \int \exp \left(\mathscr{H}_{G, T_n, \dot{z}_n} \right) v^y_n(d\underline{x}) \geq e^{-c_1 |T_n|}.$$

So

$$\mathscr{I}_2 \leq 2c_1.$$

Final Step. In order to finish the proof, all we need is to show that the term $\mathscr{I}_1$ is uniformly bounded in y for y in a compact subset of $\mathbf{R}$. Let J be a test function as before, and let $k_n(y)$ be the density of $m_{T_n}(\underline{x})$ with respect to $\rho^{T_n}(d\underline{x}) = \exp \left(-\sum_{a \in T_n} \phi(x_a) \right) d\underline{x}$. Then

$$\int k_n(y) J(y) \left(\int \exp \left(\sum_{a \in T_n} |\phi'(x_a)| \right) v^y_n(d\underline{x}) \right) dy$$

$$= \int J(m_{T_n}(\underline{x})) \exp \left(\sum_{a \in T_n} (|\phi'(x_a)| - \phi(x_a)) \right) \prod_{a \in T_n} dx_a$$

$$= M^{|T_n|} \int J(m_{T_n}(\underline{x})) \prod_{a \in T_n} \frac{e^{|\phi'(x_a)| - \phi(x_a)}}{M} dx_a$$

$$= M^{|T_n|} \int J(y) \tilde{k}_n(y) dy,$$

where $\tilde{k}_n(y)$ is the density of $m_{T_n}(\underline{x})$ with respect to the product measure

$$\prod_{a \in T_n} \frac{e^{|\phi'(x_a)| - \phi(x_a)}}{M} dx_a$$

and

$$M = \int e^{|\phi'(x)| - \phi(x)} dx.$$

Therefore

$$\frac{1}{|T_n|} \log \int \exp\left(\sum_{a \in T_n} |\phi'(x_a)|\right) v_n^y(d\underline{x}) = \frac{1}{|T_n|} \log \tilde{k}_n(y) - \frac{1}{|T_n|} \log k_n(y) + M. \quad (2.18)$$

Note that we can write

$$\prod_{a \in T_n} \frac{e^{|\phi'(x_a)| - \phi(x_a)}}{M} dx_a = \prod_{a \in T_n} e^{-\tilde{\phi}(x_a)} dx_a,$$

where

$$\tilde{\phi}(x) = -|\phi'(x)| + \phi(x) + \log M.$$

So, by (2.18) the term $\mathscr{I}_1$ is bounded if we can show $(1/|T_n|) \log k_n(y)$ is bounded and $(1/|T_n|) \log \tilde{k}_n(y)$ is bounded above. We will show these in Lemmas 2.10 and 2.11. $\square$

Lemma 2.10. $(1/|T_n|) \log k_n(y)$ *is bounded over any compact subset of* **R**.

Proof. This immediately follows from Theorem 3.4 in [4]. $\square$

Lemma 2.11. *There exists a positive constant* c_2 *such that*

$$\tilde{k}_n(y) \leqq c_2 |T_n|$$

for all $n \geqq 1$.

Proof. Let $f = e^{-\tilde{\phi}}$. First we show $f \in L^2(\mathbf{R})$. Because of (2.5), it suffices to check that $e^{-\phi}$ is uniformly bounded. This follows from the integrability of the derivative of $e^{-\phi}$,

$$\int |\phi'| e^{-\phi} \leqq \log \int e^{|\phi'| - \phi} < \infty.$$

Since $f \in L^2(\mathbf{R})$, we have $f * f \in L^\infty$. This implies that the n-fold convolution of f is bounded by $\|f * f\|_\infty$ for $n \geqq 2$. Therefore the density of $\sum_{a \in T_n} x_a$ is also uniformly bounded for all $n \geqq 1$. Thus $\tilde{k}_n(y)$ is bounded by $|T_n| \|f * f\|_\infty$. $\square$

An easy consequence of Lemma 2.9 and Corollary 2.6 is the following theorem which is a large deviation result for the densities $g_n(F, \underline{z}; y)$.

Theorem 2.12. *Let* $F \in C^1_{\mathrm{loc}}$. *Then*

$$\lim_{n \to \infty} \frac{1}{|T_n|} \log g_n(F, \underline{z}; y) = -h(y)$$

uniformly in boundary conditions $\underline{z}$ *and uniformly in* y *in compact intervals of* **R**.

Proof. It suffices to show

$$\lim_{n \to \infty} \frac{1}{|T_n|} \log g_n(F, z_n; y) = -h(y)$$

uniformly on compact y-intervals, where $\{z_n\}$ is any sequence of boundary conditions.

By Lemma 2.10 the sequence $\{(1/|T_n|) \log g_n(F, z_n; y)\}$ is equicontinuous over compact intervals. Now we show that this sequence is bounded over any compact y-intervals. By (2.15) we have

$$\frac{1}{|T_n|} \log \int \exp(\mathscr{H}_{F, T_n, z_n}(\underline{x})) v_n^y(d\underline{x})$$

$$= \frac{1}{|T_n|} \log g_n(F, \underline{z}_n; y) - \frac{1}{|T_n|} \log k_n(y) + \log Z_{F, T_n, z_n},$$

where v_n^y and k_n are as in the proof of Lemma 2.9. Since the right-hand side of the above identity is uniformly bounded in y, and since the sequence $(1/|T_n|) \log k_n(y)$ is uniformly bounded over compact y-intervals (Lemma 2.10), therefore the sequence $\{(1/|T_n|) \log g_n(F, \underline{z}_n; y)\}$ is also uniformly bounded over compact y-intervals.

If a subsequence of $\{(1/|T_n|) \log g_n(F, \underline{z}_n; y)\}$ converges to a function g, then by Corollary 2.6, g can only be $-h(y)$ and this completes the proof. $\quad\square$

Now we check that I_F^y is a rate function.

Lemma 2.13. *The set $\{\mu : I_F^y(\mu) \leq \ell\}$ is compact for all ℓ.*

Proof. Let $I_F^y(\mu) < \infty$. Then, by the basic entropy inequality

$$\int w(x_0) \mu(d\underline{x}) = \frac{1}{|T_n|} \int \left(\sum_{a \in T_n} w(x_a) \right) \mu(d\underline{x})$$

$$= \frac{1}{|T_n|} \int \left(\sum_{a \in T_n} w(x_a) \right) \mu^{T_n}(d\underline{x})$$

$$\leq \frac{1}{|T_n|} \log \int \exp \left(\sum_{a \in T_n} w(x_a) \right) \rho^{T_n}(d\underline{x})$$

$$+ \frac{1}{|T_n|} \int \log \frac{d\mu^{T_n}}{d\rho^{T_n}}(\underline{x}) \mu^{T_n}(d\underline{x}).$$

Therefore, after passing to limit as $n \to \infty$,

$$\int w(x_0) \mu(d\underline{x}) \leq \log \int e^{w(x) - \phi(x)} dx + H_\rho(\mu)$$

$$= \log \int e^{w(x) - \phi(x)} dx + I_F(\mu) + \int F d\mu - \psi(F)$$

$$\leq c_0 + I_F(\mu),$$

where c_0 is a constant depending on F.

This implies that $\int w(x_0) \mu(dx)$ is bounded if $I_F(\mu)$ is bounded. Note that $\Phi(\mu) = \int x_0 d\mu$ is continuous if it is restricted to the set $\{\mu : \int w(x_0) \mu(dx) \leq \ell_0\}$ for

any ℓ_0,

$$
\begin{aligned}
\{\mu : I_F^y(\mu) \leq \ell\} &= \{\mu : \int x_0 \, d\mu = y \text{ and } I_F(\mu) \leq \ell + h(y)\} \\
&= \{\mu : \int x_0 \, d\mu = y \text{ and } \int w(x_0) \leq c_0 + \ell + h(y)\} \\
&\quad \cap \{\mu : I_F(\mu) \leq \ell + h(y)\}.
\end{aligned}
$$

Thus the set $\{\mu : I_F^y(\mu) \leq \ell\}$ is compact, because I_F is a rate function.

Proof of Theorem 2.7. Step 1. We use Lemma 2.3, in order to show the inequality (2.2) (i.e. the upper bound estimate) for the family $\{P_{F,T_n,z_n}^{y_n}\}$. For this, we need to consider a suitable closed base for the topology of the weak convergence. Let $\mathscr{A}_0$ be the collection of the closed sets of the form

$$
\{\mu \in \mathscr{M}_\tau(X) : \int G_1 \, d\mu \in I_1, \int G_2 \, d\mu \in I_2 \cdots \int G_k \, d\mu \in I_k\},
$$

where $I_1, \ldots, I_k$ are fixed closed subsets of $\mathbf{R}$ and $G_1, \ldots, G_k$ are in C_{loc}^1. We also use the compact notation $\underline{G} = (G_1, \ldots, G_k)$ in the sequel.

Note that if $A_1, \ldots, A_k$ are in $\mathscr{A}_0$ then $A_1 \cap \cdots \cap A_m$ is also in $\mathscr{A}_0$. Let $\mathscr{A}$ be the collection of all finite unions of the elements of $\mathscr{A}_0$. That is,

$$
\mathscr{A} = \{B_1 \cup \cdots \cup B_k | \text{for some } B_1, B_2, \ldots, B_k \text{ in } \mathscr{A}_0\}.
$$

It is easy to check that $\mathscr{A}$ is a topological closed base for $\mathscr{M}_\tau(X)$. Since the space $\mathscr{M}_\tau(X)$ is separable, then for any closed subset C of $\mathscr{M}_\tau(X)$, there exists a decreasing sequence $\{A_n\}$ in $\mathscr{A}$ such that $C = \cap_n A_n$.

Step 2. Consider the continuous function $\Phi : \mathscr{M}_\tau(x) \to \mathbf{R}^k$ defined as

$$
\Phi(\mu) = \int \underline{G} \, d\mu.
$$

For ease in notation, we set $v_n^{y_n} = v_{F,T_n,z_n}^{y_n}$. Let $\underline{\theta} = (\theta_1, \ldots, \theta_k) \in \mathbf{R}^k$. We define

$$
\xi(\underline{\theta}; y) = \xi(\theta_1, \ldots, \theta_k; y) = \lim_{n \to \infty} \frac{1}{|T_n|} \log \int \exp(|T_n|(\underline{\theta} \cdot \Phi(\mu))) P_{F,T_n,z_n}^{y_n}(d\mu)
$$

$$
= \lim_{n \to \infty} \frac{1}{|T_n|} \int \exp(\mathscr{H}_{\underline{\theta} \cdot \underline{G}, T_n}) dv_n^{y_n}.
$$

The last equality needs some explanation. From the definition of $P_{F,T_n,z_n}^{y_n}$ we obtain the last term with $\mathscr{H}_{\underline{\theta} \cdot \underline{G}, T_n, z_n}$. But, after passing to limit, the contribution of the boundary condition disappears. Now we use (2.15) to conclude

$$
\xi(\underline{\theta}; y) = \lim_{n \to \infty} \frac{1}{|T_n|} \log g_n(\underline{\theta} \cdot \underline{G} + F, \underline{z}_n; y)
$$

$$
- \lim_{n \to \infty} \frac{1}{|T_n|} \log g_n(F, \underline{z}_n; y)
$$

$$
+ \Psi(\underline{\theta} \cdot \underline{G} + F) - \Psi(F).
$$

Let $\varphi(\cdot, \lambda)$ be defined as in (2.13) and let $h(\cdot; y)$ be the convex conjugate of $\varphi(\cdot; \lambda)$ in the variable λ. By Theorem 2.12,

$$
\xi(\underline{\theta}; y) = -h(\underline{\theta} \cdot \underline{G} + F; y) + h(F; y) + \Psi(\underline{\theta} \cdot \underline{G} + F) - \Psi(F).
$$

Step 3. By Lemma 2.2, the family

$$v_n^{y_n}(\int \underline{G}\,d\mu \in d\underline{t})$$

has the upper bound large deviation property with rate

$$I(\underline{t};y)$$

which is the convex conjugate of $\xi(\underline{\theta};y)$ in the variable $\underline{\theta}$. More precisely

$$I(\underline{t};y) = \sup_{\underline{\theta}} (\underline{\theta}\cdot\underline{t} - \xi(\underline{\theta};y))$$

$$= \sup_{\underline{\theta}} (\underline{\theta}\cdot\underline{t} + \sup_{\lambda}(\lambda y - \varphi(\underline{\theta}\cdot\underline{G} + F;\lambda)) - \Psi(\underline{\theta}\cdot\underline{G} + F) + \Psi(F)) - h(y)$$

$$= \sup_{\underline{\theta},\lambda} (\underline{\theta}\cdot\underline{t} + \lambda y - \varphi(\underline{\theta}\cdot\underline{G} + F;\lambda) - \Psi(F + \underline{\theta}\cdot\underline{G}) + \Psi(F)) - h(y).$$

Step 4. Let $\hat{\Phi}(\mu) = (\int \underline{G}\,d\mu, \int x_0\,d\mu)$. We have shown the Contraction Principle for Φ in Lemma 2.4 (when $\underline{G} = 0$). However, with the same argument, we have the same result for $\hat{\Phi}$ for any $\underline{G}$ in C^1_{loc}. We can also identify the rate function as in Lemma 2.5. Thus the family $\{P_{F,T_n,z_n} \circ \hat{\Phi}^{-1}\}$ has the large deviation property with the rate function

$$\hat{I}(\underline{t};y) = \inf\{I_F(\mu):\int \underline{G}\,d\mu = \underline{t} \text{ and } \int x_0\,d\mu = y\}$$
$$= \text{convex conjugate of } \hat{\varphi}(\underline{\theta};\lambda) \text{ in } \underline{\theta} \text{ and } \lambda,$$

where

$$\hat{\varphi}(\underline{\theta};\lambda) = \lim_{n\to\infty} \frac{1}{|T_n|}\log\int \exp\left(\lambda \sum_{a\in T_n} x_a + \mathscr{H}_{\underline{\theta}\cdot\underline{G}}(\underline{x})\right) v_{F,T_n,z_n}(d\underline{x})$$

$$= \lim_{n\to\infty} \frac{1}{|T_n|}\log\int \exp\left(\lambda \sum_{a\in T_n} x_a\right) v_{F+\underline{\theta}\cdot\underline{G},T_n,z_n}(d\underline{x}) - \Psi(F) + \Psi(\underline{\theta}\cdot\underline{G} + F)$$

$$= \varphi(F + \underline{\theta}\cdot\underline{G};\lambda) - \Psi(F) + \Psi(\underline{\theta}\cdot\underline{G} + F).$$

Thus

$$I(\underline{t};y) = \hat{I}(\underline{t};y) - h(y).$$

Final Step. Let $A = \{\mu:\int G_1\,d\mu \in I_1, \ldots, \int G_k\,d\mu \in I_k\}$ as in the beginning of the proof. Then

$$\limsup_{n\to\infty} \frac{1}{|T_n|}\log P^{y_n}_{F,T_n,z_n}(A) \leq -\inf\{I(\underline{t};y):t_1 \in I_1, \ldots, t_k \in I_k\} \text{ by step 3,}$$

$$= -\inf\{\hat{I}(\underline{t};y):t_1 \in I_1, \ldots, t_k \in I_k\} + h(y)$$

$$= -\inf_{\substack{\mu\in A \\ \int x_0 d\mu = y}} (I_F(\mu) - h(y))$$

$$= -\inf_{\mu\in A} I_F^y(\mu).$$

Since any set in $\mathscr{A}$ is a finite union of the sets in $\mathscr{A}_0$, we also have the above inequality for $A \in \mathscr{A}$. $\quad\square$

Let $F \in C^1_{loc}$. For each λ, set $F_\lambda(\underline{x}) = F(\underline{x}) + \lambda x_0$. For simplicity we denote $\mathscr{G}_{F_\lambda}$ by $\mathscr{G}_\lambda$.

Theorem 2.14. *Suppose that $G \in C^1_{loc}$ or $G(\underline{x}) = G_1(\underline{x}) + \phi'(x_0)$ for some $G_1 \in C^0_{loc}$. Let $\underline{z}_n$ be a sequence in X, let y_n be a sequence of real numbers such that $\lim_{n \to \infty} y_n = y$, and let $F \in C^1_{loc}$. Then the family*

$$\alpha_n(dt) = v^{y_n}_{F, T_n, z_n} \left\{ \frac{1}{|T_n|} \sum_{a \in T_n} G(\tau_a \underline{x}) \in dt \right\}$$

has the upper bound large deviation property with rate

$$J_G(t) = \inf \{ I^y_F(\mu) : \int G d\mu = t \}.$$

Moreover $J_G(t) = 0$ if and only if $t = \int G d\mu$ for some $\mu \in \mathscr{G}_{h'(y)}$ with $\int x_0 d\mu = y$.

Proof. If $G \in C^0_{loc}$, then the large deviation property is an immediate consequence of the Contraction Principle and Theorem 2.7. If $G(\underline{x}) = G_1(\underline{x}) + \phi'(x_0)$ for some $G_1 \in C^0_{loc}$, we may apply Lemma 2.1. For this we need to show that

$$\limsup_{\ell \to \infty} \limsup_{n \to \infty} \frac{1}{|T_n|} \log v^{y_n}_{F, T_n, z_n} \left(\frac{1}{|T_n|} \sum_{a \in T_n} w(\phi'(x_a)) \geq \ell \right) = -\infty. \qquad (2.19)$$

Because the function $\Phi(\mu) = \int G d\mu$, restricted to the set $\{\mu : \int w(\phi'(x_0)) \mu(d\underline{x}) \leq \ell\}$ is continuous.

Formula (2.19) follows from

$$\limsup_{n \to \infty} \frac{1}{|T_n|} \log \int \exp \left(\sum_{a \in T_n} w(\phi'(x_a)) \right) dv^{y_n}_{F, T_n, z_n} \leq +\infty \qquad (2.20)$$

and the Tchebychev inequality. For this, we can essentially repeat our argument in the final step of the proof of Lemma 2.9 (see (2.18)).

If $J_G(t) = 0$, then there exists a measure μ in $\mathscr{M}_\tau(X)$ such that

$$\int x_0 d\mu = y; \quad \int G d\mu = t; \quad \text{and} \quad I_F(\mu) = h(y). \qquad (2.21)$$

On the other hand by (2.11) we have

$$I_{F_\lambda}(\mu) = H_\rho(\mu) - \int F d\mu - \lambda \int x_0 d\mu + \Psi(F_\lambda)$$
$$= I_F(\mu) - \{\lambda y - (\Psi(F_\lambda) - \Psi(F))\}. \qquad (2.22)$$

From the definition of the function h as the convex conjugate of φ, we have

$$h(y) = \sup_\lambda (\lambda y - \varphi(F; \lambda)) = \sup_\lambda (\lambda y - \Psi(F_\lambda) + \Psi(F)),$$

and the supremum is attained at $\lambda = h'(y)$. Therefore

$$h(y) = h'(y)y - \Psi(F_{h'(y)}) + \Psi(F).$$

From this, (2.21) and (2.22), we obtain

$$I_{F_\lambda}(\mu) = 0 \quad \text{if} \quad \lambda = h'(y).$$

Finally, it follows from the *Variational Principle* (see Preston [10], p. 115) that $\mu \in \mathscr{G}_{h'(y)}$.

Remark 2.15. We have assumed that h is differentiable. Since h is convex, this is equivalent to strict convexity of $\varphi(\lambda)$ or $\Psi(F_\lambda)$ in λ. This also follows from the Variational Principle. See Preston [10], p. 158 for more details.

Corollary 2.16. *If* $G_0(\underline{x}) = \phi'(x_0) - (\partial \mathcal{H}_n/\partial x_0)(\underline{x})$, *where* $\mathcal{H}_n = \mathcal{H}_{F,T_n}$, *then* $J_{G_0}(t) = 0$ *if and only if* $t = h'(y)$.

Proof. Let $\mu \in \mathcal{G}_\lambda$ for some $\lambda \in \mathbf{R}$. Then

$$\int G_0 d\mu = - \int \left(\frac{\partial \mathcal{H}_n}{\partial x_0}(\underline{x}) - \phi'(x_0) \right) \mu(d\underline{x})$$

$$= - \int \frac{1}{|T_n|} \sum_{a \in T_n} \left(\frac{\partial \mathcal{H}_n}{\partial x_a}(\underline{x}) - \phi'(x_a) \right) \mu(d\underline{x})$$

$$= - \int \mu(d\underline{z}) \int \frac{1}{|T_n|} \sum_{a \in T_n} \left(\frac{\partial \mathcal{H}_n}{\partial x_a}(\underline{x}) - \phi'(x_a) \right) dv_{F_\lambda, T_n, \underline{z}}$$

$$= \lambda + r_n, \tag{2.23}$$

where $\lim_{n \to \infty} r_n = 0$. This is because

$$dv_{F_\lambda, T_n, \underline{z}} = \frac{1}{Z_{F_\lambda, T_n, \underline{z}_n}} \exp \left(\tilde{\mathcal{H}}_n(\underline{x}) - \sum_{a \in T_n} \phi'(x_a) + \lambda \sum_{a \in T_n} x_a \right),$$

where $\tilde{\mathcal{H}}_n = \mathcal{H}_{F,T_n,\underline{z}}$. The last equality in (2.23) follows from integration by parts. The error r_n comes from the boundary condition $\underline{z}$ when we replace $\partial \tilde{\mathcal{H}}_n/\partial x_a$ with $\partial \mathcal{H}_n/\partial x_a$.

Now let $\lambda = h'(y)$ and pass to limit $n \to \infty$ in (2.23). $\quad \square$

Theorem 2.17. *Let K be a bounded subset of* $\mathbf{R}$, *and let G_0 be as in Corollary 2.16.* *Then*

$$\lim_{n \to \infty} \sup_{\substack{\underline{z} \in X \\ y \in K}} \int \left| \frac{1}{|T_n|} \sum_{a \in T_n} G_0(\tau_a \underline{x}) - h'(y) \right| dv^y_{F,T_n,\underline{z}} = 0. \tag{2.24}$$

Proof. It follows from Theorem 2.14 and its corollary that

$$\lim_{n \to \infty} v^{y_n}_{F,T_n,\underline{z}_n} \left(\left| \frac{1}{|T_n|} \sum_{a \in T_n} G_0(\tau_a \underline{x}) - h'(y) \right| > \delta \right) = 0,$$

where $y_n \to y, \underline{z}_n$ is an arbitrary sequence in X, and $\delta > 0$. Therefore, for (2.24) we only need to check the uniform integrability of the sequence

$$\frac{1}{|T_n|} \sum_{a \in T_n} G_0(\tau_a \underline{x})$$

with respect to the measures $v^{y_n}_{F,T_n,\underline{z}_n}$. For this, it suffices to show

$$\sup_n \int w \left(\frac{1}{|T_n|} \sum_{a \in T_n} \phi'(x_a) \right) dv^{y_n}_{F,T_n,\underline{z}_n} < \infty.$$

Since w is convex, it is enough to show

$$\sup_n \int \frac{1}{|T_n|} \sum_{a \in T_n} w(\phi'(x_a)) dv_{F,T_n,z_n}^{y_n} < \infty.$$

But this is an immediate consequence of (2.20) and Jensen's inequality. $\square$

Theorem 2.18. *Let* $F \in C^1_{\text{loc}}$. *Then*

$$\frac{1}{|T_n|} \frac{g_n'(y)}{g_n(y)} \rightarrow -h'(y)$$

uniformly in $\underline{z} \in X$ *and* y *in a compact subset of* **R**, *where*

$$g_n(y) = g_n(F, \underline{z}; y).$$

Proof. This follows from Theorem 2.17 and (2.16). $\square$

The function G_0 is of very special form. That is why we have an ergodic theorem like Theorem 2.16 even when phase transition occurs (i.e. $\mathcal{G}_\lambda$ for some λ is not singleton). In general we do not expect to have such a result for any $G \in C^0_{\text{loc}}$. If we however assume that the space of Gibbs measures has a *linear structure*, we can extend Theorem 2.16 to any $G \in C^1_{\text{loc}}$. Note that this assumption does not exclude phase transition.

Definition 2.19. We say that $\mathcal{G}_\lambda$ has linear structure if $\mu_1, \mu_2 \in \mathcal{G}_\lambda$ and $\int x_0 d\mu_1 = \int x_0 d\mu_2$, then $\mu_1 = \mu_2$.

If $\mathcal{G}_\lambda$ has linear structure for all λ, then, for any $G \in C^0_{\text{loc}}$, we can define

$$A_G(y) = \int G d\mu,$$

where μ is the unique measure in $\mathcal{G}_{h'(y)}$ such that $\int x_0 d\mu = y$.

Theorem 2.20. *Under the above assumption,*

$$\lim_{n \to \infty} \sup_{\substack{\underline{z} \in X \\ y \in K}} \int \left| \frac{1}{|T_n|} \sum_{a \in T_n} G(\tau_a \underline{x}) - A_G(y) \right| dv_{F,T_n,\underline{z}}^y = 0,$$

where K *is any bounded subset of* **R**.

Proof. This follows from Theorem 2.14 and the proof to Theorem 2.17. $\square$

In Sect. 5 we will need the following estimate.

Lemma 2.21. *There exists a constant* c *such that*

$$|h'(y)| \leqq c + h(y). \tag{2.25}$$

Proof. Let G_0 be as in Corollary 2.16, and let $\mu \in \mathcal{G}_{h'(y)}$ be such that $\int x_0 d\mu = y$. Let $\lambda = h'(y)$. Using the basic entropy estimate we have

$$\int \frac{1}{|T_n|} \sum_{a \in T_n} |\phi'(x_a)| dv_{F_\lambda,T_n,\underline{z}} \leqq \frac{1}{|T_n|} \log \int \exp \left(\sum_{a \in T_n} |\phi'(x_a)| \right) d\rho^{T_n}$$

$$+ \frac{1}{|T_n|} \int (\mathcal{H}_{F_\lambda,T_n,\underline{z}} - \log Z_{F_\lambda,T_n,\underline{z}}) dv_{F_\lambda,T_n,\underline{z}}.$$

By integrating both sides in $\underline{z}$ with respect to μ and passing to the limit, we obtain

$$\limsup_{n \to \infty} \int \left(\frac{1}{|T_n|} \sum_{a \in T_n} |\phi'(x_a)| \right) \mu(d\underline{x}) \leq \log \int e^{|\phi'(x)| - \phi(x)} dx + \int F d\mu + \lambda \int x_0 d\mu - \Psi(F_\lambda)$$

$$= c_1 + \lambda \int x_0 d\mu - \Psi(F_\lambda) + \Psi(F)$$

$$\leq c_1 + h(y),$$

where c_1 is a suitable constant and $\lambda = h'(y)$. On the other hand, by our calculation in Corollary 2.16 we have

$$|h'(y)| \leq c_2 + \limsup_{n \to \infty} \int \left(\frac{1}{|T_n|} \sum_{a \in T_n} |\phi'(x_a)| \right) d\mu$$

for some constant c_2. This will give us (2.25) with $c = c_1 + c_2$. $\quad\square$

3. A Local Ergodic Theorem

Initially we start from a distribution having density f_N^0 with respect to the Gibbs measure ν_N with periodic boundary condition (see (1.12)) The initial density f_N^0 satisfies the entropy condition

$$\frac{1}{N^d} \int f_N^0 \log f_N^0 d\nu_N \leq C.$$

Then the evolution governed by the generator $\mathscr{L}_N$ (see (1.10)) gives us the density f_N^t which satisfies the forward equation

$$\frac{\partial f_N^t}{\partial t} = \mathscr{L}_N f_N^t.$$

Then for large N the measure $f_N^t d\nu_N$ locally looks somewhat like a Gibbs measure with the interaction F and a *chemical potential* $\lambda = h'(y)$, where y corresponds to the macroscopic charge density, coming from the averages m_T. To make this precise, we first state Theorem 3.1 which is some kind of *ergodic theorem*.

Let Q_N be the law of the diffusion generated by $\mathscr{L}_N$ on the space $C([0, \infty), X_N)$ where $X_N = \mathbf{R}^{S_N^d}$.

Theorem 3.1. *Let G_0 be as in Corollary 2.16, and let $T_\ell(a) = a + T_\ell$. Then*

$$\lim_{k \to \infty} \limsup_{\varepsilon \to 0} \limsup_{\ell \to \infty} \limsup_{N \to \infty} \frac{1}{N^d} \sum_{a \in S_N^d} E^{Q_N} \int_0^t \left| \frac{1}{|T_\ell(a)|} \sum_{u \in T_\ell(a)} G_0(\tau_u \underline{x}(s)) \right.$$

$$\left. - \psi_k \circ h'(m_{T_{N\varepsilon}(a)}(\underline{x}(s))) \right| ds = 0, \tag{3.1}$$

where

$$\psi_k(x) = \begin{cases} x & |x| \leq k; \\ k & x > k; \\ -k & x < -k. \end{cases}$$

If we set

$$\hat{f}_N^t(\underline{x}) = \frac{1}{t} \int_0^t f_N^s(\underline{x}) ds$$

and

$$\eta_N(\underline{x}) = \frac{1}{N^d} \sum_{a \in S_N^d} \hat{f}_N^t(\tau_a \underline{x}) \tag{3.2}$$

for $\underline{x} \in \mathbf{R}^{S_N^d}$, then (3.1) can be written as

$$\lim_{k \to \infty} \limsup_{\varepsilon \to 0} \limsup_{\ell \to \infty} \limsup_{N \to \infty} E^{\eta_N dv_N} \left| \frac{1}{|T_\ell|} \sum_{u \in T_\ell} G_0(\tau_u \underline{x}) - \psi_k \circ h'(m_{T_{N\varepsilon}}(\underline{x})) \right| = 0. \tag{3.3}$$

The proof of (3.3) is carried out in several steps. Let P^N denote $\eta_N dv_N$.

Theorem 3.2. *For any* $\delta > 0$,

$$\lim_{\ell \to \infty} \limsup_{N \to \infty} P^N \left\{ \underline{x} : \left| \frac{1}{|T_\ell|} \sum_{u \in T_\ell} G_0(\tau_u \underline{x}) - h'(m_{T_\ell}(\underline{x})) \right| > \delta \right\} = 0.$$

Before proving Theorem 3.2, we study the limit points of the sequence $\eta_N dv_N$. For this we follow [4]. First we define

$$\mathscr{I}^N(\mu) = \frac{1}{N^2} \sup_{f > 0} \left[\int \frac{-\mathscr{L}_N f}{f} d\mu \right]$$

for μ a measure on the space $\mathbf{R}^{S_N^d}$. Then

Lemma 3.3. (a)

$$\frac{1}{N^d} \int \eta_N \log \eta_N dv_N \leqq C, \tag{3.4}$$

(b)

$$\frac{1}{N^d} \mathscr{I}^N(P^N) \leqq \frac{C}{4N^2 t}. \tag{3.5}$$

We omit the proof, which follows the proof of Theorem 2.4 in [4].

Let Proj_ℓ denote the projection on the space $\mathbf{R}^{T_\ell}$. Let $\mathscr{T}$ be the set of measures $\mu \in \mathscr{M}_\tau(X)$ such that for each ℓ, $\text{Proj}_\ell(\mu)$ is a weak limit point of the sequence $\text{Proj}_\ell(P^N)$.

Lemma 3.4. (a)

$$\sup_{\mu \in \mathscr{T}} \sup_\ell E^\mu \frac{1}{|T_\ell|} \sum_{a \in T_\ell} w(x_a) < \infty;$$

(b)

$$\sup_{\mu \in \mathscr{T}} \sup_\ell E^\mu \frac{1}{|T_\ell|} \sum_{a \in T_\ell} w(\phi'(x_a)) < \infty;$$

(c)

$$\mathscr{T} \subseteq \mathscr{G}_F^c.$$

Proof. (a) is an easy consequence of (3.4) and the basic entropy estimate. (b) also follows from this estimate

$$E^{PN} \frac{1}{|T_\ell|} \sum_{a \in T_\ell} w(\phi'(x_a)) = E^{PN} \frac{1}{N^d} \sum_{a \in S_N^d} w(\phi'(x_a))$$

$$\leqq \frac{1}{N^d} \log \int \exp \left(\sum w(\phi'(x_a)) dv_N + \frac{1}{N^d} \int \eta_N \log \eta_N dv_N.$$

This is uniformly bounded in N because of (3.4) and (2.6).
(c) Consider the following diffusion generator

$$\mathcal{L}_{a,b} = \frac{1}{2} \left(\frac{\partial}{\partial x_a} - \frac{\partial}{\partial x_b} \right)^2 - \frac{1}{2} \left(\phi'(x_a) - \phi'(x_b) - \frac{\partial \mathcal{H}_N}{\partial x_a} + \frac{\partial \mathcal{H}_N}{\partial x_b} \right) \left(\frac{\partial}{\partial x_a} - \frac{\partial}{\partial x_b} \right).$$

For finite $T \subseteqq \mathbb{Z}^d$, we define

$$\mathcal{L}_T = \sum \mathcal{L}_{a,b},$$

where the sum is over bonds a, b (adjacent sites) in T. Corresponding to $\mathcal{L}_T$, we associate the form

$$\mathcal{I}_T(\mu) = \sup_{f > 0} \left[\int \frac{-\mathcal{L}_T f}{f} d\mu \right], \tag{3.6}$$

where $\mu \in \mathcal{M}(\mathbb{R}^S)$ for some S with $T + \Lambda \subseteqq S$ (Λ is the range of F). When $\mu \in \mathcal{M}(\mathbb{R}^S)$ then the supremum in (3.6) is over $f \in C^\infty(\mathbb{R}^S, \mathbb{R})$. Note that

$$\mathcal{I}^N(\mu) = \mathcal{I}_{S_N^d}(\mu) = \sum \mathcal{I}_{a,b}(\mu),$$

$$\mathcal{I}^N(\mu) = \sum \mathcal{E}^N_{a,b}(\sqrt{\eta_N}),$$

where

$$\mathcal{E}^N_{a,b}(f) = \frac{1}{2} \int \left(\frac{\partial f}{\partial x_a} - \frac{\partial f}{\partial x_b} \right)^2 dv_N. \tag{3.7}$$

From these considerations and Lemma 3.3, we obtain

$$\mathcal{E}^N_{a,b}(\sqrt{\eta_N}) \leqq \frac{C}{4N^2 td}$$

for a, b adjacent sites (this is because η_N is translation invariant in the space $\mathbb{R}^{S_N^d}$). This implies

$$\mathcal{I}_{a,b}(P^N) \leqq \frac{C}{4N^2 td}$$

(for the relation between $\mathcal{I}_{a,b}$ and $\mathcal{E}^N_{a,b}$ in general see Stroock [12], Theorem 7.44).
    Suppose that $a + \Lambda, b + \Lambda \subseteqq T_\ell$. Then, in (3.6) with $T = \{a, b\}$, if we restrict the supremum to functions $f \in C^\infty(\mathbb{R}^{T_\ell}, \mathbb{R})$ and $f > 0$, we obtain

$$\mathcal{I}_{a,b}(\text{Proj}_\ell(P^N)) \leqq \mathcal{I}_{a,b}(P^N).$$

Thus

$$\mathscr{I}_{a,b}(\mathrm{Proj}_\ell(P^N)) \leqq \frac{C}{4N^2 td}. \tag{3.8}$$

Let $\mu \in \mathscr{T}$. Then by lower semicontinuity of $\mathscr{I}_{a,b}$, we have

$$\mathscr{I}_{a,b}(\mathrm{Proj}_\ell \mu) = 0.$$

Thus

$$\mathscr{I}_T(\mathrm{Proj}_\ell \mu) = 0 \tag{3.9}$$

for any finite set $T \subseteq \mathbb{Z}^d$. In Lemma 3.5, we will show that (3.9) implies (c) and this completes the proof. $\square$

Lemma 3.5. *The following statements are equivalent:*

(a) $\mathscr{I}_T(\mathrm{Proj}_\ell \mu) = 0$ *for all large ℓ such that $T + \Lambda \subseteq T_\ell$, and all finite sets $T \subseteq \mathbb{Z}^d$;*
(b) $\int \mathscr{L}_T f \, d\mu = 0$, *for all $f \in C^\infty_{\mathrm{loc}}$ and all finite sets $T \subseteq \mathbb{Z}^d$;*
(c) $\mu \in \mathscr{G}^c_F$.

Proof. (a ⇔ b) Consider the inequalities

$$\int \frac{\mathscr{L}_T f}{f} \, d\mu \geqq 0 \tag{3.10}$$

for $f \in C^\infty_{\mathrm{loc}}$, $f > 0$ and $T \subseteq \mathbb{Z}^d$ finite. By setting $f = e^{tu}$ with $u \in C^\infty_{\mathrm{loc}}$ and $t \in \mathbf{R}$, and then varying t, it is easy to see that (3.10) is equivalent to

$$\int \mathscr{L}_T u \, d\mu = 0 \tag{3.11}$$

for all $u \in C^\infty_{\mathrm{loc}}$ and $T \subseteq \mathbb{Z}^d$ finite.

   (b ⇔ c) Let $\mathscr{F}_T$ be the σ-field generated by the variables x_a with $a \notin T$, and $m_T(\underline{x})$. Then, if $f, g \in C^\infty_{\mathrm{loc}}$ and g is $\mathscr{F}_T$-measurable, we have

$$\mathscr{L}_T(fg) = g\mathscr{L}_T f. \tag{3.12}$$

   Given $\mu \in \mathscr{M}_\tau(X)$, let $\mu^y_{T,\underline{z}}$ denote the conditional probability distribution of μ given $\mathscr{F}_T$. More precisely

$$\mu^y_{T,\underline{z}}(d\underline{x}) = \mu(d\underline{x} \mid x_a = z_a \quad \text{for} \quad a \notin T, \quad \text{and} \quad m_T(\underline{x}) = y),$$

where $y = m_T(\underline{z})$. From (3.12), we conclude that (3.11) is equivalent to

$$\int \mathscr{L}_T f \, d\mu^y_{T,\underline{z}} = 0 \tag{3.13}$$

for all $f \in C^\infty_{\mathrm{loc}}$, and almost all $\underline{z}$ with respect to μ (here, we have used the fact that we only need to verify (3.13) for a suitable countable family of functions f).

   It is not hard to see that $\mathscr{L}_T$ acting on the space of functions defined for the variables x_a, for all $a \in T$ and $m_T(\underline{x}) = y$ is elliptic with the unique invariant measure $v^y_{T,\underline{z}}$. Thus (3.13) is equivalent to

$$v^y_{T,\underline{z}} = \mu^y_{T,\underline{z}} \quad \text{a.a.} \quad \underline{z},$$

and this completes the proof. $\square$

Proof of Theorem 3.2. On account of Theorem 2.16 and Lemma 3.4 (c), we only need to check

$$\limsup_{p \to \infty} \limsup_{\ell \to \infty} \sup_{\mu \in \mathscr{T}} \int_{|m_{T_\ell}(\underline{z})| \geq p} \mu(d\underline{z}) = 0.$$

But this is an immediate consequence of Lemma 3.4 (a), because w in this lemma satisfies (2.6). $\square$

What we have shown so far is an ergodic theorem for the averages of G_0 over the microscopically large blocks of $\mathbb{Z}^d$. In Theorem 3.1, however, we need to show that these averages are close to the chemical potential of a macroscopic charge density. The gap between Theorems 3.1 and 3.2 is filled by the following result:

Theorem 3.6. *For any $\delta > 0$,*

$$\lim_{\varepsilon \to 0} \limsup_{\ell \to \infty} \limsup_{N \to \infty} P^N \{\underline{x} : |h'(m_{T_\ell}(\underline{x}) - h'(m_{T_{N\varepsilon}}(\underline{x}))| > \delta\} = 0.$$

For this, we first link a block to a *macroscopically close* block by the following lemma.

Lemma 3.7. *For any $\delta > 0$*

$$\lim_{\varepsilon \to 0} \limsup_{\ell \to \infty} \limsup_{N \to \infty} \sup_v P^N \{\underline{x} : |h'(m_{T_\ell}(\underline{x}) - h'(m_{T_\ell(v)}(\underline{x}))| > \delta\} = 0,$$

where the supremum is over sites v such that $T_\ell(v) \subseteq T_{N\varepsilon}$, and

(Λ *is the range of F*).

Proof. Let $d\beta_{N,\ell,v}$ be the projection of $\eta_N dv_N$ on the space $\mathbf{R}^{T_\ell + \Lambda} \times \mathbf{R}^{T_\ell(v) + \Lambda}$, where v may depend on N, ℓ and ε. After a change of variable we view this measure as a measure on the space $\mathbf{R}^{T_\ell + \Lambda} \times \mathbf{R}^{T_\ell(v_0) + \Lambda}$, where v_0 is now a fixed site such that

$$(T_\ell(v_0) + \Lambda) \cap (T_\ell + \Lambda) = \varnothing.$$

We abuse the notation and denote this measure by β_{N,ℓ,v_0}.

Let β_ℓ be any limit point of β_{N,ℓ,v_0} as N goes to infinity. Then by (3.9) and its consequence (3.11) we have

$$\int \mathscr{L}_{T_\ell} f \, d\beta_\ell = \int \mathscr{L}_{T_\ell(v_0)} f \, d\beta_\ell = 0$$

for all $f \in C^\infty(\mathbf{R}^{T_\ell + \Lambda} \times \mathbf{R}^{T_\ell(v_0) + \Lambda}, \mathbf{R})$. The proof of Lemma 3.5 implies that

$$d\beta_\ell = \int v^{y_1}_{F, T_{\ell,\underline{z}}} v^{y_2}_{F, T_\ell(v_0), \underline{z}} \gamma(dy_1, dy_2, d\underline{z}) \tag{3.15}$$

for some probability measure γ. We also consider $\mathscr{L}_{0,v}$ and $\mathscr{E}^N_{0,v}$, where $\Lambda \subseteq T_\ell$, $T_\ell(v) \subseteq T_{N\varepsilon}$. Then, the proof of Lemma 2.3 in [4] implies that $\mathscr{E}^N_{0,v}(\eta_N) \leq C\varepsilon^2 d/4t$. Therefore

$$\mathscr{J}_{0,v}(P^N) \leq \frac{C\varepsilon^2 d}{4t},$$

$$\mathscr{J}_{0,v}(\beta_{N,l,v}) \leq \frac{C\varepsilon^2 d}{4t}.$$

Then by lower semicontinuity of $\mathscr{I}_{0,v}$ (or $\mathscr{I}_{0,v_0}$),

$$\mathscr{I}_{0,v_0}(\beta_\ell) \leqq \frac{C\varepsilon^2 d}{4t}.$$

Let $\hat{\beta}_\ell(dy_1, dy_2)$ be the distribution of m_{T_ℓ} and $m_{T_\ell(v_0)}$ with respect to β_ℓ. Our Theorem 2.18 and (3.15) allow us to follow the proof of Theorem 4.3 in [4] and conclude that any $\hat{\beta}$, a limit point of $\hat{\beta}_\ell$ as ℓ goes to infinity, satisfies

$$\int_{\mathbf{R}^2} |h'(y_1) - h'(y_2)|^2 \hat{\beta}(dy_1, dy_2) \leqq \frac{2C\varepsilon^2 d}{t}. \tag{3.16}$$

Now let $\varepsilon \to 0$ in (3.16). This will prove (3.14). $\square$

Proof of Theorem 3.6. Let

$$B_{N,\ell,\varepsilon} = \{v \in S_N^d : T_\ell(v) \subseteq T_{N\varepsilon} \text{ and } (T_\ell + \Lambda) \cap (T_\ell(v) + \Lambda) = \varnothing\}.$$

Let $M_{N,\ell}(v)$ denote the indicator function of the set

$$\{\underline{x} : (m_{T_\ell}(\underline{x}), m_{T_\ell(v)}(\underline{x})) \in A_{\delta/2}\},$$

where

$$A_\delta = \{(x, y) \in \mathbf{R}^2 : |h'(x) - h'(y)| \geqq \delta\}.$$

We simplify the notation by dropping indices N, ℓ and ε, from B and $M(v)$. We will also denote $\lim_{\varepsilon \to 0} \lim_{\ell \to \infty} \sup \lim_{N \to \infty} \sup$ by 'Lims'. After these conventions, Lemma 3.7 says

$$\text{Lims} \sup_{v \in B} E^{P^N} M(v) = 0.$$

Since

$$E^{P^N} \frac{1}{|T|} \sum_{a \in T} w(x_a) \leqq C \tag{3.17}$$

for any $T \subseteq S_N^d$, we obtain

$$\text{Lims} \sup_{v \in B} E^{P^N} M(v) |m_{T_\ell(v)}| = 0.$$

Therefore

$$\text{Lims} \, E^{P^N} \frac{1}{|B|} \sum_{v \in B} M(v) = 0, \tag{3.18}$$

$$\text{Lims} \, E^{P^N} \frac{1}{|B|} \sum_{v \in B} M(v) |m_{T_\ell(v)}| = 0. \tag{3.19}$$

From (3.19) we conclude

$$\text{Lims} \, E^{P^N} \left| m_{T_{N\varepsilon}} - \frac{1}{|B|} \sum_{v \in B} (1 - M(v)) m_{T_\ell(v)} \right| = 0. \tag{3.20}$$

Denoting $\sum_{v \in B} M(v)$ by $\hat{M}$, we can write

$$\text{Lims } P^N\left(\left|1 - \frac{|B|}{|B| - \hat{M}}\right| > \delta'\right) = 0$$

for any $\delta' > 0$ (because of (3.18)). This and the uniform integrability property (3.17) imply

$$\text{Lims } P^N\left(\left|\frac{1}{|B|}\sum_{v \in B}(1 - M(v))m_{T_\ell(v)} - \frac{1}{|B| - \hat{M}}\sum_{v \in B}(1 - M(v))m_{T_\ell(v)}\right| > \delta'\right) = 0$$

for $\delta' > 0$. Let $W = \dfrac{1}{|B| - \hat{M}}\sum_{v \in B}(1 - M(v))m_{T_\ell(v)}$. Then by (3.20),

$$\text{Lims } P^N(|m_{T_{N_\varepsilon}} - W| > \delta') = 0. \tag{3.21}$$

Since W is a convex combination of random variables $m_{T_\ell(v)}$ such that

$$(m_{T_\ell}, m_{T_\ell(v)}) \notin A_{\delta/2},$$

we have

$$(m_{T_\ell}, W) \notin A_{\delta/2}.$$

This is simply because the set $\{y : (x, y) \notin A_{\delta/2}\}$ is an interval for fixed $x \in \mathbf{R}$. Finally (3.21) guarantees

$$\text{Lims } P^N((m_{T_\ell}, m_{T_{N_\varepsilon}}) \in A_\delta) \leqq \text{Lims } P^N((m_{T_{N_\varepsilon}}, W) \in A_{\delta/2}) = 0,$$

and this is precisely the content of the theorem. $\square$

Proof of Theorem 3.1. It follows from what we have just proved and Theorem 3.2

$$\limsup_{\varepsilon \to 0} \limsup_{\ell \to \infty} \limsup_{N \to \infty} P^N\left(\left|\psi_k\left(\frac{1}{|T_\ell|}\sum_{u \in T_\ell}G_0(t_u\underline{x})\right) - \psi_k \circ h'(m_{T_{N_\varepsilon}}(\underline{x}))\right| > \delta\right) = 0.$$

Since ψ_k is bounded, we can replace the convergence in probability with the convergence in expectation. Now all we need to complete the proof is the uniform integrability of the sequence

$$\left\{\frac{1}{|T_\ell|}\sum_{u \in T_\ell}G_0(\tau_u\underline{x})\right\}.$$

But this is just the role of Lemma 3.4 (b). $\square$

If we assume that $\mathscr{G}_\lambda$ has linear structure for all λ (see Definition 2.19), then we can improve Theorem 3.1 by replacing G_0 with any $G \in C^0_{\text{loc}}$. Essentially, this is because $A_G(y)$ is continuous in y and $A_G(y)$ is affine on the interval $\{y : h'(y) = h'(x)\}$ for each fixed x. These two properties of A_G will be shown in Lemma 3.9.

Theorem 3.8. *Let $G \in C^0_{\text{loc}}$. Then*

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} E^{\eta_N d\nu_N}\left|\frac{1}{|T_{N\varepsilon}|}\sum_{a \in T_{N\varepsilon}}G(\tau_a(\underline{x})) - A_G(m_{T_{N_\varepsilon}})\right| = 0.$$

Proof. If we go through the proof of Theorem 3.1, we will see that we only need to verify

$$\lim_{\varepsilon \to 0} \limsup_{\ell \to \infty} \limsup_{N \to \infty} P^N\left(\left|\frac{1}{|T_{N\varepsilon}|} \sum_{v \in T_{N\varepsilon}} A_G(m_{T_\ell(v)}) - A_G(m_{T_{N\varepsilon}})\right| > \delta\right) = 0 \qquad (3.22)$$

for any $\delta > 0$.

We continue to use our compact notations like B and Lims as in the proof of Theorem 3.6. We can write the content of this theorems as

$$\text{Lims} \sup_{v \in B} P^N(|h'(m_{T_{N\varepsilon}}) - h'(m_{T_\ell(v)})| > \delta) = 0 \qquad (3.23)$$

for any $\delta > 0$. Let K denote the (may be degenerate) interval $\{y : h'(y) = h'(m_{T_{N\varepsilon}})\}$ and let C_v be the closest point in K to $m_{T_\ell(v)}$. Since A_G is affine on K, we have

$$A_G\left(\frac{1}{|B|} \sum_{v \in B} C_v\right) = \frac{1}{|B|} \sum_{v \in B} A_G(C_v). \qquad (3.24)$$

It is not hard to see that (3.23) implies

$$\text{Lims} \, P^N\left(\left|\frac{1}{|B|} \sum_{v \in B} C_v - m_{T_{N\varepsilon}}\right| > \delta\right) = 0, \qquad (3.25)$$

$$\text{Lims} \, P^N\left(\left|\frac{1}{|B|} \sum_{v \in B} A(C_v) - \frac{1}{|B|} \sum_{v \in B} A(m_{T_\ell(v)})\right| > \delta\right) = 0 \qquad (3.26)$$

for any $\delta > 0$. We omit the proof of (3.25) and (3.26) which is very much like the proof of Theorem 3.6. Now (3.22) follows from (3.24), (3.25) and (3.26). $\square$

Lemma 3.9. (a) A_G *is a continuous function, and*
(b) A_G *is affine on the interval* $\{y : h'(y) = h'(x)\}$.

Proof. (a) Using the basic entropy estimate, it is not hard to see

$$\sup_{\substack{\mu \in \mathscr{G}_\lambda \\ \lambda \in J}} \int w(x_0) d\mu < \infty,$$

where J is a bounded subset of $\mathbf{R}$. This implies that $\Phi(\mu) = \int x_0 d\mu$ is a continuous function on the space $\bigcup_{\lambda \in J} \mathscr{G}_\lambda$. Now let $y_n \to y$ and $y_n = \int x_0 d\mu_n$ for some $\mu_n \in \bigcup_\lambda \mathscr{G}_\lambda$. Then if μ is any weak limit point of the sequence μ_n, we have

$$y = \int x_0 d\mu \quad \text{and} \quad \mu \in \bigcup_\lambda \mathscr{G}_\lambda.$$

Thus

$$\lim_{n \to \infty} A_G(y_n) = \lim_{n \to \infty} \int G d\mu_n = \int G d\mu = A_G(y).$$

(b) This follows from linearity of $\int G d\mu$ in μ. $\square$

4. Large Deviations for Empirical Measures

In Sect. 5 we will use Theorem 3.1 to derive the hydrodynamic equation. This requires to replace back ψ_k with h'. This is the object of Theorem 4.1.

Let f_N be the density relative to dv_N of a random variable on $\mathbf{R}^{S_N^d}$ such that

$$\sup_N \frac{1}{N^d} \int f_N \log f_N dv_N \leqq C.$$

Let Q_N be the law of the empirical measure

$$\mu = \frac{1}{N^d} \sum_{a \in S_N^d} x_a \delta_{a/N},$$

where x_a are distributed according to $f_N dv_N$.

Theorem 4.1. *For any limit Q of Q_N, we have*

$$Q\{\mu : \mu \ll d\theta\} = 1$$

and

$$E^Q\left[\int h\left(\frac{d\mu}{d\theta}\right)\right] \leqq C.$$

Here, $d\theta$ denotes the probability Haar measure on the d-dimensional torus.
The proof of this follows Lemma 6.3 in [4] which uses the large deviation property of the empirical measures. When the interaction F is not present (i.e. v_N is a product measure), the large deviation property is shown in [13]. In Theorem 4.2 we show the upper bound large deviation property for empirical measures with respect to v_N. This is all we need to carry out the proof of Theorem 4.1.

Theorem 4.2. *The family Q_N has the upper bound large deviation property with the rate function*

$$J(\mu) = \begin{cases} \int h\left(\frac{d\mu}{d\theta}\right) d\theta & \text{if } \mu \ll d\theta \\ +\infty & \text{otherwise} \end{cases},$$

We recall the function φ defined by

$$\varphi(\lambda) = \lim_{n \to \infty} \frac{1}{|T_n|} \log \int \exp\left(\lambda \sum_{a \in T_n} x_a\right) dv_N.$$

We first prove

Theorem 4.3. *Let f be a continuous function on the d-dimensional torus. Then*

$$\lim_{N \to \infty} \frac{1}{N^d} \log E^{v_N} \exp\left(\sum_{a \in S_N^d} f(a/N)\right) = \int \varphi(f(\theta)) d\theta. \tag{4.1}$$

Proof. Let $\zeta(f)$ denote the left-hand side of (4.1). The idea is dividing S_N^d into small pieces on which of each $f(a/N)$ is almost a constant. Then we can use φ to calculate $\zeta(f)$.

Let k be a fixed positive integer and let $\mathscr{C}_k^N$ be a collection of cubes of length k, all separated by corridors of width ℓ, where ℓ is the length of Λ (Λ is the range of F). Let Γ denote the union of the cubes in $\mathscr{C}_k^N$ and let M_k^N be the total number of

them. We may construct $\mathscr{C}_k^N$ such that

$$\lim_{N\to\infty} \frac{k^d M_k^N}{N^d} = 1. \tag{4.2}$$

Since f and F are bounded and since $\int e^{c|x|}\rho(dx) < \infty$ for all $c\in\mathbf{R}(\rho(dx) = e^{-\phi(x)}dx)$, we may drop all terms with x_a, $a\in S_N^d - \Gamma$ in (4.1), and obtain

$$\zeta(f) = \lim_{N\to\infty} \frac{1}{N^d} \log \int \prod_{T\in\mathscr{C}_R^N}\left[\exp\left(\sum_{a\in T} f(a/N)x_a + \mathscr{H}_{F,T}(\underline{x})\right)\right]\rho^T(d\underline{x}) - \psi(F). \tag{4.3}$$

Let $\omega(\delta) = \sup|f(t+h) - f(t)|$, where the supremum is over t and over $|h| < \delta$. Let $b(T)$ denote the center of T. In (4.3) we like to replace $f(a/N)$ with $f(b(T)/N)$. Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then by the Schwartz inequality

$$\int \exp\left[\sum_{a\in T}\left(f\left(\frac{a}{N}\right)x_a + \mathscr{H}_{F,T}(\underline{x})\right)\right]d\rho^T \leq \mathscr{A}(N,T,p)\mathscr{B}(N,T,p), \tag{4.4}$$

where

$$\mathscr{A}(N,T,p) = \left[\int\exp\left(f\left(\frac{b(T)}{N}\right)\sum_{a\in T}x_a + p\mathscr{H}_{F,T}(\underline{x})\right)d\rho^T\right]^{1/p},$$

$$\mathscr{B}(N,T,p) = \left[\int\left(q\omega\left(\sqrt{d}\frac{k}{N}\right)\sum_{a\in T}|x_a|\right)\right]^{1/q}.$$

Furthermore $\mathscr{A}(N,T,p) \leq \mathscr{A}'(N,T,p)\exp(|T|(p-1)/p\|F\|_\infty)$, where

$$\mathscr{A}'(N,T,p) = \left[\int\exp\left(pf\left(\frac{b(T)}{N}\right)\sum_{a\in T}x_a + \mathscr{H}_{F,T}(\underline{x})\right)d\rho^T\right]^{1/p},$$

and

$$\limsup_{N\to\infty}\frac{1}{N^d}\log\prod_{T\in\mathscr{C}_k^N}\mathscr{B}(N,T,p) = \limsup_{N\to\infty}\frac{M_k^N k^d}{N^d q}\log\int e^{q\omega\sqrt{k}/N)|x|}\rho(dx) = 0. \tag{4.5}$$

Let $\varphi_T(\lambda) = \frac{1}{|T|}\log E^{v_F,T}\exp\left(\lambda\sum_{a\in T}x_a\right)$. Note that $\varphi_T(\lambda)$ only depends on k the length of T. From (4.4) and (4.5) we conclude

$$\zeta(f) \leq \limsup_{p\to 1}\limsup_{k\to\infty}\limsup_{N\to\infty}\frac{1}{N^d}\sum_{T\in\mathscr{C}_k^N}\log\mathscr{A}'(N,T,p) - \Psi(F).$$

On the other hand

$$\frac{1}{|T|}\log\mathscr{A}'(N,T,p) = \frac{1}{p|T|}\log\int\exp\left(pf\left(\frac{b(T)}{N}\right)\sum_{a\in T}x_a\right)dv_{F,T} + \frac{1}{|T|}\log Z_{F,T}$$

$$= \frac{1}{p}\varphi_T\left(pf\left(\frac{b(T)}{N}\right)\right) + \frac{1}{|T|}\log Z_{F,T}.$$

Thus

$$\zeta(f) \leqq \limsup_{p \to 1} \limsup_{k \to \infty} \limsup_{N \to \infty} \left\{ \frac{1}{p} \frac{1}{N^d} \sum_{T \in \mathscr{C}_k^N} \varphi_T \left(pf\left(\frac{b(T)}{N} \right) \right) + \frac{1}{p} \frac{M_k^N}{N^d} \log Z_{F,T} \right\} - \psi(F)$$

$$= \limsup_{p \to 1} \limsup_{k \to \infty} \frac{1}{p} \int \varphi_T(pf(\theta)) d\theta$$

$$= \int \varphi(f(\theta)) d\theta.$$

We can prove $\zeta(f) \geqq \int \varphi(f(\theta)) d\theta$ in the same way. First we apply the Schwarz inequality to write

$$\int \exp\left(\sum_{a \in T} f\left(\frac{b(T)}{N} \right) x_a + \mathscr{H}_{F,T}(\underline{x}) \right) d\rho^T$$

$$\leqq \left[\int \exp\left(p \sum_{a \in T} f\left(\frac{a}{N} \right) x_a + p\mathscr{H}_{F,T}(\underline{x}) \right) d\rho^T \right]^{1/p} \mathscr{B}(N, T, p).$$

The rest of the proof is analogous to what we did in the above and we omit it.

Proof of Theorem 4.2. Let K be the convex conjugate of ζ. That is

$$K(\mu) = \sup_f \left(\int f d\mu - \zeta(f) \right), \tag{4.6}$$

where the supremum is over all bounded continuous functions. We can obtain the upper bound large deviation estimate (2.1) for the family $\{Q_N\}$ when C is compact by the same argument used in [14] (Lemma 11.3). To extend this to closed sets, we have to establish some kind of exponential tightness (see Stroock [12], page 46). More precisely, we need to show that for any large ℓ there exists a compact set C_ℓ such that

$$\lim_{\ell \to \infty} \limsup_{N \to \infty} \frac{1}{N^d} \log Q_N(C_\ell^c) = -\infty. \tag{4.7}$$

Set $C_\ell = \{\mu: \|\mu\| \leqq \ell\}$, where $\|\mu\|$ denotes the total variation of μ. C_ℓ is compact, and

$$Q_N(C_\ell^c) = v_N\left(\frac{1}{N^d} \sum_a |x_n| \geqq \ell \right) \leqq e^{-\ell N^d} E^{v_N} \exp\left(\sum_a |x_a| \right).$$

Now (4.7) follows from (2.4).

So far we have shown the upper bound large deviation property for the family $\{Q_N\}$ with rate K. We will show in Lemma 5.4 that K is equal to J and J is a rate function. $\quad\square$

Lemma 4.4. $K = J$ *and J is a rate function.*

Proof. Step 1. Let f be a bounded measurable function. By Lusin's theorem, for any $\varepsilon > 0$, there is a closed subset of torus D_ε and a bounded continuous function f_ε such that $\mu(D_\varepsilon^c) + |D_\varepsilon^c| < \varepsilon$ ($|\cdot|$ denotes the Haar measure on torus) and $f_\varepsilon|_{D_\varepsilon} = f|_{D_\varepsilon}$. From this we see that the supremum in (4.6) can be extended to all bounded measurable functions.

Step 2. Let $d\mu$ be absolutely continuous with respect to $d\theta$, $g = d\mu/d\theta$. Set $g_n = (-n) \vee (g \wedge n)$ and $f_n = h'(g_n)$. If $-n < g(x) < n$, then $g_n(x) = g(x)$ and $f_n(x) = h'(g(x))$. Therefore

$$\int_{-n < g < n} h(g) = \int_{-n < g < n} [g f_n - \varphi(f_n)] \leqq \sup_f \int (gf - \varphi(f)) = K(\mu).$$

On the other hand, it is easy to see $K(\mu) \leqq \int h(g)$. Thus

$$K(\mu) = J(\mu) \quad \text{if} \quad \mu \ll d\theta.$$

Final Step. We will finish the proof by showing $K(\mu) < \infty$ implies that $\mu \ll d\theta$. Since K is convex and translation invariant, then

$$K(k_\varepsilon * \mu) \leqq K(\mu),$$

where $\{k_\varepsilon\}$ is an *approximation to identity*. Therefore, if $K(\mu) < \infty$, then by step 2,

$$\sup_{\varepsilon > 0} \int h(k_\varepsilon * \mu) < \infty.$$

On the other hand $\lim_{|x| \to \infty} (h(x))/|x| = +\infty$. In fact if $h(x_n) \leqq C x_n$ for some sequence $x_n \uparrow + \infty$, and some positive constant C, then $h(x) \leqq Cx$ for large x, because h is convex. But this implies that $\varphi(y)$ the convex conjugate of h is infinite for $y > C$, and this is impossible. Therefore the sequence $\{k_\varepsilon * \mu\}$ is uniformly integrable, and this implies that μ is absolutely continuous.

J is a rate function because

$$\lim_{|x| \to \infty} \frac{h(x)}{|x|} = + \infty. \quad \square$$

5. Hydrodynamic Limit

Let f_N^0 be the initial density with respect to the Gibbs measure v_N. The measure v_N is the Gibbs measure with periodic boundary conditions corresponding to the interaction field F, where F is bounded, continuously differentiable and has bounded first derivatives. Then the evolution governed by $\mathscr{L}_N$ and starting from f_N^0 at time $t = 0$, will give us a probability measure Q_N on the space $C([0, \infty), \mathbf{R}^{N^d})$. Let $\mathscr{M}_\ell$ denote the compact metric space (under weak convergence) of all signed measures of total variation at most ℓ. For each N, and the set of trajectories $\{x_a(t) : a \in S_N^d\}$ we define

$$\mu(t) = \frac{1}{N^d} \sum_{a \in S_N^d} x_a(t) \delta_{a/N} \tag{5.1}$$

in the space $\cup_\ell \mathscr{M}_\ell$. This, as a function of $t \in [0, t_0]$, is an element of $\Omega = \cup_\ell C([0, t_0], \mathscr{M}_\ell)$, where $C([0, t_0], \mathscr{M}_\ell)$ is the space of weakly continuous maps from $[0, t_0]$ into $\mathscr{M}_\ell$. Then (5.1) introduces a probability measure $\hat{Q}_N$ on the space Ω. It is not hard to see that the sequence $\hat{Q}_N$ is tight. This follows Lemmas 6.1 and 6.2 in [4] adapted to our setting. Let Q be any limit point of the sequence Q_N. Then

Theorem 5.1. Q *is concentrated on the single path* $\mu(t, d\theta) = m(t, \theta)d\theta$ *that is the unique weak solution of*

$$\frac{\partial m}{\partial t} = \frac{1}{2}\Delta h'(m), \quad m(0, \theta) = m_0(\theta) \tag{5.2}$$

satisfying the estimates

$$\int h(m(t, \theta))d\theta \leq C, \tag{5.3}$$

$$\frac{1}{2}\int \int_0^\infty |\nabla h'(m(s, \theta))|^2 ds \, d\theta \leq C. \tag{5.4}$$

for all $t \geq 0$.

Proof. Since the entropy is decreasing, we have

$$\frac{1}{N^d}\int f_N^t \log f_N^t dv_N \leq C$$

for all $t \geq 0$. Therefore, by Theorem 4.1 we have $\mu(t, d\theta) = m(t, \theta)d\theta$ and

$$E^Q \int h(m(t, \theta))d\theta \leq C.$$

This and the uniqueness result (Theorem 5.2) imply (5.3). The inequality (3.16) allows us to follow the proof of Theorem 5.1 and Lemma 6.6 in [4] to conclude (5.4). The uniqueness of the solution of (5.2) under the conditions (5.3) and (5.4) will be shown in Theorem 5.2 below.

We now turn to the proof of (5.2). Let J be a smooth test function. Consider the functional

$$V(t, N, \varepsilon, k; \mu) = \int J(\theta)\mu(t, d\theta) - \int J(\theta)\mu(0, d\theta)$$

$$- \frac{1}{2N^d}\int_0^t \sum_{a \in S_N^d} \Delta J(a/N)\psi_k \circ h'(m_{T_{N_\varepsilon}}(\tau_a \underline{x}(s)))ds.$$

By Itô's formula

$$\frac{1}{N^d}\sum_a J(a/N)x_a(t) - \frac{1}{N^d}\sum_a J(a/N)x_a(0)$$

$$= \frac{1}{2N^d}\sum_a \int_0^t \Delta J(a/N)G_0(\tau_a \underline{x}(s))ds + M_N(t) + r_N,$$

where M_N is a Martingale and $\lim_{N \to \infty} r_N = 0$ in probability (see the proof of Lemma 3.4.(b) for the way we treated the unboundedness of G_0). r_N is an error coming from replacing $(1/N^2)\sum_j [J(a/N + e_j/N) + J(a/N - e_j/N) - 2J(a/N)]$ with $\Delta J(a/N)$ in Itô's formula. An explicit calculation shows $\lim_{N \to \infty} E^{Q_N}|M_N(t)|^2 = 0$. This and Theorem 3.1 imply

$$\lim_{k \to \infty} \lim_{\varepsilon \to 0} \sup \lim_{N \to \infty} \sup E^{Q_N}|V(t, N, \varepsilon, k; \mu)| = 0.$$

Therefore

$$\lim_{k \to \infty} E^Q |V(t, k; \mu)| = 0,$$

where

$$V(t, k; \mu) = \int J(\theta)\mu(t, d\theta) - \int J(\theta)\mu(0, d\theta)$$

$$- \frac{1}{2} \int_0^t \int \Delta J(\theta)\psi_k \circ h'(m(s, \theta))\,d\theta\,ds.$$

This is because when a/N is close θ_0, then $m_{T_{N\varepsilon}}(\tau_a \underline{x}(s))$ is almost the average of $\mu(s, d\theta)$ over the cube of side 2ε and center θ_0 divided by its volume. This converges to $m(s, \theta_0)$ as ε tends to zero.

Finally, using Lemma 2.21 and Theorem 4.1 we can pass to the limit as $k \to \infty$ and yield

$$E^Q \left| \int J(\theta)\mu(t, d\theta) - \int J(\theta)\mu(0, d\theta) - \frac{1}{2} \int_0^t \int \Delta J(\theta) h'(m(s, \theta))\,ds\,d\theta \right| = 0,$$

and this completes the proof.   □

Theorem 5.2. *The weak solution of (5.2) is unique within the class of functions $m(t, \theta)$ satisfying (5.3) and (5.4).*

Proof. Consider the nonlinear operator $A(m) = -\Delta h'(m)$ with

$$\mathscr{D}(A) = \{m \in H^{-1} \cap L^1 : h'(m) \in H^1\}.$$

The operator A is *maximal and monotone* in the sense that

$$\langle m_1 - m_2, Am_1 - Am_2 \rangle_{H^{-1}} \geqq 0$$

for $m_1, m_2 \in H^{-1}$, and the graph of A as a subset of $H^{-1} \times H^{-1}$ is maximal within the graph monotone operators (see Barbu [1] for more details). Thus we have uniqueness of the solution of (5.2) in the H^{-1}-sense (this follows from the monotonicity property). For the rest of the proof, we verify that if the weak solution of (5.2) satisfies (5.4), then this solution is also a strong solution in the space of H^{-1}.

Let $v(t) = \int_0^t \nabla h'(m(s))\,ds$. Then v is absolutely continuous as a function from $[0, \infty)$ into L^2, and $v'(t) = \nabla h'(m(t))$ for almost all t. Since $m(t, \cdot)$ is a weak solution of (5.2), we have $\frac{1}{2} \operatorname{div} v(t) = m(t) - m(0)$ in distribution. Therefore $\|m(t) - m(s)\|_{H^{-1}}^2 \leqq (d/4)\|v(t) - v(s)\|_{L^2}^2$ and this implies that $m : [0, \infty] \to H^{-1}$ is absolutely continuous and $dm/dt = \frac{1}{2}\Delta h'(m)$ for almost all t, in the H^{-1}-sense.   □

When h' is not strictly increasing, the solutions of (5.2) might be singular. If, however, $m(0) = m_0$ is in a single phase (for example m_0 is in a region on which h'' is uniformly positive), then $m(t)$ will stay in that phase for all $t \geqq 0$. This is evident in the following theorem.

Theorem 5.3. (Maximum Principle). *Suppose that $m_1(t)$ and $m_2(t)$ are two solutions of (5.2) such that $m_1(0), m_2(0) \in L^\infty$ and*

$$m_1(0, \theta) \leqq m_2(0, \theta)$$

for almost all θ, then

$$m_1(t, \theta) \leqq m_2(t, \theta) \tag{5.5}$$

for all t and almost all θ.

Proof. Let $A^\varepsilon u = -\Delta \lambda^\varepsilon(u)$, where

$$\lambda^\varepsilon(x) = \frac{1}{\varepsilon} \int h'(y) k\left(\frac{x-y}{\varepsilon}\right) dy + \varepsilon x,$$

where $k: \mathbf{R} \to \mathbf{R}^+$ is a smooth function with support in the interval $[-1, 1]$. Then $\lambda^\varepsilon(\cdot)$ is smooth with $(d/dx)\lambda^\varepsilon(x) \geqq \varepsilon$ for all x. We also have

$$\lambda^\varepsilon(x) \to h'(x) \tag{5.6}$$

uniformly in bounded subsets of $\mathbf{R}$.

Let $m_i^\varepsilon(t)$ be the unique solution (in H^{-1}-sense) of $(dm/dt) + \frac{1}{2}A^\varepsilon m = 0$ such that $m_i^\varepsilon(0) = m_i(0)$ for $i = 1$ or 2. We then have

$$\frac{d}{dt} \| m_i(t) - m_i^\varepsilon(t) \|_{H^{-1}}^2 = \langle m_i(t) - m_i^\varepsilon(t), A(m_i(t)) - A^\varepsilon(m_i^\varepsilon(t)) \rangle_{H^{-1}}$$

$$= \langle m_i(t) - m_i^\varepsilon(t), h'(m_i(t)) - \lambda^\varepsilon(m_i^\varepsilon(t)) \rangle_{L^2}. \tag{5.7}$$

On the other hand, we can use the maximum principle for solutions m_i^ε to obtain

$$m_1^\varepsilon(t, \theta) \leqq m_2^\varepsilon(t, \theta) \tag{5.8}$$

for all t and θ. We also have $\sup_{t,\theta} |m_i^\varepsilon(t, \theta)| < \infty$ for $i = 1, 2$. Therefore, by (5.6),

$$\lim_{\varepsilon \to 0} \sup_t \| h'(m_i^\varepsilon(t)) - \lambda^\varepsilon(m_i^\varepsilon(t) \|_{L^2} = 0.$$

This allows us to replace $\lambda^\varepsilon(m_i^\varepsilon(t))$ with $h'(m_i^\varepsilon(t))$ in (5.7). So

$$\lim_{\varepsilon \to 0} \sup_t \frac{d}{dt} \| m_i^\varepsilon(t) - m_i(t) \|_{H^{-1}}^2 \leqq 0$$

for $i = 1, 2$. Therefore $m_i^\varepsilon(t)$ converges to $m_i(t)$ in H^{-1}, for all t. This and (5.8) imply (5.5). $\quad \square$

It is of interest to study the asymptotic behavior of the solution $m(t)$ when t goes to infinity. For an account of this problem, see [5], Lecture 26. Here, we confine ourselves to the case $m_0 \in L^\infty$, and we show that the limit of $m(t)$ as t goes to infinity, exists.

Theorem 5.4. *If $m_0 \in L^\infty$, then $\lim_{t \to \infty} m(t) = m(\infty)$ exists in H^{-1}, and we have*

$$h'(m(\infty, \theta_0)) = h'(\int m_0(\theta) d\theta) \tag{5.9}$$

for almost all θ_0.

Proof. Since $m_0 \in L^\infty$, we have $\sup_t \| m(t) \|_{L^\infty} < \infty$. Therefore $\{m(t): t \geqq 0\}$ is a precompact subset of H^{-1}. Taking into account the condition (5.4), we may choose

a sequence of $t_n \uparrow \infty$, such that

$$m(t_n) \to m(\infty) \quad \text{in } H^{-1}, \tag{5.10}$$

$$\nabla h'(m(t_n)) \to 0 \quad \text{in } L^2, \tag{5.11}$$

for some $m(\infty) \in H^{-1}$. Since $\int m(t_n, \theta) d\theta = \int m_0(\theta) d\theta$, we have $\int m(\infty, \theta) d\theta = \int m_0(\theta) d\theta$ and (5.11) guarantees that we also have (5.9).

Finally $\|m(t) - m(\infty)\|_{H^{-1}}$ is nonincreasing in t (because of (5.9), $v(t) = m(\infty)$ is a steady solution of (5.2)), thus $\lim_{t \to \infty} m(t) = m(\infty)$ in H^{-1}.

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