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Quantum 2-Spheres and Big q-Jacobi Polynomials

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Abstract. Orthogonal bases for the algebras of functions of Podles' quantum 2-spheres are explicitly determined in terms of big q-Jacobi polynomials. This gives a group-theoretic interpretation of the symmetric big q-Jacobi polynomials and the symmetric q-Hahn polynomials.

Quantum groups, introduced by Drinfeld [D], Jimbo [J] and Woronowicz [W1], are now realized to provide a good framework for q-analogues of special functions. The little q-Jacobi polynomials were the first example of q-orthogonal polynomials to be understood by quantum groups. It was found by Vaksman–Soibelman [VS], Masuda et al. [M0] and Koornwinder [K1]that they naturally appear as matrix elements of the irreducible unitary representations of the quantum group $SU_q(2)$ (see also [K0] and [M1]). Up to now, it is also known by Kirillov–Reshetikhin [KR] and Koelink–Koornwinder [KK] that the Clebsch–Gordan coefficients for $SU_q(2)$ are expressed in terms of the q-Hahn polynomials.

In this paper, we will show that the *big q-Jacobi polynomials* $P_n^{(\alpha,\alpha)}(x;c,d;q)$ *of symmetric type* appear as spherical functions on the *quantum 2-spheres of Podles.* Quantum 2-spheres are studied by P. Podles [P] from the viewpoint of operator algebra theory. He also gives the irreducible decomposition of their algebras of functions. We will determine explicitly their orthogonal bases in terms of the big q-Jacobi polynomials.

Throughout this paper, we denote by G the quantum group $SU_q(2)$, where q is a *real number* with $0<q<1$. The algebra of functions $A(G)$ is a Hopf algebra over $\mathbb{C}$ with a $*$-operation. As for quantum groups, we will follow the notation and the terminology of [M1].

Podles' quantum 2-spheres are a family of quantum G-spaces. A *quantum G-space X* is a quantum space "on which the quantum group G acts." This means that the algebra of functions $A(X)$ on X has the structure of a left (or right) $A(G)$-comodule such that the structure mapping $L_G: A(X)\to A(G)\otimes_{\mathbb{C}} A(X)$ is a $\mathbb{C}$-algebra homomorphism. (When we consider the real structure of X, we also require that $A(X)$ has a $*$-operation and that L_G is compatible with the $*$-structure.)

It is natural to expect that a typical example of a quantum G-space will appear as a quotient space G/K by some quantum subgroup K of G. However, Podles' quantum 2-spheres are not necessarily realized as quotient spaces of the form G/K while they have some properties of "homogeneous spaces." This phenomenon is peculiar to quantum geometry, as is closely related to the fact that quantum groups cannot contain so many quantum subgroups as in the classical case. From this point of view, it is remarkable that Podles' quantum 2-spheres allow us a geometric interpretation of big q-Jacobi polynomials. Big q-Jacobi polynomials are also characteristic of q-analysis in the sense that they are connected with the essential gap between the origin and the other points.

For details on the quantum group $G = SU_q(2)$ and its unitary representations, we refer the reader to [M1].

1. Quantum 2-Spheres $S_q^2 = S_q^2(c, d)$

First we give a definition of the quantum 2-sphere $S_q^2 = S_q^2(c, d)$. Hereafter, we fix two *real constants* c and d.

The algebra of functions $A(S_q^2)$ on $S_q^2 = S_q^2(c, d)$ is defined as the $\mathbb{C}$-algebra generated by three elements ξ, z and η with the relations

$$z\xi = q^2\xi z, \quad \eta z = q^2 z\eta, \quad q\xi\eta = -(c-z)(d+z), \quad q\eta\xi = -(c-q^2z)(d+q^2z). \tag{1.1}$$

We endow this algebra with a $*$-structure such that

$$\xi^* = -q^{-1}\eta, \quad \eta^* = -q\xi, \quad z^* = z. \tag{1.2}$$

Our definition of quantum 2-spheres is apparently different from that of Podles [P]. But a suitable change of generator systems for the algebra $A(S_q^2)$ shows that the two definitions make no essential differences (see Remark 2 below).

Recall that the algebra of functions $A(SU_q(2))$ on the quantum group $G = SU_q(2)$ is a $*$-Hopf algebra $\mathbb{C}[x, u, v, y]$ with the defining relations

$$\begin{cases} ux = qxu, \ vx = qxv, \ yu = quy, \ yv = qvy, \\ vu = uv \quad \text{and} \quad xy - q^{-1}uv = yx - quv = 1. \end{cases} \tag{1.3}$$

The coproduct $\Delta\colon A(SU_q(2)) \to A(SU_q(2)) \otimes_{\mathbb{C}} A(SU_q(2))$ is defined by $\Delta(x) = x \otimes x + u \otimes v$, $\Delta(u) = x \otimes u + u \otimes y$, $\Delta(v) = v \otimes x + y \otimes v$ and $\Delta(y) = v \otimes u + y \otimes y$. The $*$-structure of $A(SU_q(2))$ is given by $x^* = y$, $u^* = -q^{-1}v$ (see [M1]).

The quantum sphere $S_q^2(c, d)$ has the structure of a quantum $SU_q(2)$-space. By straightforward verification, one can show that there exists a $\mathbb{C}$-algebra homomorphism $L_G\colon A(S_q^2) \to A(SU_q(2)) \otimes_{\mathbb{C}} A(S_q^2)$ such that

$$L_G \begin{pmatrix} \varphi_{-1} \\ \varphi_0 \\ \varphi_1 \end{pmatrix} = (W^{(1)} \otimes 1) \cdot \left(1 \otimes \begin{pmatrix} \varphi_{-1} \\ \varphi_0 \\ \varphi_1 \end{pmatrix} \right) \tag{1.4}$$

for $\varphi_{-1} = \xi$, $\varphi_0 = (1+q^2)^{-1/2}\{c - d - (1+q^2)z\}$ and $\varphi_1 = \eta$. Here $W^{(1)}$ denotes the following representation matrix of the irreducible unitary $A(SU_q(2))$-comodule V_1^L

of spin $l=1$ (see [M1], Theorem 2.8):

$$W^{(1)} = \begin{pmatrix} x^2 & (1+q^2)^{1/2}xu & u^2 \\ (1+q^2)^{1/2}xv & 1+(q+q^{-1})uv & (1+q^2)^{1/2}uy \\ v^2 & (1+q^2)^{1/2}vy & y^2 \end{pmatrix}. \tag{1.5}$$

The homomorphism $L_G: A(S_q^2) \to A(SU_q(2)) \otimes_{\mathbb{C}} A(S_q^2)$ of (1.4) endows the algebra $A(S_q^2)$ with the structure of a left $A(G)$-comodule. Note that L_G is compatible with the $*$-structure.

In the following sections, we also use the action of the quantum universal enveloping algebra $U_q(sl(2;\mathbb{C})) = \mathbb{C}[e, f, k, k^{-1}]$ on $A(S_q^2)$. Recall that the left $A(SU_q(2))$-comodule structure $L_G: A(S_q^2) \to A(SU_q(2)) \otimes_{\mathbb{C}} A(S_q^2)$ gives rise to a right $U_q(sl(2;\mathbb{C}))$-module structure (see [M1]). The action of an element a of $U_q(sl(2;\mathbb{C}))$ is defined by $\hat{a} = (a \otimes \mathrm{id}) \circ L_G: A(S_q^2) \to A(S_q^2)$. By (1.4) and (1.5), one sees that the elements k, e and f act on $A(S_q^2)$ as follows:

$$\begin{cases} \hat{k}(\xi) = q\xi,\ \hat{k}(z) = z,\ \hat{k}(\eta) = q^{-1}\eta, \\ \hat{e}(\xi) = q^{-1/2}\{c - d - (1+q^2)z\},\ \hat{e}(z) = -q^{-1/2}\eta,\ \hat{e}(\eta) = 0, \\ \hat{f}(\xi) = 0,\ \hat{f}(z) = -q^{-1/2}\xi,\ \hat{f}(\eta) = q^{-1/2}\{c - d - (1+q^2)z\}. \end{cases} \tag{1.6}$$

Note that $\hat{k}$ is an algebra automorphism of $A(S_q^2)$ and that $\hat{e}$ and $\hat{f}$ are twisted derivations on $A(S_q^2)$:

$$\hat{e}(ab) = \hat{e}(a)\hat{k}(b) + \hat{k}^{-1}(a)\hat{e}(b), \quad \hat{f}(ab) = \hat{f}(a)\hat{k}(b) + \hat{k}^{-1}(a)\hat{f}(b) \tag{1.7}$$

for any $a, b \in A(S_q^2)$. This follows from the fact that L_G is an algebra homomorphism.

Remark 1. If $(c, d) = (1, 0)$, the algebra $A(S_q^2)$, together with the left $A(SU_q(2))$-comodule structure, is isomorphic to the subalgebra of $A(SU_q(2))$ consisting of all right K-invariants with respect to the diagonal subgroup $K = U(1)$ of $SU_q(2)$. In this sense, the quantum 2-sphere $S_q^2(1,0)$ is identified with the quotient space $SU_q(2)/K$ (cf. [M0]).

Remark 2. In order to see the connection between Podles' definition and ours, it suffices to take the following system X_{-1}, X_1, X_0 of generators for $A(S_q^2)$:

$$X_{-1} = (1+q^2)^{1/2}\xi, \quad X_0 = c - d - (1+q^2)z, \quad X_1 = (1+q^2)^{1/2}\eta. \tag{1.8}$$

In this generator system, the defining relations (1.1) are rewritten in the form

$$X_0X_{-1} - q^2X_{-1}X_0 = (1-q^2)(c-d)X_{-1}, \tag{1.9}$$

$$X_1X_0 - q^2X_0X_1 = (1-q^2)(c-d)X_1, \tag{1.10}$$

$$q(X_1X_{-1} - X_{-1}X_1) = -(1-q^2)X_0^2 + (1-q^2)(c-d)X_0, \tag{1.11}$$

$$qX_{-1}X_1 + q^{-1}X_1X_{-1} = X_0^2 - (c+q^2d)(c+q^{-2}d). \tag{1.12}$$

The $*$-structure of $A(S_q^2)$ is then given by

$$X_{-1}^* = -q^{-1}X_1, \quad X_0^* = X_0, \quad X_1^* = -qX_{-1}. \tag{1.13}$$

This coincides with Podles' definition up to normalization. The substitutions

$X_{-1} = -(\mu + \mu^{-1})e_{-1}$, $X_0 = e_0$, $X_1 = e_1$ and $q = \mu$ recover the defining relations (2) of [P]. It should be remarked here that the coaction of $A(SU_q(2))$ taken in [P] is a right coaction while ours is a left coaction. The conditions of Theorem 1 and Theorem 2 of [P] correspond to the following (1.14.1) and (1.14.2), respectively:

$$\begin{cases} \text{a) } cd \geqq 0 \quad \text{and} \quad (c,d) \neq (0,0), \quad \text{or} \\ \text{b) } c + q^{2k}d = 0 \quad \text{for some} \quad k \in \mathbb{Z} \quad \text{with} \quad |k| > 1. \end{cases} \tag{1.14.1}$$

$$cd \geqq 0 \quad \text{and} \quad (c,d) \neq (0,0). \tag{1.14.2}$$

2. Invariant Functional on $A(S_q^2)$

Let $K = U(1)$ be the diagonal subgroup of $SU_q(2)$ ([M1]). Recall that the algebra of functions on K is the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ with the $*$-structure $t^* = t^{-1}$. The natural epimorphism $\pi_K: A(SU_q(2)) \to A(K)$ is defined by $\pi_K(x) = t$, $\pi_K(u) = \pi_K(v) = 0$ and $\pi_K(y) = t^{-1}$.

For any integer n, we denote by ${}^K A(S_q^2)[n]$ the subspace of $A(S_q^2)$ of all K-invariants relative to the character t^n of K:

$${}^K A(S_q^2)[n] = \{\varphi \in A(S_q^2); L_K(\varphi) = t^n \otimes \varphi\}, \tag{2.1}$$

where $L_K = (\pi_K \otimes \mathrm{id}) \circ L_G: A(S_q^2) \to A(K) \otimes A(S_q^2)$. Then, by an argument similar to that of [M1], one can show that the algebra of functions $A(S_q^2)$ is decomposed into the direct sum

$$A(S_q^2) = \bigoplus_{m \in \mathbb{Z}} {}^K A(S_q^2)[2m]. \tag{2.2}$$

The subalgebra $A(K \backslash S_q^2) = {}^K A(S_q^2)[0]$ of all K-invariant elements is a polynomial ring $\mathbb{C}[z]$ in one variable z. Moreover the subspace ${}^K A(S_q^2)[2m]$ is a free left (or right) $\mathbb{C}[z]$-module of rank one for each $m \in \mathbb{N}$:

$${}^K A(S_q^2)[2m] = \begin{cases} \xi^m \mathbb{C}[z] & \text{if} \quad m \geqq 0, \\ \mathbb{C}[z]\eta^{-m} & \text{if} \quad m \leqq 0. \end{cases} \tag{2.3}$$

Proposition 1 ([P]). *For each $l \in \mathbb{N}$, there exists a unique irreducible $A(SU_q(2))$-subcomodule V_l of $A(S_q^2)$ containing η^l. Moreover the algebra of functions $A(S_q^2)$ is decomposed into the direct sum*

$$A(S_q^2) = \bigoplus_{l \in \mathbb{N}} V_l. \tag{2.4}$$

Proof. For each $d \in \mathbb{N}$, set $F_d = \sum_{i+j+k \leqq d} \mathbb{C}\xi^i z^j \eta^k$. Then, by (1.4), it is easily seen that F_d is a left $A(SU_q(2))$-subcomodule. Since F_d is finite dimensional, F_d is completely reducible. By (1.6) and (1.7), one can show that any element $\varphi \in A(S_q^2)$ satisfying the condition $\hat{e}(\varphi) = 0$ and $\hat{k}(\varphi) = \mathrm{const}\,\varphi$ must have the form $\varphi = a\eta^l$ for some $a \in \mathbb{C}$, $l \in \mathbb{N}$. This shows that F_d contains a unique irreducible $A(SU_q(2))$-subcomodule V_l of dimension $2l+1$ for each $0 \leqq l \leqq d$ and that F_d has a unique irreducible decomposition $F_d = \bigoplus_{l=0}^{d} V_l$. Since d is arbitrary, we obtain the irreducible decomposition (2.4) of $A(S_q^2)$. ■

We say that a linear functional $h: A(S_q^2) \to \mathbb{C}$ is $SU_q(2)$-invariant if $1.h(a) = (\mathrm{id} \otimes h) \circ L_G(a)$ for all $a \in A(S_q^2)$.

Corollary. *There exists a unique $SU_q(2)$-invariant linear functional $h: A(S_q^2) \to \mathbb{C}$ with $h(1) = 1$.*

In fact, the projection $A(S_q^2) \to V_0 = \mathbb{C} \cdot 1$ in the irreducible decomposition (2.4) gives the invariant functional h.

The invariant functional h on $A(S_q^2)$ is represented by the *Jackson integral* on the q-interval $[-d, c]$. Recall that the Jackson integral on the q-interval $[-d, c]$ is defined by

$$\int_{-d}^{c} F(z) d_q z = \int_{0}^{c} F(z) d_q z - \int_{0}^{-d} F(z) d_q z, \tag{2.5}$$

where

$$\int_{0}^{c} F(z) d_q z = c(1-q) \sum_{k \geqq 0} F(cq^k) q^k. \tag{2.6}$$

Proposition 2. *The $SU_q(2)$-invariant functional $h: A(S_q^2) \to \mathbb{C}$ is factored through the projection $A(S_q^2) \to {}^K A(S_q^2)[0] = \mathbb{C}[z]$ of the decomposition (2.2). Suppose that $c + d \neq 0$. Then, for any polynomial $F(z) \in \mathbb{C}[z]$, the value of h is represented by the Jackson integral*

$$h(F(z)) = \frac{1}{c+d} \int_{-d}^{c} F(z) d_{q^2} z. \tag{2.7}$$

Proof. The former half of the Proposition is clear. The invariance of h implies $h(\hat{e}(a)) = 0$ for any $a \in A(S_q^2)$. On the other hand, by (1.6) and (1.7) we have

$$q^{1/2}(1-q^2)\hat{e}(z^n \xi) = -(1-q^{2(n+2)})z^{n+1} + (1-q^{2(n+1)})(c-d)z^n + (1-q^{2n})cdz^{n-1}.$$

Hence the following equality holds for $n = 0, 1, 2, \ldots$:

$$(1-q^{2(n+2)})h(z^{n+1}) - (c-d)(1-q^{2(n+1)})h(z^n) - cd(1-q^{2n})h(z^{n-1}) = 0.$$

Solving this recurrence formula under the initial condition $h(1) = 1$, we have

$$h(z^n) = \frac{c^{n+1} - (-d)^{n+1}}{c+d} \frac{1-q^2}{1-q^{2(n+1)}} \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

This is equivalent to expression (2.7). ■

The invariant functional h gives rise to the following hermitian from $\langle \ , \ \rangle_R$ on $A(S_q^2)$:

$$\langle a, b \rangle_R = h(ab^*) \quad \text{for} \quad a, b \in A(S_q^2). \tag{2.8}$$

By an argument similar to that in Sect. 3 of [M1], one can show that the hermitian form $\langle \ , \ \rangle_R$ is $SU_q(2)$-invariant.

Note that

$$\eta^l(\eta^l)^* = c^l d^l (q^2 z/c; q^2)_l (-q^2 z/d; q^2)_l, \tag{2.9}$$

$$\xi^l(\xi^l)^* = q^{-2l} c^l d^l (z/c; q^{-2})_l (-z/d; q^{-2})_l. \tag{2.10}$$

By these formulae, Proposition 2 implies that the hermitian form $\langle\ ,\ \rangle_R$ is positive definite, if the couple (c, d) satisfies condition (1.14.2).

3. Spherical Functions

As we have seen in Proposition 1, there exists a unique irreducible $A(SU_q(2))$-subcomodule V_l of $A(S_q^2)$ containing η^l for each $l \in \mathbb{N}$. Since V_l is isomorphic to the unitary left $A(SU_q(2))$-comodule V_l^L of spin l of [M1], one can find a unique basis $(\varphi_i^{(l)})_{-l \leqq i \leqq l}$ for V_l such that $\varphi_l^{(l)} = \eta^l$ and

$$L_G(\varphi_i^{(l)}) = \Sigma_{-l \leqq j \leqq l} w_{i,j}^{(l)} \otimes \varphi_j^{(l)} \quad \text{for} \quad -l \leqq i \leqq l. \tag{3.1}$$

Here $w_{i,j}^{(l)} (-l \leqq i, j \leqq l)$ are the matrix elements of the irreducible unitary $A(SU_q(2))$-comodule V_l^L. Note that $\varphi_i^{(l)} \in {}^K A(S_q^2)[-2i]$ for each $-l \leqq i \leqq l$.

Now we give an explicit formula for the *spherical functions* $\varphi_i^{(l)}$ in terms of big q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x; c, d; q)$ $(\alpha, \beta, n \in \mathbb{N})$ defined by

$$P_n^{(\alpha,\beta)}(x; c, d; q) = \sum_{k \geqq 0} \frac{(q^{-n}; q)_k (q^{\alpha+\beta+n+1}; q)_k (q^{\alpha+1} x/c; q)_k}{(q; q)_k (q^{\alpha+1}; q)_k (-q^{\alpha+1} d/c; q)_k} q^k, \tag{3.2}$$

where $(a; q)_k = \prod_{0 \leqq i \leqq k-1} (1 - aq^i)$ for $k \in \mathbb{N}$, $a \in \mathbb{C}$. Our definition of $P_n^{(\alpha,\beta)}(x; c, d; q)$ is different from that of [AA0] by a constant factor.

Theorem 3. *For each $l \in \mathbb{N}$, the spherical functions $\varphi_i^{(l)}$ $(-l \leqq i \leqq l)$ are expressed in terms of the big q-Jacobi polynomials in z as follows.*

Case (I) $-l \leqq i \leqq 0$:

$$\varphi_i^{(l)} = (-c)^{l+i} q^{-1/2(l+i)(l-3i+3)} \begin{bmatrix} 2l \\ l+i \end{bmatrix}_{q^2}^{-1/2} \begin{bmatrix} l \\ l+i \end{bmatrix}_{q^2}$$
$$\cdot (-q^{2(1-i)} d/c; q^2)_{l+i} \xi^{-i} P_{l+i}^{(-i,-i)}(z; c, d; q^2). \tag{3.3}$$

Case (II) $0 \leqq i \leqq l$:

$$\varphi_i^{(l)} = (-c)^{l-i} q^{-1/2(l-i)(l+3i+3)} \begin{bmatrix} 2l \\ l-i \end{bmatrix}_{q^2}^{-1/2} \begin{bmatrix} l \\ l-i \end{bmatrix}_{q^2}$$
$$\cdot (-q^{2(1+i)} d/c; q^2)_{l-i} P_{l-i}^{(i,i)}(z; c, d; q^2) \eta^i. \tag{3.4}$$

Here the symbol $\begin{bmatrix} m \\ n \end{bmatrix}_q$ *stands for the Gauss bionomial coefficient*

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{(q; q)_m}{(q; q)_n (q; q)_{m-n}}. \tag{3.5}$$

Proof. First we consider Case (II). For each $0 \leqq i \leqq l$, we set $\varphi_i^{(l)} = F_{l-i}^{(l)}(z) \eta^i$, where $F_{l-i}^{(l)}(z) \in \mathbb{C}[z]$. Since the $A(SU_q(2))$-comodule V_l is isomorphic to V_l^L of [M1], we have

$$\hat{f}(F_{l-i}^{(l)}(z)\eta^i) = q^{(1/2)-l} \frac{(1 - q^{2(l-i+1)})^{1/2} (1 - q^{2(l+i)})^{1/2}}{1 - q^2} F_{l-i+1}^{(l)}(z) \eta^{i-1}$$

from (4.20) of [M1]. Since $\hat{f}(\xi)=0$, we have

$$\hat{f}(F^{(l)}_{l-i}(z)\eta^i\xi^i)=q^{(1/2)-l+i}\frac{(1-q^{2(l-i+1)})^{1/2}(1-q^{2(l+i)})^{1/2}}{1-q^2}F^{(l)}_{l-i+1}(z)\eta^{i-1}\xi^i.$$

Hence we obtain

$$\mathrm{cd}\,\hat{f}(F^{(l)}_{l-i}(z)(q^2z/c;q^2)_i(-q^2z/d;q^2)_i)=-q^{(3/2)-l+i}\frac{(1-q^{2(l-i+1)})^{1/2}(1-q^{2(l+i)})^{1/2}}{1-q^2}$$
$$\cdot F^{(l)}_{l-i+1}(z)(q^2z/c;q^2)_{i-1}(-q^2z/d;q^2)_{i-1}\xi.$$

As for the action of f, (1.6) and (1.7) implies that $\hat{f}(F(z))=-q^{-1/2}\xi D_{q^2}F(z)$ for any $F(z)\in\mathbb{C}[z]$. Here D_q is the operator defined by $D_qF(z)=(F(z)-F(qz))/(z(1-q))$. Hence we obtain the following recurrence formula for $F^{(l)}_i(z)$:

$$\mathrm{cd}\,D_{q^2}(F^{(l)}_{l-i}(z)(q^2z/c;q^2)_i(-q^2z/d;q^2)_i)=q^{2-l+i}\frac{(1-q^{2(l-i+1)})^{1/2}(1-q^{2(l+i)})^{1/2}}{1-q^2}$$
$$\cdot T_{q^2}\{F^{(l)}_{l-i+1}(z)(q^2z/c;q^2)_{i-1}(-q^2z/d;q^2)_{i-1}\},$$

where the symbol T_q means $T_qF(z)=F(qz)$. From this we have

$$(\mathrm{cd})^{l-i}(T_{q^2}^{-1}D_{q^2})^{l-i}\{(q^2z/c;q^2)_l(-q^2z/d;q^2)_l\}=q^{(1/2)(l-i)(5-l+i)}\frac{(q^2;q^2)^{1/2}_{l-i}(q^2;q^2)^{1/2}_{2l}}{(1-q^2)^{l-i}(q^2;q^2)^{1/2}_{l+i}}$$
$$\cdot F^{(l)}_{l-i}(z)(q^2z/c;q^2)_i(-q^2z/d;q^2)_i.$$

Comparing this with the Rodrigues formula for the big q-Jacobi polynomials (see Appendix), we obtain the desired formula for Case (II). The formula for Case (I) can be derived from (I) by $\varphi^{(l)}_{-i}=(-q)^{-i}(\varphi^{(l)}_i)^*$. The last equality follows from the property of the matrix elements that $(w^{(l)}_{-i,-j})^*=(-q)^{j-i}w^{(l)}_{i,j}$ (see [M1], Theorem 2.8). ■

Remark 3. The *q-Hahn polynomials* $Q_n(x;q^\alpha,q^\beta,N;q)$ are defined by

$$Q_n(x;q^\alpha,q^\beta,N;q)=\sum_{k\geqq 0}\frac{(q^{-n};q)_k(q^{\alpha+\beta+n+1};q)_k(x;q)_k}{(q;q)_k(q^{\alpha+1};q)_k(q^{-N};q)_k}q^k, \tag{3.6}$$

where $n,N\in\mathbb{N}$ and $0\leqq n\leqq N$. If the couple (c,d) satisfies the condition $c+q^{2k}d=0$ for some $k\in\mathbb{N}$, Theorem 3 means that q-Hahn polynomials appear as spherical functions on the quantum 2-sphere $S^2_q(c,d)$. For example, if $0\leqq l<k$ and $0\leqq i\leqq l$, then one has

$$\varphi^{(l)}_i=(-c)^{l-i}q^{-1/2(l-i)(l+3i+3)}\begin{bmatrix}2l\\ l-i\end{bmatrix}^{-1/2}_{q^2}\begin{bmatrix}l\\ l-i\end{bmatrix}_{q^2}$$
$$\cdot(q^{2(1+i-k)};q^2)_{l-i}Q_{l-i}(q^{2(i+1)}z/c;q^{2i},q^{2i},k-i-1;q^2)\eta^i. \tag{3.7}$$

4. Orthogonality Relations and q-Difference Equations

First we show the orthogonality of spherical functions $\varphi^{(l)}_i$ and determine the square length of them.

Proposition 4 (*Orthogonality of Spherical Functions*).

$$\langle \varphi_i^{(l)}, \varphi_j^{(m)} \rangle_R = 0 \quad \text{if} \quad (l,i) \neq (m,j), \tag{4.1}$$

$$\langle \varphi_i^{(l)}, \varphi_i^{(l)} \rangle_R = \frac{1-q^2}{1-q^{2(2l+1)}} \begin{bmatrix} 2l \\ l \end{bmatrix}_{q^2}^{-1} \prod_{r=1}^{l} (c+q^{2r}d)(q^{2r}c+d). \tag{4.2}$$

Proof. Fix two non-negative integers $l, m \in \mathbb{N}$ and consider the $(2l+1) \times (2m+1)$ matrix $J = (\langle \varphi_i^{(l)}, \varphi_j^{(m)} \rangle_R)_{i,j}$. Since the hermitian form $\langle \ , \ \rangle_R$ is $SU_q(2)$-invariant, one has $W^{(l)}J = JW^{(m)}$, where $W^{(l)} = (w_{i,j}^{(l)})_{-l \leqq i,j \leqq l}$. By the irreducibility of V_l^L and V_m^L, one has $J=0$ if $l \neq m$. This proves the orthogonality (4.1) for $l \neq m$. If $l = m$, then J is a scalar matrix. This means that $\langle \varphi_i^{(l)}, \varphi_j^{(l)} \rangle_R = 0$ $(i \neq j)$ and that the square length $\langle \varphi_i^{(l)}, \varphi_i^{(l)} \rangle_R$ $(-l \leqq i \leqq l)$ does not depend on i. The square length $\langle \varphi_l^{(l)}, \varphi_l^{(l)} \rangle_R = \langle \eta^l, \eta^l \rangle_R$ is determined by (2.9), (2.10) and a direct calculation as follows:

$$\begin{aligned}\langle \eta^l, \eta^l \rangle_R &= c^l d^l h((q^2 z/c; q^2)_l (-q^2 z/d; q^2)_l) \\ &= \frac{c^l d^l}{c+d} \int_{-d}^{c} (q^2 z/c; q^2)_l (-q^2 z/d; q^2)_l d_{q^2} z \\ &= \frac{(1-q^2)(q^2;q^2)_l^2}{(q^2;q^2)_{2l+1}} c^l d^l (-q^2 d/c; q^2)_l (-q^2 c/d; q^2)_l,\end{aligned}$$

which proves (4.2). The last equality is a special case of the generalized Beta integral in [AA1]. ■

From Proposition 4, we see that the hermitian form $\langle \ , \ \rangle_R$ is non-degenerate if $c + q^{2k}d \neq 0$ for any $k \in \mathbb{Z} \backslash \{0\}$. If $c + q^{2k}d = 0$ for $k \in \mathbb{Z} \backslash \{0\}$, then $\langle \ , \ \rangle_R$ is non-degenerate on V_l with $l < |k|$ (cf. Theorem 1 and Theorem 2 of [P].)

Combining Proposition 2, Theorem 3 and Proposition 4, we have the orthogonality relation for the big q-Jacobi polynomials $P_n^{(\alpha,\alpha)}(x; c, d; q)$ of symmetric type ([AA0]):

$$\begin{aligned}&\int_{-d}^{c} P_m^{(\alpha,\alpha)}(x;c,d;q) P_n^{(\alpha,\alpha)}(x;c,d;q)(qx/c;q)_\alpha(-qx/d;q)_\alpha d_q x \\ &= \delta_{m,n} q^{1/2n(n+3)+2n\alpha} \frac{(1-q)(q;q)_\alpha^2 (q;q)_n}{(1-q^{2n+2\alpha+1})(q;q)_{n+2\alpha}} \\ &\cdot (c+d) \frac{d^n(-qd/c;q)_\alpha(-qc/d;q)_{n+\alpha}}{c^n(-q^{\alpha+1}d/c;q)_n}.\end{aligned} \tag{4.3}$$

We remark that, if $c + q^k d = 0$ for some $k \in \mathbb{Z} \backslash \{0\}$, the Jackson integral reduces to a finite sum, so that (4.3) implies the orthogonality relation for the q-Hahn polynomials $Q_n(x; q^\alpha, q^\alpha, N; q)$.

The q-difference equation for $P_n^{(\alpha,\alpha)}(x; c, d; q)$ can be also deduced from the action of the Casimir element of $U_q(sl(2;\mathbb{C}))$ as in [M1]. Let C be the *Casimir element* of $U_q(sl(2))$:

$$C = \frac{qk^2 + q^{-1}k^{-2} - 2}{(q-q^{-1})^2} + fe. \tag{4.4}$$

Then the action $\hat{C}: A(S_q^2) \to A(S_q^2)$ of C is given by

$$\hat{C} = \frac{q\hat{k}^2 + q^{-1}\hat{k}^{-2} - 2}{(q-q^{-1})^2} + \hat{e}\hat{f}.$$

It is known by [M1] that the Casimir operator $\hat{C}$ has eigenvalue $(q^{2l+1} + q^{-2l-1} - 2)/(q-q^{-1})^2$ over the irreducible representation of spin l. Hence one has

$$\hat{C}\varphi_i^{(l)} = \frac{q^{2l+1} + q^{-2l-1} - 2}{(q-q^{-1})^2}\varphi_i^{(l)} \quad \text{for} \quad i = -l, -l+1, \dots, l. \tag{4.5}$$

On the other hand, lengthy calculations show the following.

Proposition 5. *For each* $\alpha \in \mathbb{N}$, *define a* q*-difference operator* $Q_\alpha: \mathbb{C}[z] \to \mathbb{C}[z]$ *by*

$$\begin{aligned} Q_\alpha = \frac{-q^{-2\alpha-1}}{(1-q^2)^2 z^2}\{&(c - q^{2(\alpha+1)}z)(d + q^{2(\alpha+1)}z)T_{q^2} \\ &- (1+q^2)cd - (1+q^{2\alpha})q^2(c-d)z + 2q^{2\alpha+3}z^2 + q^2(c-z)(d+z)T_{q^2}^{-1}\}, \end{aligned} \tag{4.6}$$

where $T_q F(z) = F(qz)$. *Then the action of the Casimir element on* $K_A(S_q^2)[2m]$ $(m \in \mathbb{Z})$ *is given by*

$$\hat{C}(\xi^m F(z)) = \xi^m(Q_m F(z)) \qquad \text{if} \quad m \geqq 0 \tag{4.7}$$

and

$$\hat{C}(F(z)\eta^{-m}) = (Q_{-m}F(z))\eta^{-m} \quad \text{if} \quad m \leqq 0. \tag{4.8}$$

By Theorem 3 and Proposition 5, we see that the big q-Jacobi polynomial $P_{l-\alpha}^{(\alpha,\alpha)}(z; c, d; q^2)$ satisfies the q-difference equation

$$\left(Q_\alpha - \frac{q^{2l+1} + q^{-2l-1} - 2}{(q-q^{-1})^2}\right)P_{l-\alpha}^{(\alpha,\alpha)}(z; c, d; q^2) = 0 \tag{4.9}$$

for $0 \leqq \alpha \leqq l$. This gives a group-theoretic interpretation of the following q-difference equation for the big q-Jacobi polynomial $P_n^{(\alpha,\alpha)}(x; c, d; q)$:

$$\begin{aligned} &[(c - q^{\alpha+1}x)(d + q^{\alpha+1}x)T_q - (1+q)cd - q(1+q^\alpha)(c-d)x \\ &\quad + q^{-n+1}(1 + q^{2\alpha+2n+1})x^2 + q(c-x)(d+x)T_q^{-1}]P_n^{(\alpha,\alpha)}(x; c, d; q) = 0. \end{aligned} \tag{4.10}$$

Remark 4. The big q-Jacobi polynomials of general type $P_n^{(\alpha,\beta)}(x; c, d; q)$ can be also realized as spherical functions on a certain quantum $SU_q(2)$-space. A group-theoretic interpretation of them will be discussed in detail in our forthcoming paper [NM].

Appendix

Here we present the Rodrigues formula for the big q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x; c, d; q)$ $(n \in \mathbb{N}, 0 < q < 1)$. It is probably known by experts but we could not find the explicit formula in the literature. We also give a direct proof of it. We use the notation of the q-shifted factorials $(x; q)_\infty = \prod_{k \geqq 0}(1 - xq^k)$ and $(x; q)_\alpha = (x; q)_\infty / (q^\alpha x; q)_\infty$ for $\alpha \in \mathbb{C}$.

Proposition A.1. (*Rodrigues Formula for Big q-Jacobi Polynomials*).

$$(D_qT_q^{-1})^n\{(qx/c;q)_{\alpha+n}(-qx/d;q)_{\beta+n}\}$$
$$=\frac{(-1)^n}{(1-q)^nd^n}q^{-n(n+1)/2-\alpha n}(q^{\alpha+1};q)_n(-q^{\alpha+1}d/c;q)_n$$
$$\cdot(qx/c;q)_\alpha(-qx/d;q)_\beta P_n^{(\alpha,\beta)}(x;c,d;q), \tag{A.1}$$

where $D_qF(x)=(F(x)-F(qx))/x(1-q)$ and $T_qF(x)=F(qx)$.

Setting $a=q^\alpha$ and $b=q^\beta$, we consider the following q-hypergeometric series;

$$P_n(x;a,b;c,d;q)=\sum_{k=0}^{n}\frac{(q^{-n};q)_k(abq^{n+1};q)_k(axq/c;q)_k}{(q;q)_k(aq;q)_k(-adq/c;q)_k}q^k \tag{A.2}$$

for each $n\in\mathbb{N}$. We define the corresponding weight function $w(x;a,b;c,d)$ by

$$w(x;a,b;c,d;q)=\frac{(xq/c;q)_\infty(-xq/d)_\infty}{(axq/c;q)_\infty(-bxq/d)_\infty}. \tag{A.3}$$

Then the Rodrigues formula for $P_n(x;a,b;c,d;q)$ is written as follows:

$$(D_qT_q^{-1})^nw(x;aq^n,bq^n;c,d;q)$$
$$=q^{-n(n+1)/2}\frac{(aq;q)_n(-adq/c;q)_n}{(1-q)^n(-ad)^n}w(x;a,b;c,d;q)P_n(x;a,b;c,d;q). \tag{A.4}$$

Lemma A.2 (*Recurrence formula for* (*A.4*)).

$$D_qT_q^{-1}\{w(x;a,b;c,d;q)P_n(x;a,b;c,d;q)\}$$
$$=-\frac{(1-a)(1+ad/c)}{(1-q)ad}w(x;aq^{-1},bq^{-1};c,d;q)P_{n+1}(x;aq^{-1},bq^{-1};c,d;q). \tag{A.5}$$

Proof. Check first that

$$D_qT_q^{-1}\{w(x;a,b;c,d;q)(axq/c;q)_kq^k\}$$
$$=\frac{1}{(1-q)ad}w(x;aq^{-1},bq^{-1};c,d;q)$$
$$\cdot\{(ax/c;q)_{k+1}(1-abq^k)-(ax/c;q)_k(1-aq^k)(1+adq^k/c)\}.$$

Then we have

$$w(x;aq^{-1},bq^{-1};c,d;q)^{-1}D_qT_q^{-1}\{w(x;a,b;c,d;q)P_n(x;a,b;c,d;q)\}$$
$$=\frac{1}{(1-q)ad}\left\{\sum_k\frac{(q^{-n};q)_k(abq^{n+1};q)_k}{(q;q)_k(aq;q)_k(-adq/c;q)_k}(ax/c;q)_{k+1}(1-abq^k)\right.$$
$$\left.-\sum_k\frac{(q^{-n};q)_k(abq^{n+1};q)_k}{(q;q)_k(aq;q)_{k-1}(-adq/c;q)_{k-1}}(ax/c;q)_k\right\}$$
$$=\frac{(1-a)(1+ad/c)}{(1-q)ad}\left\{\sum_k\frac{(q^{-n};q)_{k-1}(abq^{n+1};q)_{k-1}}{(q;q)_{k-1}(a;q)_k(-ad/c;q)_k}(ax/c;q)_k(1-abq^{k-1})\right.$$
$$\left.-\sum_k\frac{(q^{-n};q)_k(abq^{n+1};q)_k}{(q;q)_k(a;q)_k(-ad/c;q)_k}(ax/c;q)_k\right\}$$

$$= \frac{(1-a)(1+ad/c)}{(1-q)ad} \sum_k \frac{(q^{-n};q)_{k-1}(abq^{n+1};q)_{k-1}}{(q;q)_k(a;q)_k(-ad/c;q)_k}(ax/c;q)_k$$

$$\cdot\{(1-q^k)(1-abq^{k-1})-(1-q^{-n+k-1})(1-abq^{n+k})\}$$

$$= -\frac{(1-a)(1+ad/c)}{(1-q)ad} \sum_k \frac{(q^{-n};q)_k(abq^n;q)_k}{(q;q)_k(a;q)_k(-ad/c;q)_k}(ax/c;q)_k q^k$$

$$= -\frac{(1-a)(1+ad/c)}{(1-q)ad} P_n(x;aq^{-1},bq^{-1};c,d;q). \quad \blacksquare$$

Starting from $P_0(x;aq^n,bq^n;c,d;q)\ w(x;aq^n;bq^n;c,d;q)$, one can use Lemma A.2 successively to prove (A.4) and also Proposition A.1.

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