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Commutator Anomalies and the Fock Bundle

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Abstract. We show that the anomalous finite gauge transformations can be realized as linear operators acting on sections of the bundle of fermionic Fock spaces parametrized by vector potentials, and more generally, by splittings of the fermionic one-particle space into a pair of complementary subspaces. On the Lie algebra level we show that the construction leads to the standard formula for the relevant commutator anomalies.

1. Introduction

We shall study the structure of the Fock bundle arising from a system of massless Weyl fermions coupled to an external non-Abelian Yang–Mills field in Hamiltonian framework. Let M be the physical space of odd dimension. We are mainly concerned with the case $\dim M = 3$ but we shall make some remarks about the (easier) case $\dim M = 1$; our discussion of the case $\dim M = 3$ can, without essential complications, be generalized to higher dimensions. We shall assume that M is an oriented compact spin manifold with a given spin structure. Let $\mathscr{A}$ be the space of smooth $\mathbf{g}$ valued vector potentials on M, where $\mathbf{g}$ is the Lie algebra of a compact Lie group G.

Let E be the tensor product (with a fixed inner product in the fibers) of the Dirac spin bundle and a trivial vector bundle over M, with a unitary representation ρ of the gauge group G in the fibers of the latter bundle. Let H be the Hilbert space obtained as a completion from the space of smooth sections of E with respect to the inner product

$$(\psi, \psi') = \int_M \langle \psi(x), \psi'(x) \rangle d(\mathrm{vol}) \tag{1.1}$$

defined by a given volume form on M.

For each $A \in \mathscr{A}$ denote by $W_{A,\lambda}$ the plane in H spanned by eigenvectors of the chiral massless Dirac (Weyl) operator D_A corresponding to eigenvalues greater or equal to $\lambda \in \mathbb{R}$. Each plane $W = W(A, \lambda)$ determines a fermionic Fock space $\mathscr{F}_W$. The Fock space is generated from the vacuum by creation operators a_i^*

corresponding to an orthonormal basis in W labelled by integers $i>0$, and by annihilation operators a_i corresponding to an orthonormal basis in the complement $W^{\perp}$ labelled by integers $i \leq 0$.

Note that in this way we obtain a bundle of Fock spaces labelled by the pairs (A, λ) and not by the vector potentials A. But, since $\mathscr{A}$ is a vector space, we can always choose a plane $W(A)$ as a smooth function of $A \in \mathscr{A}$ such that $W(A)$ does not differ too much from any of the planes $W(A, \lambda)$, $\lambda \in \mathbb{R}$; we shal explain this more precisely in the next section, but essentially "not too much" means that the different Fock representations of the canonical anticommutation relations are equivalent. However, it should be noted that there is *no natural way* to choose the function $W(A)$. On the other hand, the Fock representations corresponding to given A but different λ's are naturally isomorphic up to a phase, [Se].

An important property of the planes $W(A, \lambda)$ is that they belong to an infinite-dimensional Grassmannian manifold Gr_p modelled by *Schatten ideals* L_{2p}, where $2p = 1 + \dim M$. Thus we are lead to examine the structure of a Fock bundle over the manifold Gr_p. The Fock bundle over $\mathscr{A}$ is then obtained as a pull-back with respect to a chosen map $A \mapsto W(A)$. However, we shall see that there is a better way to define the Fock bundle over $\mathscr{A}$ which makes the construction independent of the choice of $W(A)$.

The action of the group $\mathscr{G}$ of gauge transformations on Dirac spinor field defines an action of $\mathscr{G}$ on the Grassmannian Gr_p. The question is this: Can the action be lifted to the Fock bundle $\mathscr{F}$ over Gr_p? We shall see in fact that there are two different natural candidates for the Fock bundle; the difference is that one of the bundles contains an everywhere nonvanishing *vacuum section* whereas the vacuum sector of the second bundle is twisted and there is no everywhere nonvanishing vacuum section. The origin for "commutator anomalies", or Schwinger terms, is the twisting of the vacuum section, [NA], in the second bundle: The group $\mathscr{G}$ acts only through a nontrivial Abelian extension $\hat{\mathscr{G}}$ in the Fock bundle. The Lie algebra of the extension in the case $\dim M = 1$ is essentially an affine Kac–Moody algebra (central extension of the loop algebra L_g) whereas in higher dimensions one has operator valued Schwinger terms, [JJ, M1, F, Si].

The structure and consequences of the Schwinger terms in non-Abelian gauge theory in $3+1$ dimensions has been discussed, after the revival of the subject in [M1, F], in several papers in both space-time and Hamilton formulation; see, e.g., [BG, CS, DT, FHK, HS, J, NS, R, RSF, Se, Y], and references in those papers. The existence of this type of Schwinger terms was indicated in perturbative calculations already in [JJ].

The novel aspects of the present paper as compared to other discussions in the literature are the following. As already mentioned, we show that the appearance of the Schwinger terms can be understood in a simple "universal way" using the theory of infinite-dimensional Grassmannian manifolds. One might as well-study Fock space parametrized by other fields (e.g., a Higgs field) as long as the one-particle spaces are "comparable" (they are elements of the same Grassmannian Gr_p). The second important feature is that we are able to show that the gague transformations, including the Schwinger terms, really act as well-defined operators between different fibers of a Fock bundle. In the earlier approaches the (generators of the) gauge transformations are well-defined in some chosen regularization and the Schwinger terms remain finite when the regularization is removed; however, the gauge operators themselves have not been defined as true Hilbert space transformations. Finally, we stress that our method is completely unperturbative.

We shall use the machinery of determinant bundles over infinite-dimensional Grassmannians developed in [PS] and later generalized in [MR] for discussing the commutator anomalies in $3+1$ space-time dimensions. Generalized Fock bundles were introduced in [M2] using the determinant bundle formalism for discussing the gauge group action; however, the Fock spaces studied in [M2] were not the standard ones of the canonical formalism. In this paper we want to show that essentially the same results can be obtained using the canonical formalism.

2. Fock Bundle from Determinant Bundles

Let H be a complex separable Hilbert space (the one-particle space) and $H = H_+ \oplus H_-$ a splitting into a pair of closed infinite-dimensional subspaces. Fix an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ of H such that $e_n \in H_+$ for $n>0$ and $e_n \in H_-$ for $n \leqq 0$. Denote by Gr_p $(p=1,2,\ldots)$ the Grassmannian consisting of closed infinite-dimensional subspaces $W \subset H$ such that

1. the projection $\mathrm{pr}_{H_+}: W \to H_+$ is a Fredholm operator,
2. the projection $\mathrm{pr}_{H_-}: W \to H_-$ belongs to the Schatten ideal L_{2p}.

The Schatten ideal L_{2p} consists by definition of those bounded operators A for which $(A^*A)^p$ has a converging trace.

The manifold Gr_p is an union of disjoint connected components $\mathrm{Gr}_p^{(k)}$ $(k\in\mathbb{Z})$ consisting of planes W such that the Fredholm index dim ker–dim coker of the projection pr_{H_+} is equal to k.

Each linear operator g in H can be written in the block form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.1}$$

with respect to the splitting $H = H_+ \oplus H_-$. Let GL_p be the group of continuous invertible operators g such that b, c are in L_{2p}. The group GL_p acts in a natural way in Gr_p; we can write $\mathrm{Gr}_p = GL_p/B_+$, where B_+ consists of the operators g which $c=0$.

A basis $w = \{w_n\}_{n=1,2,\ldots}$ of $W \in \mathrm{Gr}_p^{(0)}$ is said to be *p-admissible* (with respect to the basis $\{e_n\}_{n>0}$ of H_+) if $w_+ - 1 \in L_p$, where w_+ is the infinite matrix defined by

$$\mathrm{pr}_{H_+} w_i = \sum_{j>0} (w_+)_{ji} e_j. \tag{2.2}$$

The set of all 1-admissible basis of all the planes $W \in \mathrm{Gr}_p^{(0)}$ forms an infinite-dimensional manifold $\mathrm{St}_p^{(0)}$. Any $W \in \mathrm{Gr}_p^{(0)}$ has such a basis, since a Fredholm operator of index zero is of the form invertible + finite rank operator. For the other components $\mathrm{Gr}_p^{(k)}$ the set of admissible basis is defined similarly except that

H_+ is replaced by the plane spanned by the vectors e_n, $n > -k$. (The manifold St_p defined here is different from the St_p employed in [MR].)

We recall from [PS] the geometric construction of the fermionic Fock space as the space of holomorphic sections of a complex line bundle DET_1^* over Gr_1. Let GL^1 be the group of invertible $\mathbb{N} \times \mathbb{N}$ matrices of the type $1 + L_1$. A section of DET_1^* is by definition of a map $\psi: \mathrm{St}^1 \to \mathbb{C}$ such that $\psi(wt) = \psi(w) \cdot \det t$ for each $t \in GL^1$. The vacuum is represented by the holomorphic section ψ_0 which is nonzero only on $\mathrm{St}_1^{(0)}$ and is there given by $\psi_0(w) = \det w_+$. A Fock basis is obtained as follows. Let $S = \{i_1, i_2, \ldots\}$ be an increasing sequence of integers such that $i_n - n = -k$ for $n \gg 0$. Denote by ψ_S the section which is zero on $\mathrm{St}_1^{(m)}$ for $m \neq k$ and on $\mathrm{St}_1^{(k)}$ it is given by $\psi_S(w) = \det w_S$, where w_S is the matrix obtained from the matrix $w_{ij} = \langle w_j, e_i \rangle$ by selecting rows labelled by the integers S. The inner product is defined such that the ψ_S's form an orthonormal basis. In a more standard language

$$\psi_S = a_{j_1}^* \cdots a_{j_s}^* a_{i_1} \cdots a_{i_r} |\mathrm{vac}\rangle, \tag{2.3}$$

where $j_1, \ldots, j_s$ are missing positive integers and $i_1, \ldots, i_r$ are the nonpositive integers in S.

We generalize the discussion above. Each plane $W \in \mathrm{Gr}_p (p \geqq 1)$ determines a Grassmannian $\mathrm{Gr}_1(W)$ consisting of planes W' such that

1. the projection of W' to $W^\perp$ is a Hilbert–Schmidt operator,
2. the projection of W' to W is a Fredholm operator.

We define the Stiefel manifold $\mathrm{St}_1(W)$ for $W \in \mathrm{Gr}_p$ as follows. Choose a basis w of W such that $w \in \mathrm{St}_p$. A basis w' of a plane $W' \in \mathrm{Gr}_1(W)$ belongs to $\mathrm{St}_1(W)$ if it is 1-admissible with respect to w; this property does not depend on the choice of w.

The Fock space $\hat{\mathscr{F}}_W$ consists of complex valued holomorphic functions ψ on $\mathrm{St}_1(W)$ transforming $\psi(wt) = \psi(w) \cdot \det t$ under a change $t \in GL^1$ of basis. The different Fock spaces form a Fock bundle $\hat{\mathscr{F}}$ over the base Gr_p. A section of this bundle is a smooth map $\mathrm{Gr}_p \ni W \mapsto \psi_W \in \hat{\mathscr{F}}_W$, that is, ψ is a function of $W \in \mathrm{Gr}_p$ and $f \in \mathrm{St}_1(W)$ such that $\psi_W(ft) = \psi_W(f) \cdot \det t$ for $t \in GL^1$. However, as we shall see, this is not the correct "physical" Fock bundle. The physical Fock bundle $\mathscr{F}$ is a tensor product of $\hat{\mathscr{F}}$ with a line bundle DET_p over Gr_p, to be defined below.

Given an admissible basis w of $W \in \mathrm{Gr}_p$ the Fock vacuum in $\hat{\mathscr{F}}_W$ is the holomorphic section $\psi(f) = \det f^{(w)}$, $f \in \mathrm{St}_1^{(0)}(W)$, where $f^{(w)}$ is the matrix representing the projection of f to the basis w,

$$f_i = \sum f_{ji}^{(w)} w_j \bmod W^\perp.$$

The construction of the vacuum depends on the choice of the basis w by a multiplicative factor $= \det t^{-1}$, where t is a transformation of basis. This means that, although we have a well-defined *vacuum line bundle* Vac over Gr_p, there is no everywhere nonvanishing global section of Vac.

The total space of the determinant bundle DET_p consists of all pairs $(w, \lambda) \in \mathrm{St}_p \times \mathbb{C}$ modulo the right action of the group GL^1 defined by

$$(w, \lambda) \cdot t = (wt, \lambda \det t^{-1}). \tag{2.4}$$

A section is a complex valued function on St_p satisfying $\psi(wt)=\psi(w)\cdot\det t^{-1}$. The projection to the base Gr_p is given by $(w,\lambda)\mapsto\pi(w)$, where $\pi:\mathrm{St}_p\to\mathrm{Gr}_p$ is the canonical projection.

Let $\mathscr{F}=\hat{\mathscr{F}}\otimes\mathrm{DET}_p$. A section of $\mathscr{F}$ is then a complex valued function $\psi(f,w)$ of $w\in\mathrm{St}_p$ and $f\in\mathrm{St}_1(W)$, where W is the linear span of w, such that

$$\psi(ft,wt')=\psi(f,w)\cdot\det t\cdot\det t'^{-1} \tag{2.5}$$

for $t,t'\in GL^1$, and which is holomorphic in the variable f. $\mathscr{F}$ is the "physical" Fock bundle over Gr_p. It has an everywhere nonvanishing vacuum section given by $\psi(f,w)=\det f^{(w)}$.

Let us now return to the problem of defining the Fock bundle over the space $\mathscr{A}$ of vector potentials. We set $2p=1+\dim M$. Let now $W=W(A,\lambda)$ and $W'=W(A,\lambda')$ for a given $A\in\mathscr{A}$ and for some $\lambda,\lambda'\in\mathbb{R}$. We claim that

$$\mathrm{St}_1(W)=\mathrm{St}_1(W'). \tag{2.6}$$

Let w,w' be 1-admissible basis of W,W', respectively. Since a product of operators of type $1+L_1$ is again in $1+L_1$, it is sufficient to show that w' is 1-admissible relative to w. Suppose for example that $\lambda>\lambda'$. Then $W'=W\oplus V$, where V is finite-dimensional. It follows that

$$w_i'=\sum_j\alpha_{ji}w_j+\sum_j\beta_{ji}w_j^{\perp}, \tag{2.7}$$

where $w^{\perp}$ is a basis of $W^{\perp}$ and β is a matrix of finite rank.

Let $-k,-k'$ be the Fredholm indices of the planes W,W', respectively. Then

$$w_i=\sum_{j>k}a_{ji}e_j+\sum_{j\leqq k}b_{ji}e_j,\quad w_i'=\sum_{j>k'}a_{ji}'e_j+\sum_{j\leqq k'}b_{ji}'e_j.$$

The matrices a,a' belong to $1+L_1$ and $b,b'\in L_{2p}$. On the other hand, from (2.7) it follows that

$$a'=\alpha a+\text{a finite rank matrix}.$$

Since both a' and a are in $1+L_1$ it follows that $\alpha\in 1+L_1$.

Thus we have a well-defined Stiefel manifold $\mathrm{St}_1(A)=\mathrm{St}_1(W(A,\lambda))$ for each $A\in\mathscr{A}$ which does not depend on the choice of λ. The Fock space $\hat{\mathscr{F}}_A$ is now defined as the space of holomorphic functions $\psi:\mathrm{St}_1(A)\to\mathbb{C}$ with $\psi(wt)=\psi(w)\cdot\det t$, $t\in GL^1$, in the same way as the Fock spaces $\hat{\mathscr{F}}_W$ previously.

Given a map $A\mapsto W(A)\in\mathrm{Gr}_p$ we can pull back the determinant bundle over Gr_p to a line bundle over $\mathscr{A}$; taking a tensor product of this line bundle with the bundle of Fock spaces $\mathscr{F}_A$ we obtain a bundle of "physical" Fock spaces $\hat{\mathscr{F}}_A$ over $\mathscr{A}$; however, as already mentioned in the introduction, there is no natural way to choose the function $W(A)$.

3. Gauge Group Action in Fock Bundles

The group GL_p does not act in DET_p: If $w\in\mathrm{St}_p$ then in general $g\cdot w$ is not in St_p. However, we can always find a transformation $q=q(W)$ $(W=\pi(w))$ of the basis such that $gwq^{-1}\in\mathrm{St}_p$. If $p=1$ then $q(W)$ can be chosen such that it depends only on g and not on W. When $p>1$ $q(W)$ depends on W also. If q' is another matrix with the same property as q then $q'=qt$ for some $t\in GL^1$. Let (g,q,λ) be a triple consisting of $g\in GL_p$, of a function $q=q(W)$ such that $gwq(W)^{-1}\in\mathrm{St}_p$ for each $w\in\mathrm{St}_1(W)$, and a function $\lambda:\mathrm{Gr}_p\to\mathbb{C}^\times$. We define an action of (g,q,λ) on DET_p by

$$(g,q,\lambda)\cdot(w,\mu)=(gwq(W)^{-1},\mu\lambda(W)),\tag{3.1}$$

where W is the plane spanned by w. These transformations form a group with the multiplication rule

$$(g,q,\lambda)\cdot(g',q',\lambda)=(gg',q'',\lambda''),\tag{3.2}$$

where $q''(W)=q(g'W)q'(W)$ and $\lambda''(W)=\lambda(g'W)\lambda'(W)$. The normal subgroup N consisting of triples $(1,q(W),\det q(W))$, with $q:\mathrm{Gr}_p\to GL^1$, acts trivially on DET_p and therefore the group $\widehat{GL}_p$ obtained by taking the quotient of the whole group by N acts on DET_p.

The group $\widehat{GL}_p$ is a principal bundle over $\widehat{GL}_p$ with fiber $\mathrm{Map}(\mathrm{Gr}_p,\mathbb{C}^\times)$. The projection into the base is the mapping $(g,q,\lambda)\mapsto g$. GL_p acts in $\hat{\mathscr{F}}$ as follows. On the base Gr_p we have the natural action of GL_p. Let $\psi\in\hat{\mathscr{F}}_W$ for some $W\in\mathrm{Gr}_p$. If $w\in\mathrm{St}_1(W)$ then $gwq(W)^{-1}\in\mathrm{St}_1(gW)$ for any $(g,q,\lambda)\in\widehat{GL}_p$ and therefore we can define an element $\psi'\in\hat{\mathscr{F}}_{gW}$ by

$$\psi'(w')=\psi(g^{-1}w'q(W))\cdot\lambda(g^{-1}W)^{-1},\tag{3.3}$$

where $w'\in\mathrm{St}_1(gW)$. This formula gives a homomorphism of $\widehat{GL}_p$ into the group of invertible bundle maps in $\hat{\mathscr{F}}$.

Let us compute explicitly the commutation relations of the Lie algebra extension $\widehat{gl}_p$ corresponding to the group $\widehat{GL}_p$ in the case $p=2,\dim M=3$. For that we need a more explicit form of the group law near the identity element. If $g\in GL_2$ is near 1 then the block a has an inverse a^{-1}. For $W\in\mathrm{Gr}_2$ let $F:H\to H$ be the linear operator such that

$$F|_W=+1,\quad F|_{W^\perp}=-1.\tag{3.4}$$

Writing

$$F=\begin{pmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{pmatrix}\tag{3.5}$$

with respect to the decomposition $H=H_+\oplus H_-$, the off-diagonal blocks belong to L_4 whereas $F_{11}-1\in L_2$, $F_{22}+1\in L_2$, [MR].

Lemma 3.6. *Let* $w\in\mathrm{St}_1(W)$, $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, *$F$ the operator describing the plane* $W\in\mathrm{Gr}_2$, *and* $q=a+\frac{1}{2}bF_{21}$. *Then for g in a small neighborhood of* 1 (*which does not depend on* W,w) *the operator q is invertible and gwq^{-1} belongs to* $\mathrm{St}_1(gW)$.

Proof. Since F is unitary, the operator norm of F_{21} is smaller or equal to one. Since $\|A\|\leqq\|A\|_p$ for $p\geqq 1$, the operator norm of bF_{21} is small when b is near zero in L_4 norm; on the other hand, a is near 1 when g is in a small neighborhood of the unity, and therefore $a+\frac{1}{2}bF_{21}$ is invertible when g is near unity.

For proving the second statement let us assume first that w is an admissible basis of W. Now we have

$$(gwq^{-1})_+ = (aw_+ + bw_-)q^{-1} = a(w_+ + a^{-1}bw_-)(1 + \tfrac{1}{2}a^{-1}bF_{21})^{-1}a^{-1}. \tag{3.7}$$

This operator is in $1 + L_1$ iff the operator $(1 + a^{-1}bw_-)(1 + \frac{1}{2}a^{-1}bF_{21})^{-1}$ is $1 +$ a trace class operator. When b is small we can expand the second factor as a power series in b,

$$(1 + \tfrac{1}{2}a^{-1}bF_{21})^{-1} = 1 - \tfrac{1}{2}a^{-1}bF_{21} + \cdots,$$

where the rest is a trace class operator, since bF_{21} is a Hilbert–Schmidt operator. Thus, modulo a trace class operator,

$$\begin{aligned}&(1 + a^{-1}bw_-)(1 + \tfrac{1}{2}a^{-1}bF_{21})^{-1}\\ &\quad = 1 + a^{-1}b(w_- - \tfrac{1}{2}F_{21}) - a^{-1}bw_- \cdot \tfrac{1}{2}a^{-1}bF_{21} + \cdots.\end{aligned}$$

The last term is in L_1 since $w_-, b, F_{21} \in L_4$. In the second term $b(w_- - \frac{1}{2}F_{21}) \in L_1$ by a result in [MR].

Let then $w' \in \mathrm{St}_1(W)$ be arbitrary. Let α be the matrix representing the projection of w' to the admissible basis w of W. We have to show that $gw'q^{-1}$ is admissible relative to gwq^{-1}. But the matrix representing the projection of the former to the latter is equal to $q\alpha q^{-1}$. Since $\alpha - 1 \in L_1$, we have $q\alpha q^{-1} \in 1 + L_1$, and so $gw'q^{-1} \in \mathrm{St}_1(W)$.

From the lemma follows especially that $(g, q, 1)$ belongs to $\widehat{GL}_p$ when g is near 1 and $q = q(W)$ is given by the lemma. Let $g_1, g_2 \in GL_2$ and denote by q_{12} the operator valued function corresponding to the product g_1g_2. We can write

$$(g_1, q_1, 1)(g_2, q_2, 1) = (g_1g_2, q, 1) \equiv (g_1g_2, q_{12}, \omega(g_1, g_2)) \bmod N, \tag{3.8}$$

where $q(W) = q_1(g_2^{-1}W)q_2(W)$ and $\omega(g_1, g_2)$, a *local 2-cocycle* on GL_2, is a $\mathbb{C}^\times$ valued function on Gr_2, given by

$$\begin{aligned}\omega(g_1, g_2)(W) = \det\{&[a_1a_2 + b_1c_2 + \tfrac{1}{2}(a_1b_2 + b_1d_2)F_{21}]\\ &\cdot(a_2 + \tfrac{1}{2}b_2F_{21})^{-1}2(2a_1 + b_1c_2F_{11}\alpha_2 + b_1c_2F_{12}\gamma_2\\ &+ b_1d_2F_{21}\alpha_2 + b_1d_2F_{22}\gamma_2)^{-1}\},\end{aligned} \tag{3.9}$$

where

$$\begin{pmatrix}\alpha_2 & \beta_2\\ \gamma_2 & \delta_2\end{pmatrix} = g_2^{-1}.$$

For any Lie group the commutator of a pair X, Y of Lie algebra elements is obtained from

$$[X_1, X_2] = \frac{1}{2}\frac{d^2}{dtds}e^{tX_1}e^{sX_2}e^{-tX_1}e^{-sX_2}|_{s=t=0}, \tag{3.10}$$

and therefore the Lie algebra cocycle is

$$c(X_1, X_2) = \frac{d^2}{dtds}\omega(e^{tX_1}, e^{sX_2})|_{s=t=0}. \tag{3.11}$$

When $X_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, we obtain

$$\begin{aligned} c(X_1, X_2) &= \operatorname{tr}(b_1c_2 - b_2c_1 + \tfrac{1}{2}b_2c_1F_{11} - \tfrac{1}{2}b_1c_2F_{11} + \tfrac{1}{2}c_2b_1F_{22} - \tfrac{1}{2}c_1b_2F_{22}) \\ &= \tfrac{1}{8}\operatorname{tr}(F-\varepsilon)[[\varepsilon, X_1], [\varepsilon, X_2]] \end{aligned} \tag{3.12}$$

where $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This cocycle was derived earlier in [MR] using a different approach. We have now proven the following theorem:

Theorem 3.13. *The group GL_2 acts in the bundle* DET_2 *over* Gr_2 *such that the action on the base* Gr_2 *is the natural action of GL_2. The Lie algebra $\widehat{\mathbf{gl}}_2$ of $\widehat{GL}_2$, which as a vector space is the direct sum of* $\mathbf{gl}_2$ *with the Abelian Lie algebra* $\operatorname{Map}(\mathrm{Gr}_2, \mathbb{C})$, *is defined by the commutator*

$$[(X_1, \lambda_1), (X_2, \lambda_2)] = ([X_1, X_2], X_1 \cdot \lambda_2 - X_2 \cdot \lambda_1 + c(X_1, X_2)),$$

where $X \cdot \lambda$ is the Lie derivative of the function λ in the direction of the vector field X on the manifold Gr_2 (*with respect to the canonical action of GL_2 on* Gr_2.)

The cocycle (3.12) is a "universal cocycle" corresponding to the 2-cocycle

$$c_0(X, Y, A) = \text{const.} \cdot \int_M \operatorname{tr}(XdY - YdX) \wedge dA \tag{3.14}$$

in three space dimensions, where $X, Y: M \to \mathbf{g}$ are infinitesimal gauge transformations and $A \in \mathscr{A}$, see [M1] and [F].

In the case $p = 1$, corresponding to $\dim M = 1$, we can set $q(W) = a$ in the computations above and the cocycle is in that case simply

$$-\tfrac{1}{8}\operatorname{tr}\varepsilon[[\varepsilon, X_1], [\varepsilon, X_2]] = \operatorname{tr}(b_1c_2 - b_2c_1),$$

which was derived in [L], [PS].

We are now able to discuss the action of GL_2 in the Fock bundles $\mathscr{F}$ and $\hat{\mathscr{F}}$. Let $\psi = \psi(f, w)$ be a section of $\mathscr{F}$. For $g \in GL_2$ define $\psi' = T(g)\psi$ by

$$\psi'(f, \omega) = \psi(g^{-1}fq(W), g^{-1}wq(W)), \tag{3.15}$$

where $W = \pi(w)$ and q is chosen such that $g^{-1}wq$ is admissible. The section ψ' *does not* depend on the choice of q by the formula (2.5). Thus GL_2 acts properly, without any projective factors, in $\mathscr{F}$. However, this is not so for the bundle $\hat{\mathscr{F}}$. By definition of $\mathscr{F}$, $\hat{\mathscr{F}} = \mathscr{F} \otimes \mathrm{DET}_2^*$, and therefore we have a natural action of $\widehat{GL}_2$, not of GL_2, in the bundle $\hat{\mathscr{F}}$.

Theorem 3.16. *The group $\widehat{GL}_2$ acts in the Fock bundle $\hat{\mathscr{F}}$ such that the action on the base is the canonical action of GL_2 and the corresponding Lie algebra cocycle (Schwinger term) is $(-1) \times$ the cocycle (3.12).*

Finally we shall discuss the action of the gauge group $\mathscr{G}$ in the Fock bundle $\hat{\mathscr{F}}$ over $\mathscr{A}$, as defined in the Sect. 2. The group $\mathscr{G}$ acts in H through pointwise multiplication of spinor fields. It is known that $\mathscr{G} \subset GL_2$, when the space M is three-dimensional, see, e.g., [MR]. The extension $\widehat{GL}_2$ of GL_2 defines thus also an

Abelian extension $\hat{\mathscr{G}}$ of $\mathscr{G}$. The action in the Fock bundle is defined in a similar way as was done in the case of Fock spaces parametrized by elements of Gr_2.

Let $g \in \mathscr{G}$. For $A \in \mathscr{A}$ we can choose an infinite matrix $q(A)$ such that $gwq(A)^{-1} \in \mathrm{St}_1(A^{g^{-1}})$ for $w \in \mathrm{St}_1(A)$, where $A^g = g^{-1}Ag + g^{-1}dg$. The action of a triple (g, q, λ), where $\lambda: \mathscr{A} \to \mathbb{C}^\times$, on sections $\psi(A, w)$ of the Fock bundle is then given by

$$[T(g, q, \lambda)\psi](A, w) = \lambda(A^g)^{-1}\psi(A^g, g^{-1}wq(A)). \tag{3.17}$$

The normal subgroup consisting of triples $(1, q, \det q)$, where $q: \mathscr{A} \to GL^1$, acts trivially in $\hat{\mathscr{F}}$ and therefore we have an action of the quotient group $\hat{\mathscr{G}}$ in $\hat{\mathscr{F}}$. The group $\hat{\mathscr{G}}$ is a principal bundle with fiber $\mathrm{Map}(\mathscr{A}, \mathbb{C}^\times)$ over $\mathscr{G}$. The group $\mathrm{Map}(\mathscr{A}, \mathbb{C}^\times)$ is an Abelian normal subgroup of $\hat{\mathscr{G}}$.

Of course, the action of $\mathscr{G}$ in the bundle $\mathscr{F}$ over $\mathscr{A}$ does not involve any commutator anomally, but, moving from $\hat{\mathscr{F}}$ to $\mathscr{F}$ requires a choice of the function $W(A)$.

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