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An Index Theorem for Super Derivations*

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Dedicated to Roland Dobrushin

Abstract. We show that the Chern character given by a super-KMS functional on a quantum algebra can be interpreted in terms of the index of a super derivation on a projection of the algebra.

I. Introduction

The heat kernel representation of the index of a Fredholm operator provides a natural connection between statistical physics (through the partition function) and geometry (regarding the index as a topological invariant). The equivalence in certain statistical mechanics models between the Gibbs variational principles (as expressed through the equation of Dobrushin, Lanford and Ruelle) and the KMS condition provided a fundamental interpretation of trace invariants for operator algebras. Recently the importance of the KMS property (in a super or graded setting) has emerged as a fundamental starting point for the definition of a class of geometric invariants. This subject unifies Connes' noncommutative differential geometry, analysis in an infinite dimensional setting, and ideas from statistical and particle physics. Thus it is especially appropriate to dedicate this note to Roland Dobrushin.

Our purpose here is to investigate an aspect of index theory for a super derivation without assuming that it is generated by a Fredholm operator, without assuming compactness or a bound on the dimension of an underlying manifold, and without necessarily obtaining integral invariants. In place of the standard assumptions we suppose that we are given a super-KMS functional on a quantum algebra $\mathscr{A}$ (defined below). This elementary assumption leads to a Chern character τ for the quantum algebra, and a pairing between τ and the K_0 group of the even part of the algebra. We show here that this pairing can be interpreted in terms of the

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index theory for a super derivation on a projection of the quantum algebra. This is a generalization of a related result when the square of the super derivation is inner and arises from a trace-class heat kernel [C2]. This yields an interpretation of the pairing between $K_0(\mathscr{A}_+)$ and τ in the super-KMS setting, related of course to Connes' theory for entire cyclic cohomology.

2. The Projection of a Quantum Algebra

Let $(\mathscr{A}, \Gamma, \alpha, d)$ be a quantum algebra. Thus $\mathscr{A}$ is a unital C^*-algebra with a $\mathbb{Z}_2$ grading Γ. Also α_t is a continuous, one-parameter automorphism group of $\mathscr{A}$, which commutes with Γ. The infinitesimal generator of α_t is $d^2 = -i \left.\frac{d}{dt} \alpha_t\right|_{t=0}$, where d is a super derivation on $\mathscr{A}$. Let e be an even, self-adjoint, idempotent in the domain of d,

$$e^2 = e = e^* = e^\Gamma \in \mathscr{D}(d) \subset \mathscr{A} \ . \tag{1}$$

Define $\mathscr{A}^e$ as the unital C^* algebra $e\mathscr{A}e$ with the unit element e.

Below we establish the connection between the Chern character τ defined by a super-KMS functional on the quantum algebra $(\mathscr{A}, \Gamma, \alpha, d)$ and the index of a super derivation d^e which we define on $\mathscr{A}^e$.

We define d^e on the domain $\mathscr{D}(d) \cap \mathscr{A}^e$ by

$$d^e a = e(da)e \ . \tag{2}$$

Lemma 1. *d^e is a super derivation on $\mathscr{A}^e$ and $d^e e = 0$.*

Proof. A super derivation has two properties: it anticommutes with the grading Γ and $d^e(ab) = (d^e a)b + a^\Gamma(d^e b)$. The first property is a consequence of (1) and the fact that d is a super derivation. The graded Leibniz rule follows from these properties as well. To establish $d^e e = 0$, we remark that $e^2 = e$ ensures

$$(de)e + ede = de \ . \tag{3}$$

Multiplying on the right by e we obtain

$$d^e e = e(de)e = 0 \ , \tag{4}$$

as claimed.

Let us define an element q, $q \in \mathscr{A}_-$, the odd part of $\mathscr{A}$, by the equation

$$q = ede - (de)e \ . \tag{5}$$

Define a super derivation δ_q on $\mathscr{A}$ by

$$\delta_q(a) = qa - a^\Gamma q \ , \tag{6}$$

which yields a bounded perturbation d_q of d given by

$$d_q = d + \delta_q \ . \tag{7}$$

In [JLW] we established that $D_q = (d_q)^2$ is the infinitesimal generator of a contiuous, even, one parameter automorphism group α_t^q of $\mathscr{A}$. Here $\alpha_t^q = \exp(it\,\mathrm{ad}(D_q))$, and α_t^q commutes with Γ.

Proposition 2. *The group* α_t^q *acts on* $\mathscr{A}^e$ *and* $\alpha_t^q(e)=e$. *Furthermore,* $d_q e=0$.

Proof. Using (4), we note that

$$d_q e = de + qe - eq = de - (de)e - ede \ .$$

By (3), we have $d_q e=0$. It follows that $\alpha_t^q(e)=e$. For $a=eae$, we have

$$\alpha_t^q(a)=\alpha_t^q(eae)=e\alpha_t^q(a)e \ ,$$

so α_t^q maps $e\mathscr{A}e$ into itself.

Proposition 3. *On* $\mathscr{A}^e \cap D(d)$,

$$d_q = d^e \ . \tag{8}$$

Proof. For $a=eae\in\mathscr{D}(d)$ we have by Proposition 2,

$$d_q a = d_q(eae) = e(d_q a)e \ .$$

But $eqe=e(ede-(de)e)e=0$. Thus

$$eqae=eqeae=0 \ ,$$

and likewise $eaqe=0$. Hence

$$d_q a = e(da+qa-aq)e = e(da)e = d^e a \ ,$$

as claimed. Thus we have

Corollary 4. *The derivation* $D_e=(d^e)^2$ *on* $\mathscr{A}^e$ *can be exponentiated to the continuous automorphism group*

$$\alpha_t^e = \exp(itD_e) = \exp(itD_q) \upharpoonright \mathscr{A}^e \ . \tag{9}$$

Furthermore, we now have verified

Theorem 5. $(\mathscr{A}^e, \Gamma, \alpha_t^e, d^e)$ *is a quantum algebra.*

3. The Projection of a Super-KMS Functional

Let ω be a super-KMS functional on the quantum algebra $(\mathscr{A}, \Gamma, \alpha_t, d)$. In other words, ω is a continuous linear functional on $\mathscr{A}$ such that on the subalgebra $\mathscr{A}_\alpha$ of entire elements for $t\to\alpha_t$,

$$\omega(da)=0 \quad \text{and} \quad \omega(ab)=\omega(b^\Gamma \alpha_i(a)) \ . \tag{10}$$

Let ω_e be a functional on $\mathscr{A}^e$ defined by

$$\omega_e(a)=\omega^q(a) \ , \tag{11}$$

where ω^q is the super-KMS functional for $(\mathscr{A}, \Gamma, \alpha_t^q, d_q)$ constructed in [JLW] by perturbation theory and analytic continuation of the wave operator.

Theorem 6. ω_e *is a super-KMS functional for* $(\mathscr{A}^e, \Gamma, \alpha_t^e, d^e)$.

Proof. Clearly ω_e is continuous on $\mathscr{A}^e$. Thus we need only check the analog of (10). Clearly

$$\omega_e(d^e a) = \omega^q(ed^e ae) = \omega^q(ed_q ae) = \omega^q(d_q(eae)) = 0 \ .$$

Also

$$\omega_e(ab) = \omega^q(eabe) = \omega^q(eae^2 be) = \omega^q(eb^\Gamma e\alpha_i^q(a)e)$$
$$= \omega_e(b^\Gamma \alpha_i^q(a)) = \omega_e(b^\Gamma \alpha_i^e(a)) \ .$$

In the last step we use Corollary 4 to replace α_t^q by α_t^e, and we then analytically continue $\omega_e(b^\Gamma \alpha_t^q(a)) = \omega_e(b^\Gamma \alpha_t^e(a))$ to the point $t = i$.

4. The Index of a Super Derivation

The index of the super derivation d^e on a quantum algebra $(\mathscr{A}^e, \Gamma, \alpha^e, d^e)$ is defined with respect to a super-KMS functional ω_e by

$$i_{\omega_e}(d^e) = \omega_e(\mathbf{1}) = \omega_e(e) \ . \tag{12}$$

The existence of the super-KMS property in part replaces a "Fredholm condition;" it ensures the existence of (12), although the index defined by (12) may not be integral.

An even cochain $\{f_n\}$ is *cyclic* if for each n,

$$f_n(a_0, \ldots, a_n) = (-1)^n f_n(a_n^\Gamma, a_0, \ldots, a_{n-1}) \ .$$

Furthermore, and entire cocycle τ is said to be *normalized* if

$$f_{n-1}(a_0, \ldots, a_{n-1}) = \tau_n(\mathbf{1}, a_0, \ldots, a_{n-1}) + (-1)^{n+1} \tau_n(a_0, \ldots, a_{n-1}, \mathbf{1})$$

is cyclic. Connes has proved that every entire cocycle τ is cohomologous to a normalized entire cocycle $\hat{\tau}$ with the property that $\tau_0 = \hat{\tau}_0$, see Lemma 6 of [C2] and its proof. If e is an even projection in $\mathscr{A}$ and τ is an entire cocycle, define

$$\langle \tau, e \rangle = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{n!} \hat{\tau}_{2n}(e, e, \ldots, e) \ , \tag{13}$$

where τ corresponds to $\hat{\tau}$ as above. Now let us take for τ the Chern character on $\mathscr{A}$ constructed from ω in [JLO1, K, JLO2].

Theorem 7. *With the above definitions, $\langle \tau, e \rangle$ is well-defined and*

$$\langle \tau, e \rangle = i_{\omega_e}(d^e) \ . \tag{14}$$

Proof. We remark that $\langle \tau, e \rangle$ is independent of the choice of normalized, cohomologous cocycle $\hat{\tau}$. In fact, any two choices $\hat{\tau}_1$ and $\hat{\tau}_2$ differ by a coboundary ∂G, which is normalized. It follows from Lemma 7 of [C2] that $\langle \partial G, e \rangle = 0$, showing that the pairing $\langle \tau, e \rangle$ is well defined. Similarly, for given e, $\langle \tau, e \rangle$ depends only on the cohomology class of τ.

We deform τ by deforming d to d_q. By [JLW], this deformation yields τ^q cohomologous to τ. But $\tau^q \upharpoonright \mathscr{A}^e$ is the Chern character constructed from ω_e, which by $d^e e = 0$ and $\tau_0^q = \hat{\tau}_0^q = \omega_e$ yields (14).

Remark. Consider the case where $\mathscr{A}$ is an algebra of operators on a Hilbert space $\mathscr{H}$, and where da is given by the graded commutator

$$da=[Q,a]\equiv Qa-a^{\Gamma}Q \ ,$$

with Q an odd, self-adjoint, Fredholm operator on $\mathscr{H}$. Then for $a\in\mathscr{A}^{e}$,

$$d^{e}a=[eQe,a] \ .$$

In this case, Connes established

$$\langle\tau,e\rangle=i_{\omega_e}(d^{e})=\text{Index}\,((eQe)_{+}) \ ,$$

where Index denotes the Atiyah-Singer index and + denotes the component of eQe which maps $\frac{1}{2}(I+\Gamma)\mathscr{H}$ to $\frac{1}{2}(I-\Gamma)\mathscr{H}$.

Let $\mathscr{A}_k=\text{Mat}_k(\mathscr{A})$ denote the algebra of $k\times k$ matrices with entries in $\mathscr{A}$, and let $\mathscr{A}_k^e=e\mathscr{A}_k e$, where $e\in(\mathscr{A}_k)_+$ is a self-adjoint projection. One can perform the above construction as well on $\mathscr{A}_k^e$, as follows. The super derivation $\tilde{d}^e=e\tilde{d}ae$ can be defined, using $(\tilde{d}a)_{ij}=da_{ij}$. Furthermore, a super-KMS functional ω on $(\mathscr{A},\Gamma,\alpha,d)$ extends to $\tilde{\omega}$ on $(\text{Mat}_k(\mathscr{A}),\tilde{\Gamma},\tilde{\alpha},\tilde{d})$, where $\tilde{\Gamma}=\Gamma\otimes\text{Id}$, $\tilde{\alpha}=\alpha\otimes\text{Id}$, by

$$\tilde{\omega}(a)=\sum_{i=1}^{n}\omega(a_{ii}) \ .$$

Clearly $\tilde{\omega}$ is super-KMS, as follows from

$$\tilde{\omega}(\tilde{d}a)=\sum_{i}\omega(\tilde{d}a_{ii})=0 \ ,$$

and

$$\tilde{\omega}(ab)=\sum_{j}\omega((ab)_{jj})=\sum_{j,k}\omega(a_{jk}b_{kj})=\sum_{j,k}\omega(b_{kj}^{\Gamma}\alpha_i(a_{jk}))=\tilde{\omega}(b^{\Gamma}\tilde{\alpha}_i(a)) \ .$$

More generally, if f is an entire cyclic cochain on $\mathscr{A}$, then $\tilde{f}=f\#\text{Tr}$ is an entire cyclic cochain on $\mathscr{A}_k$ [C2]. Note that Connes' growth condition guarantees that the series (13) defining $\langle\tilde{f},e\rangle$ converges. By the previous arguments, we have

Corollary 8. (General Index Formula).

$$\langle\tilde{\tau},e\rangle=\tilde{\omega}_2(\tilde{d}^{e}) \ . \tag{15}$$

This formula is our general index theorem representing the value of τ evaluated on an element of $K_0(\mathscr{A}_+)$ as the index of a super derivation.

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