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Stretched Exponential Decay in a Kinetic Ising Model with Dynamical Constraint

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Dedicated to Roland Dobrushin

Abstract. We show that for the standard nearest neighbor spin-flip dynamics in one dimension with the constraint of constant energy the spin-spin correlation function decays as $\exp[-c\sqrt{t}]$ for large t. We prove an upper and lower bound. The coefficient c of the lower bound is given as the solution of a variational problem and is conjectured to be exact.

1. Introduction

Experimentally it is found that in many materials the decay to equilibrium is not exponentially fast but better fitted by a stretched exponential of the form $\exp[-(t/\tau)^\beta]$, $0<\beta<1$. This is known as the Kohlrausch-Williams-Watts law. In a well known paper Palmer, Stein, Abrahams, and Anderson [1] tried to find a general, material-independent explanation. They argued that in systems consisting of many components which decay in parallel the decay has to be exponential, provided of course each component individually relaxes exponentially. However if there are dynamical constraints, which they imagine to be of a hierarchical nature, then stretched exponential decay is likely to occur.

Presumably the simplest model for parallel decay are independent spin-flips. Each component has then only two possible states and flips between them at random times. This yields exponential decay on the average. We are interested in how the relaxation is modified when a *local* dynamical constaint is imposed. Such models have been used in order to understand properties of glassy dynamics and of the glass transition [2, 3]. We study here the one-dimensional case only. It is physically of its own interest as describing relaxation in solid amorphous polymers [4, 5].

Let us consider then a one-dimensional spin flip process. The spin variable $\sigma(x)$ at site x takes the values ± 1. The state space is therefore $\{-1,1\}^{\mathbb{Z}}$. A spin configuration is denoted by σ. σ^x is the configuration σ with $\sigma(x)$ replaced by $-\sigma(x)$ ($=$spin flip at x). We make a specific choice of nearest neighbor flip rates, namely

$$\begin{aligned} c(x,\sigma) = {} & \tfrac{1}{2}(1-\sigma(x-1)\sigma(x+1)) + \kappa(1+e^{2\beta})(1+\sigma(x-1)\sigma(x+1)) \\ & + \kappa(1-e^{2\beta})\sigma(x)(\sigma(x-1)+\sigma(x+1)), \end{aligned} \tag{1.1}$$

$\kappa \geqq 0$, $\beta \in R$. The (pre-)generator, L, of the spin flip dynamics is given by

$$Lf(\sigma)=\sum_x c(x,\sigma)\,[f(\sigma^x)-f(\sigma)] \tag{1.2}$$

acting on local functions f. e^{Lt}, $t \geqq 0$, is a strongly continuous Markov semigroup on $C(\{-1,1\}^Z, R)$, the space of bounded, continuous functions on $\{-1,1\}^Z$. Therefore the full stochastic jump process corresponding to L can be constructed by standard methods [6]. This spin flip process is denoted by σ_t.

The flip rates in (1.1) were chosen in such a way that an invariant measure of σ_t is the Gibbs measure of the nearest neighbor Ising model with inverse temperature β and zero magnetic field and such that σ_t is reversible with respect to this measure. We prescribe the nearest neighbor energy by

$$H_N(\sigma)=-\tfrac{1}{2}\sum_{x=-N}^{N-1}\sigma(x)\sigma(x+1)\,. \tag{1.3}$$

Then the Gibbs measure is defined by

$$\frac{1}{Z}\exp[-\beta H_N(\sigma)] \tag{1.4}$$

in the limit $N\to\infty$. This Gibbs measure is denoted by $\langle\cdot\rangle_\beta$. Note that $\langle\cdot\rangle_\beta$ does not depend on κ. In fact, if $\kappa>0$, then $\langle\cdot\rangle_\beta$ is the unique invariant measure of σ_t. For $\kappa=0$ the flip rates are independent of β. The measures $\langle\cdot\rangle_\beta$, $-\infty<\beta<\infty$, are then precisely the extreme invariant measures of σ_t [6, Chap. VII, Example 1.48]. [In addition to the $\langle\cdot\rangle_\beta$'s there are four invariant measures which are concentrated on a single configuration. These configurations are $\sigma_1(x)=1$ for all x, $\sigma_2(x)=-\sigma_1(x)$, $\sigma_3(x)=1$ for x even and $\sigma_3(x)=-1$ for x odd, $\sigma_4(x)=-\sigma_3(x)$.] We are interested only in the *stationary* process, i.e. the initial distribution of σ_t is $\langle\cdot\rangle_\beta$ and σ_t is extended to $t<0$ by reversibility. Averages with respect to this stationary σ_t are also denoted by $\langle\cdot\rangle_\beta$.

One of the most basic dynamical quantity of a spin process is the spin-spin correlation function defined by

$$\langle\sigma_t(0)\sigma_0(0)\rangle_\beta \equiv S_{\beta,\kappa}(t)\,. \tag{1.5}$$

Of particular interest is its long time behavior. We note that by reversibility there exists a measure, ν, with total mass one such that

$$S_{\beta,\kappa}(t)=\int_0^\infty \nu(d\lambda)e^{-\lambda|t|}\,. \tag{1.6}$$

Therefore $S_{\beta,\kappa}>0$ and $S_{\beta,\kappa}(t)=S_{\beta,\kappa}(-t)$. Equivalent to the long time behavior is then the small λ behavior of the spectral measure ν.

Our specific model is covered by a theorem of Holley [7]. Provided $\kappa>0$ he establishes that the spin-spin correlation decays exponentially, i.e. there exists a positive constant c such that

$$S_{\beta,\kappa}(t)\leqq e^{-c|t|}\,. \tag{1.7}$$

The problem we want to investigate here is the long time decay of $S_{\beta,\kappa}$ in the border case $\kappa=0$. $\kappa=0$ means that only energy conserving flips are allowed. In this sense the dynamics is *constrained*. The simplest example to see the importance of the constraint is to consider a long segment of +spins. Without constraint it dissolves by spin flips in the interior. However with energy constraint, $\kappa=0$, only the two boundaries of the segment can move. They do so random walk like hence slow. Of course, in the stationary measure such a long segment is exponentially unlikely. We have to understand then how these two effects balance. A crude argument runs as follows: In equilibrium a segment of +spins around the origin of length L has the probability $e^{-f(\beta)L}$ with $f(\beta)$ a free energy. On the other hand the time, t, it takes until the spin at the origin is allowed to flip is typically of order L^2. Hence $S_{\beta,\kappa=0}(t)\cong\exp[-\sqrt{t/\tau}]$. It is not clear how to fix the coefficient τ. We would need the (typical) time of absorption at the origin of a random walk starting at L. Unfortunately, this time has an average which is infinite.

We will prove that

$$S_\beta(t)\equiv S_{\beta,\kappa=0}(t)\cong e^{-c(\beta)\sqrt{t}} \tag{1.8}$$

for long times (a precise statement is given below). The long time decay is a stretched exponential with exponent 1/2.

In passing we briefly have to mention another example which beautifully illustrates the concept of locally constrained dynamics. We consider the two-dimensional Ising model at low temperatures in the +phase with the usual spin-flip dynamics. We would like to know the long time decay of the spin-spin correlation $\langle\sigma_t(0)\sigma_0(0)\rangle_+-\langle\sigma_0(0)\rangle_+^2$. We approximate this expectation by only taking single Peierls contours, γ, around the origin into account. The equilibrium distribution of the contours is $Z^{-1}e^{-\beta|\gamma|}$ with β sufficiently large. $|\gamma|$ denotes here the length of the contour. Thus a very long contour is exponentially unlikely. But, as in our model, its shrinking to zero [which would cause a spin flip of $\sigma_t(0)$] is heavily constrained: the deformation of the contour must go through single spin flips. As before the competition between the unlikely occurrence in equilibrium and the dynamical constraint has to be understood. A rough argument of the same structure as above suggests that the decay is $\exp[-\sqrt{t/\tau}]$ [8]. Sokal and Thomas [9] prove that the generator of the contour dynamics (with rate growing slightly slower with $|\gamma|$ as the physical rates) has no spectral gap at zero.

2. Result

We will prove upper and lower bounds on $\log S_\beta(t)$ proportional to $\sqrt{t}$ but with differing prefactors.

Theorem. *Let $S_\beta(t)$ be the spin-spin correlation function* (1.5) *with* $\kappa=0$. (i) *(upper bound) We have*

$$\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\log S_\beta(t)=-c_+(\beta) \tag{2.1}$$

with

$$c_+(\beta)=2(\sqrt{\pi}(1+\cosh\beta))^{-1}. \tag{2.2}$$

(ii) *(lower bound) We have*

$$\liminf_{t\to\infty} \frac{1}{\sqrt{t}} \log S_\beta(t) = -c_-(\beta). \tag{2.3}$$

$c_-(\beta)$ is defined through a variational formula to be explained below. In particular $c_-(\beta) \leqq 3$ and $c_-(\beta)$ agrees with $c_+(\beta)$ for $|\beta| \to \infty$.

Remark. We conjecture that $c_-(\beta)$ is actually not only a lower bound but the precise asymptotics. A proof would require an extension from an interval to the full line of the large deviation result by Kipnis, Olla, and Varadhan [10] for the weakly asymmetric exclusion process. At present this is not available.

We explain the variational formula for $c_-(\beta)$ and prove boundedness. Let us consider mass transport in one dimension. The density, $\varrho_t(q)$, is governed by the equation of conservation type

$$\frac{\partial}{\partial t} \varrho_t(q) + \frac{\partial}{\partial q}\left[\varrho_t(q)(1-\varrho_t(q))F(q,t) - \frac{\partial}{\partial q}\varrho_t(q)\right] = 0. \tag{2.4}$$

The current, $\varrho(1-\varrho)F - \frac{\partial}{\partial q}\varrho$, has two contributions. There is a diffusive current proportional to the density gradient (Fick's law) and there is a current of the form σF induced by the space-time dependent force field F. The conductivity, σ, is a quadratic function of ϱ which vanishes for $\varrho = 0, 1$. We assume that

$$F(q,t) = -\frac{\partial}{\partial q} V(q,t) \tag{2.5}$$

with $V(q,t)$ once continuously differentiable and of compact support in q. Let $\mathscr{F}$ be the set of all such sorce fields. We solve (2.4) with the initial condition

$$\varrho_0(q) = \begin{cases} 1 & \text{for} \quad q < 0, \\ 0 & \text{for} \quad q \geqq 0. \end{cases} \tag{2.6}$$

The solution is considered only over the time interval $[0, 1]$. With our assumption on F, ϱ_t is well defined, $0 \leqq \varrho_t(q) \leqq 1$, and $\varrho_t(q)$ is continuous in both variables, $t > 0$. Qualitatively mass leaks into the half line $[0, \infty)$. In particular $\varrho_t(q) \to 1$ as $q \to -\infty$ and $\varrho_t(q) \to 0$ as $q \to \infty$.

The constant $c_-(\beta)$ is defined by

$$c_-(\beta) = \inf_{F \in \mathscr{F}} \left\{ (2\log|\coth(\beta/2)|) \int_0^\infty dq\, \varrho_1(q) + \tfrac{1}{4} \int_0^1 dt \int dq F(q,t)^2 \varrho_t(q)(1-\varrho_t(q)) \right\}. \tag{2.7}$$

To understand the structure of the variational problem (2.7) let us first set $F = 0$. Then the initial step diffuses out. The total mass in $[0, \infty)$ (times the prefactor) is an upper bound for $c_-(\beta)$. Imaging now that F is pointing to the left, $F \leqq 0$. This suppresses mass transport into $[0, \infty)$ and makes therefore the first term smaller. On the other hand we have to pay a price through the second term.

We note that the prefactor of $\int_0^\infty d\varrho_1(q)$ diverges logarithmically at $\beta=0$. Thus, in principle, also $c_-(\beta)$ could diverge leaving undecided the long time decay at $\beta=0$. An upper bound on $c_-(\beta)$ independent of β would be achieved, if there exists a solution ϱ_t to (2.4), (2.6) such that $\varrho_1=\varrho_0$ and such that the price to be paid, i.e. the second term, remains finite. [I am grateful to S. R. S. Varadhan for pointing out this argument.] We set then

$$\varrho_t(q)=\begin{cases}\psi(q/\sqrt{t}) & \text{for } 0\leqq t\leqq 1/2,\\ \psi(q/\sqrt{1-t}) & \text{for } 1/2\leqq t\leqq 1.\end{cases} \tag{2.8}$$

We choose ψ such that $\psi(q)=1-\psi(-q)$ and such that $\psi(q)\to 1$ for $q\to-\infty$, $\psi(q)\to 0$ for $q\to\infty$ sufficiently rapidly. The force field is then also of scaling form

$$F(q,t)=\begin{cases}(1/\sqrt{t})F_1(q/\sqrt{t}) & \text{for } 0\leqq t\leqq 1/2,\\ (1/\sqrt{1-t})F_2(q/\sqrt{1-t}) & \text{for } 1/2\leqq t\leqq 1,\end{cases} \tag{2.9}$$

with F_1, F_2 determined by ψ through (2.4), (2.8). By a straightforward computation we obtain

$$c_-(\beta)\leqq\sqrt{2}\int_0^\infty dq(\psi(q)(1-\psi(q)))^{-1}\left\{\psi'(q)^2+\left(\tfrac{1}{2}\int_q^\infty dq_1 q_1\psi'(q_1)\right)^2\right\}. \tag{2.10}$$

We make the ansatz $\psi(q)=\frac{1}{2}e^{-\gamma q}$, $q\geqq 0$, and optimize with respect to γ. This yields

$$c_-(\beta)\leqq 3. \tag{2.11}$$

For comparison, the maximum of $c_+(0)$ at $\beta=0$ is approximately 0.6.

3. Mapping to the Symmetric Exclusion Process

The symmetric exclusion process on Z is a particle dynamics where particles jump with equal probability to a neighboring site provided it is vacant. The occupation variables are $\eta(x)=0,1$. The state space is $\{0,1\}^Z$. η stands for a particle configuration and η^{xy} is the configuration η with occupancies at x and y interchanged ($\equiv$jump either from x to y or from y to x). The (pre-)generator is then

$$Lf=\sum_x(f(\eta^{xx+1})-f(\eta)). \tag{3.1}$$

The exclusion process is denoted by η_t.

From now on we set $\kappa=0$ and fix β. Let us define

$$\eta_t(x)=\tfrac{1}{4}(\sigma_t(x-1)-\sigma_t(x))^2. \tag{3.2}$$

Then, in law, the so defined η_t is the symmetric exclusion process. [Note that by (3.2) σ and $-\sigma$ are mapped to the same particle configuration η.] Expectations are denoted by $\mathbb{E}$. Of course, η_t is stationary and reversible. Under the mapping (3.2) the Gibbs measure (1.4) goes over to the Bernoulli measure for η_t with $\mathbb{E}(\eta_t(x))=(1+e^\beta)^{-1}$.

Let $J([0,t])$ be the current through the bond $(0,1)$ integrated over the time interval $[0,t]$, i.e.

$J([0,t])=$ number of particles which jump from 0 to 1

$-$number of particles which jump from 1 to 0 during $[0,t]$. (3.3)

Then

$$\sigma_t(0)\sigma_0(0)=(-1)^{J([0,t])} \tag{3.4}$$

and

$$S_\beta(t)=\mathbb{E}((-1)^{J([0,t])}). \tag{3.5}$$

(3.5) is not yet a tractable form. Using the stirring process [6, Chap. VIII.4] it becomes possible to average explicitly over the initial measure.

Lemma 3.1. *Let*

$$\zeta(x)=\begin{cases}1 & \text{for} \quad x\leqq 0,\\ 0 & \text{for} \quad x\geqq 1,\end{cases}$$

and let E_ζ be the expectation for symmetric exclusion with $\eta_0=\zeta$. Let $n_t=\sum_{x\geqq 1}\eta_t(x)$ be the number of particles in $[1,\infty)$ at time t. Then

$$S_\beta(t)=\mathbb{E}_\zeta(\exp[-\alpha n_t]) \tag{3.6}$$

with

$$\alpha=2\log|\coth(\beta/2)|. \tag{3.7}$$

For $\beta=0$,

$$S_0(t)=\mathbb{E}_\zeta(\chi(\{n_t=0\})), \tag{3.8}$$

where $\chi(\{\cdot\})$ denotes the indicator function of the set $\{\cdot\}$.

Proof. We represent symmetric exclusion in terms of the stirring process. Initially, there is a particle at each site (labeled by that site). To each bond $(x,x+1)$ we assign independent Poisson processes with rate 1/2. At the event times of the Poisson process for $(x,x+1)$ the particles located at x and $x+1$ are interchanged. Let ξ_t^x be the position of the particle initially at x (i.e. with label x). Then, in law

$$\eta_t(x)=\sum_y \eta(y)\chi(\{\xi_t^x=y\}), \tag{3.9}$$

$\eta_0=\eta$. Thus if in (3.9) the η's are distributed according to a Bernoulli measure, then η_t is a realization of the stationary symmetric exclusion process. Expectation for the stirring process is denoted by $\mathbb{E}_s$ and the initial Bernoulli measure by μ.

We note that a stirring particle initially and finally to the left (or the right) of the origin makes no contribution to the current (3.3). Therefore

$$\mathbb{E}((-1)^{J([0,t])})=\int\mu(d\eta)\mathbb{E}_s((-1)^{I([0,t])}), \tag{3.10}$$

where

$$I([0,t])=\sum_{x\leqq 0}\eta(x)\chi(\{\xi_t^x\geqq 1\})-\sum_{x\geqq 1}\eta(x)\chi(\{\xi_t^x\leqq 0\}). \tag{3.11}$$

Let N_t be the number of stirring particles which are at time t to the right and initially to the left of the origin. This must be the same as the number of stirring particles which are at time t to the left and initially to the right of the origin. We condition on N_t and average over the initial Bernoulli measure μ. Then

$$\begin{aligned}
&\int \mu(d\eta)\mathbb{E}_s((-1)^{I([0,t])}|N_t) \\
&= \int \mu(d\eta)\mathbb{E}_s\Big(\exp\Big[i\pi\Big(\sum_{x\leqq 0}\eta(x)\chi(\{\xi_t^x \geqq 1\}) - \sum_{x\geqq 1}\eta(x)\chi\{\xi_t^x \leqq 0\}\Big)\Big]|N_t\Big) \\
&= \mathbb{E}_s\Big(\int \mu(d\eta)\exp\Big[i\pi\Big(\sum_{x=-N_t+1}^{0}\eta(x) - \sum_{x=1}^{N_t}\eta(x)\Big)\Big]|N_t\Big) \\
&= \exp[-\alpha N_t]\,,
\end{aligned} \tag{3.12}$$

provided $\beta \neq 0$. We conclude that

$$\mathbb{E}((-1)^{J([0,t])}) = \mathbb{E}_s(\exp[-\alpha N_t])\,. \tag{3.13}$$

For $\beta = 0$ the average with respect to μ is different from zero only if $N_t = 0$.

We use again (3.9), this time with $\eta = \zeta$. Then $N_t = n_t$ in distribution and our claim follows. □

4. Upper Bound

We use an inequality of Liggett, which estimates symmetric exclusion in terms of independent random walks. Let

$$f(x_1, \ldots, x_n) = \prod_{j=1}^{n} \exp[-\alpha\chi(\{x_j \geqq 1\})]\,, \tag{4.1}$$

$x_j \in Z$. Clearly f is bounded, symmetric and positive definite in any pair of variables. Therefore [6, VIII, Proposition 1.7] is applicable and implies, in the limit $n \to \infty$,

$$\mathbb{E}_\zeta(e^{-\alpha n_t}) \leqq \mathbb{E}_\zeta^*(e^{-\alpha n_t})\,. \tag{4.2}$$

The right side is the following process. Initially every site $x \leqq 0$ is occupied and every site $x \geqq 1$ is vacant. The particles move then as independent random walks (nearest neighbor jumps with rate one) on Z with transition probability $p_t(x, y)$. n_t is the number of particles in $[1, \infty)$ at time t. Then, with x_t a single random walk,

$$\begin{aligned}
\mathbb{E}_\zeta^*(e^{-\alpha n_t}) &= \prod_{x\leqq 0} \mathbb{E}_x(e^{-\alpha\chi(\{x_t \geqq 1\})}) \\
&= \exp\Big[\sum_{x\leqq 0}\log\Big(1 + (e^{-\alpha} - 1)\sum_{y\geqq 1} p_t(x, y)\Big)\Big] \\
&\leqq \exp\Big[(e^{-\alpha} - 1)\sum_{x\leqq 0}\sum_{y\geqq 1} p_t(x, y)\Big] \\
&= \exp\Big[(e^{-\alpha} - 1)\sum_{x\geqq 1}^{\infty} x p_t(0, x)\Big]\,.
\end{aligned} \tag{4.3}$$

For long times $\sum_{x\geqq 1}^{\infty} x p_t(0, x) \cong \sqrt{t/\pi}$.

5. Lower Bound

Using Jensen's inequality the easy lower bound is

$$S_\beta(t) \geqq \exp[-\alpha \mathbb{E}_\zeta(n_t)]. \tag{5.1}$$

By duality the average is computable in terms of a single random walk as

$$\mathbb{E}_\zeta(n_t) = \sum_{x \leqq 0} \sum_{y \geqq 1} p_t(x, y). \tag{5.2}$$

Comparing with (4.3) this is the same asymptotics with a prefactor α instead of $1 - e^{-\alpha}$. Thus for small α, i.e. $|\beta| \to \infty$, both bounds coincide. On the other hand for $\alpha \to \infty$, i.e. $\beta \to 0$, the lower bound diverges leaving undecided the question of the actual long time behavior.

We improve the lower bound by choosing a new reference measure. The idea is to regulate the mass flowing into $[0, \infty)$ through external forces and to optimize afterwards. To our own surprise this leads to the hydrodynamics of the weakly asymmetric exclusion process. We modify our notation slightly to conform with the standard usage. We have

$$\liminf_{t \to \infty} \frac{1}{\sqrt{t}} \log \mathbb{E}_\zeta(\exp[-\alpha n_t]) = \liminf_{\varepsilon \to 0} \frac{1}{\sqrt{t}} \varepsilon \log \mathbb{E}_\zeta(\exp[-\alpha n_{\varepsilon^{-2}t}]), \tag{5.3}$$

and we may set $t = 1$ at the end.

Let V be the potential of (2.5) with compact support in q. We perturb the symmetric exclusion dynamics by a weak drift of order ε, slowly varying in space (order ε) and time (order ε^2). The new jump rates are

$$\begin{aligned} c_{V,s}(x, x+1, \eta) = {} & \exp[(V(\varepsilon x, \varepsilon^2 s) - V(\varepsilon(x+1), \varepsilon^2 s))/2]\eta(x)(1 - \eta(x+1)) \\ & + \exp[(-V(\varepsilon x, \varepsilon^2 s) + V(\varepsilon(x+1), \varepsilon^2 s))/2](1 - \eta(x))\eta(x+1). \end{aligned} \tag{5.4}$$

As before, the time-dependent generator is

$$L_{V,s} f(\eta) = \sum_x c_{V,s}(x, x+1, \eta)[f(\eta^{xx+1}) - f(\eta)] \tag{5.5}$$

acting on local functions. The path measure for the asymmetric process with initial configuration ζ and over the time interval $[0, \varepsilon^{-2}t]$ is denoted by $\mathbb{P}^V$, the path measure for $V = 0$ by $\mathbb{P}$.

In terms of the new reference measure,

$$\mathbb{E}_\zeta(\exp[-\alpha n_{\varepsilon^{-2}t}]) = \mathbb{E}_\zeta^V\left(\frac{d\mathbb{P}}{d\mathbb{P}^V} \exp[-\alpha n_{\varepsilon^{-2}t}]\right). \tag{5.6}$$

The Radon-Nikodym derivative is given by

$$\begin{aligned} \frac{d\mathbb{P}}{d\mathbb{P}^V} = \exp\Bigg[\int_0^{\varepsilon^{-2}t} \sum_x \{ & -\tfrac{1}{2}(V(\varepsilon x, \varepsilon^2 s) - V(\varepsilon(x+1), \varepsilon^2 s)) J_{(x,x+1)}(ds) \\ & + (\exp[(V(\varepsilon x, \varepsilon^2 s) - V(\varepsilon(x+1), \varepsilon^2 s))/2] - 1)\eta_s(x)(1 - \eta_s(x+1))ds \\ & + (\exp[(-V(\varepsilon x, \varepsilon^2 s) + V(\varepsilon(x+1), \varepsilon^2 s))/2] - 1)(1 - \eta_s(x))\eta_s(x+1)ds \} \Bigg]. \end{aligned} \tag{5.7}$$

$J_{(x,x+1)}(ds)$ is the actual current through the bond $(x, x+1)$. This is a sum of δ-functions with weight ± 1 depending on the direction of the jump. Note that by assumption the sum over x is finite. In the first term we use the conservation law

$$\frac{1}{2}\int_0^{\varepsilon^{-2}t}\sum_x V(\varepsilon x, \varepsilon^2 s)(J_{(x-1,x)}(ds) - J_{(x,x+1)}(ds))$$

$$= \frac{1}{2}\sum_x V(\varepsilon x, t)\eta_{\varepsilon^{-2}t}(x) - \frac{1}{2}\sum V(\varepsilon x, 0)\eta_0(x)$$

$$-\frac{1}{2}\int_0^t \sum_x \left(\frac{\partial}{\partial s} V(\varepsilon x, s)\right)\eta_{\varepsilon^{-2}s}(x). \tag{5.8}$$

In the second term we expand the potential difference $V(\varepsilon x, x) - V(\varepsilon x + \varepsilon, s)$ up to second order and regroup. Then, using again Jensen's inequality,

$$\varepsilon \log \mathbb{E}_\zeta(\exp[-\alpha n_{\varepsilon^{-2}t}])$$

$$\geqq \mathbb{E}_\zeta\Bigg[-\alpha\varepsilon \sum_{x \geqq 1} \eta_{\varepsilon^{-2}t}(x) + \frac{1}{2}\varepsilon \sum_x V(\varepsilon x, t)\eta_{\varepsilon^2 t}(x) - \frac{1}{2}\varepsilon\sum_x V(\varepsilon x, 0)\eta_0(x)$$

$$+\int_0^t ds\,\varepsilon \sum_x \left\{\left(-\frac{1}{2}\frac{\partial}{\partial s}V(\varepsilon x, s) + \frac{1}{2}\frac{\partial}{\partial q}F(\varepsilon x, s) + \frac{1}{4}F(\varepsilon x, s)^2\right)\eta_{\varepsilon^{-2}s}(x)\right.$$

$$\left.\left. -\frac{1}{4}F(\varepsilon x, s)^2 \eta_{\varepsilon^{-2}s}(x)\eta_{\varepsilon^{-2}s}(x+1)\right\}\right] + O(\varepsilon). \tag{5.9}$$

As anticipated (5.9) is of the scaling form for the weakly asymmetric exclusion process. DeMasi et al. [11] prove in particular that for the weakly asymmetric exclusion process with initial condition ζ in the limit $\varepsilon \to 0$ the density is almost surely governed by (2.4) with initial condition (2.6). They also establish local equilibrium which is needed for the last term in (5.9). Thus we conclude that

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_\zeta(\exp[-\alpha n_{\varepsilon^{-2}t}])$$

$$\geqq -\alpha \int_0^\infty dq\,\varrho_t(q) + \frac{1}{2}\int dq\, V(q,t)\varrho_t(q) - \frac{1}{2}\int dq\, V(q,0)\varrho_0(q)$$

$$+\int_0^t ds \int dq \left\{\left(-\frac{1}{2}\frac{\partial}{\partial s}V(q,s) + \frac{1}{2}\frac{\partial}{\partial q}F(q,s)\right.\right.$$

$$\left.\left. +\frac{1}{4}F(q,s)^2\right)\varrho_s(q) - \frac{1}{4}F(q,s)^2\varrho_s(q)^2\right\}.$$

Using (2.4) the desired bound (2.7) follows.

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References

1. Palmer, R., Stein, D., Abrahams, E., Anderson, P.W.: Models of hierarchically constrained dynamics for glassy relaxation. Phys. Rev. Lett. **53**, 958–961 (1984)
2. Fredrickson, G.H., Andersen, H.C.: Kinetic Ising model of the glass transition. Phys. Rev. Lett. **53**, 1244–1247 (1984)
3. Fredrickson, G.H.: Linear and nonlinear experiments for a spin model with cooperative dynamics. Ann. NY Acad. Sci. **484**, 185–205 (1987)
4. Skinner, J.L.: Kinetic Ising model for polymer dynamics. J. Chem. Phys. **79**, 1955–1963 (1983)
5. Budimir, J., Skinner, J.L.: Kinetic Ising model for polymer dynamics. II. J. Chem. Phys. **82**, 5232–5241 (1985)
6. Liggett, T.M.: Interacting particle systems. Berlin, Heidelberg, New York: Springer 1985
7. Holley, R.: Rapid convergence to equilibrium in one-dimensional stochastic Ising models. Ann. Probab. **13**, 72–89 (1985)
8. Huse, D., Fisher, D.: Dynamics of droplet fluctuations in pure and random Ising systems. Phys. Rev. B **35**, 6841–6846 (1987)
9. Sokal, A., Thomas, L.: Absence of a mass gap for a class of stochastic contour models. J. Stat. Phys. **51**, 907–947 (1988)
10. Kipnis, C., Olla, S., Varadhan, S.R.S.: Hydrodynamics and large deviation for simple exclusion processes (preprint)
11. DeMasi, A., Presutti, E., Scacciatelli, E.: The weakly asymmetric simple exclusion process. Ann. Inst. H. Poincaré (to appear)

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