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A New Proof of Localization in the Anderson Tight Binding Model

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Abstract. We give a new proof of exponential localization in the Anderson tight binding model which uses many ideas of the Frohlich, Martinelli, Scoppola and Spencer proof, but is technically simpler-particularly the probabilistic estimates.

1. Introduction

The Anderson tight binding model is given by the random Hamiltonian $H = -\Delta + V$ on $l^2(\mathbf{Z}^d)$, where $\Delta(x,y)=1$ if $|x-y|=1$ and zero otherwise, and $V(x)$, $x\in\mathbf{Z}^d$, are independent identically distributed random variables with common probability distribution μ. This model was introduced by Anderson [1] to describe the motion of a quantum-mechanical electron in a crystal with impurities.

It is well known that the spectrum of the Hamiltonian H is given by

$$\sigma(H)=\sigma(-\Delta)+\sigma(V)=[-2d,2d]+\operatorname{supp}\mu$$

with probability one [2, 3]. The spectrum of H can be decomposed into pure point spectrum, $\sigma_{\mathrm{pp}}(H)$, absolutely continuous spectrum, $\sigma_{\mathrm{ac}}(H)$, and singular continuous spectrum, $\sigma_{\mathrm{sc}}(H)$. There exist sets Σ_{pp}, Σ_{ac}, $\Sigma_{\mathrm{sc}}\subset\mathbf{R}$ such that $\sigma_{\mathrm{pp}}(H)=\Sigma_{\mathrm{pp}}$, $\sigma_{\mathrm{ac}}(H)=\Sigma_{\mathrm{ac}}$ and $\sigma_{\mathrm{sc}}(H)=\Sigma_{\mathrm{sc}}$ with probability one [3].

In this article we are concerned with localization. We say that the random operator H exhibits localization in an energy interval I if H has pure point spectrum in I with probability one, i.e., if $\Sigma_{\mathrm{ac}}\cap I=\Sigma_{\mathrm{sc}}\cap I=\phi$. We have exponential localization in I if we have localization and all the eigenfunctions corresponding to eigenvalues in I have exponential decay.

Exponential localization for the Anderson tight binding Hamiltonian is well understood in one dimension [3–6], where it was first established in the continuum by Gol'dsheid, Molchanov and Pastur [20]. In higher dimensions, the first results toward localization, for either high disorder or low energy, were due to Fröhlich

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and Spencer [7], who proved exponential decay for the Green's functions. These were followed by a proof of localization for a hierarchical version of H by Jona-Lasinio, Martinelli and Scoppola [8], and by a proof of the absence of absolutely continuous spectrum at higher disorder or low energy by Martinelli and Scoppola [9]. Subsequently, proofs of exponential localization, at high disorder or low energy, were given by Fröhlich, Martinelli, Scoppola and Spencer [4], Delyon, Levy and Souillard [10], and Simon and Wolff [11]. All of these higher dimensional results relied on methods or results of [7].

Recently, von Dreifus and Spencer [12, 13] introduced a new proof of the original Fröhlich and Spencer results in [7], which uses the same basic ideas, but is technically much simpler—particularly the probabilistic estimates. The key new idea is a scaling argument previously used in the study of bond percolation [14].

In this article we show how the methods of von Dreifus and Spencer can be used to give a direct proof of exponential localization. This proof uses the basic ideas behind the Fröhlich, Martinelli, Scoppola and Spencer proof [4, 5], but has much simpler probabilistic estimates. As in [6], we can allow singular distributions for the potential not permitted in [10, 11].

This article is organized as follows: We state our results in Sect. 2. Theorem 2.1 is our result on localization, it follows from Theorems 2.2 and 2.3. Theorem 2.2 is our basic technical result. Theorems 2.3 and 2.2 are proved in Sects. 3 and 4, respectively. The Appendix contains a discussion of when the hypotheses of Theorem 2.1 can be proven so we can conclude localization.

2. Statement of Results

We start with some notations and definitions.

If $\Lambda \subset \mathbf{Z}^d$, we denote by H_Λ the operator H restricted to $l^2(\Lambda)$ with zero boundary conditions outside Λ. The corresponding Green's function is $G_\Lambda(z) = (H_\Lambda - z)^{-1}$, defined for $z \notin \sigma(H_\Lambda)$. We will write

$$G_\Lambda(z; x, y) = (H_\Lambda - z)^{-1}(x, y) \quad \text{for} \quad x, y \in \Lambda.$$

If $\Lambda = \mathbf{Z}^d$ we simply write $G(z; x, y)$. Notice that we omit the dependence of H_Λ and G_Λ on the potential V.

If $x \in \mathbf{Z}^d$, $x = (x_1, \ldots, x_d)$, let $\|x\| = \max\{|x_1|, \ldots, |x_d|\}$. It will be convenient to use this norm in $\mathbf{Z}^d$. The distances in $\mathbf{Z}^d$ will always be taken with respect to this norm.

If $L > 0$, $x \in \mathbf{Z}^d$, we will denote by $\Lambda_L(x)$ the cube centered at x with sides of length L, i.e.,

$$\Lambda_L(x) = \left\{ y \in \mathbf{Z}^d;\ \|y - x\| \leqq \frac{L}{2} \right\}.$$

By $\partial\Lambda_L(x)$ we will denote its boundary, i.e.,

$$\partial\Lambda_L(x) = \left\{ \langle y, y' \rangle;\ y \in \Lambda_L(x), y' \notin \Lambda_L(x), \sum_{i=1}^{d} |y_i - y'_i| = 1 \right\}.$$

We will abuse the notation and write $y \in \partial\Lambda_L(x)$ to mean $\langle y, y' \rangle \in \partial\Lambda_L(x)$ for some

y'. We will also use

$$\sum_{y\in\partial\Lambda_L(x)} \quad \text{to denote} \quad \sum_{\langle y,y'\rangle\in\partial\Lambda_L(x)}$$

and

$$\partial\Lambda_L^+(x)=\{y'\in\mathbf{Z}^d;\langle y,y'\rangle\in\partial\Lambda_L(x) \text{ for some } y\in\Lambda_L(x)\}.$$

If $\Lambda\subset\mathbf{Z}^d$, $|\Lambda|$ will denote the number of points in Λ. Notice that

$$|\Lambda_L(x)|\leqq(L+1)^d,\quad |\partial\Lambda_L(x)|\leqq s_dL^{d-1},$$

where s_d is a constant depending only on d.

By $\mathbf{P}(A)$ we will denote the probability of the event A.

Definition. *Let $m>0$, $E\in\mathbf{R}$. A cube $\Lambda_L(x)$ is (m,E)-regular (for a fixed potential) if $E\notin\sigma(\mathrm{H}_{\Lambda_L(x)})$ and*

$$|G_{\Lambda_L(x)}(E;x,y)|\leqq e^{-mL/2}$$

for all $y\in\partial\Lambda_L(x)$. Otherwise we say that $\Lambda_L(x)$ is (m,E)-singular.

We will say that $\psi\in l^2(\mathbf{Z}^d)$ decays exponentially fast with mass $m>0$ if

$$\varlimsup_{\|x\|\to\infty}\frac{\log|\psi(x)|}{\|x\|}\leqq -m.$$

Our results on localization is:

Theorem 2.1. *Let $E_0\in\mathbf{R}$. Suppose that, for some $L_0>0$, we have:*

(P1) *$\mathbf{P}\{\Lambda_{L_0}(0)$ is (m_0,E_0)-regular$\}\geqq 1-1/L_0^p$ for some $p>d$, $m_0>0$.*
(P2) *$\mathbf{P}\{d(E,\sigma(H_{\Lambda_L(0)}))<e^{-L^\beta}\}\leqq 1/L^q$ for some β and q, $0<\beta<1$, $q>4p+6d$, all E with $|E-E_0|\leqq\eta$, where $\eta>0$, and all $L\geqq L_0$.*

Then, given m, $0<m<m_0$, there exists $B=B(p,d,\beta,q,m_0,m)<\infty$, such that if $L_0>B$, we can find $\delta=\delta(L_0,m_0,m,\beta,\eta)>0$, so, with probability one, the spectrum of H in $(E_0-\delta,\ E_0+\delta)$ is pure point and the eigenfunctions corresponding to eigenvalues in $(E_0-\delta,\ E_0+\delta)$ decay exponentially fast at infinity with mass m.

The validity of (P1) and (P2) are discussed in the Appendix. Notice that B and δ do not depend on E_0.

By the resolvent equation,

$$G_{\Lambda_{L_0}(0)}(E)=G_{\Lambda_{L_0}(0)}(E_0)+(E-E_0)G_{\Lambda_{L_0}(0)}(E)G_{\Lambda_{L_0}(0)}(E_0).$$

If

$$d(E_0,\sigma(H_{\Lambda_{L_0}(0)}))\geqq e^{-L_0^\beta},\quad |E-E_0|\leqq\tfrac{1}{2}e^{-L_0^\beta},$$

we have that $d(E,\sigma(H_{\Lambda_{L_0}(0)}))\geqq\frac{1}{2}e^{-L_0^\beta}$. If in addition $\Lambda_{L_0}(0)$ is (m_0,E_0)-regular and $y\in\partial\Lambda_{L_0}(0)$, we have that

$$|G_{\Lambda_{L_0}(0)}(E;0,y)|\leqq e^{-m_0L_0/2}+2|E-E_0|e^{2L_0^\beta}.$$

So, given any m_0' and p', $0<m_0'<m_0$ and $d<p'<p$, if we let

$$\delta=\tfrac{1}{2}e^{-2L_0^\beta}(e^{-m_0'L_0/2}-e^{-m_0L_0/2}),$$

it follows from (P1) and (P2) that

(P1') $\mathbf{P}\{$for any $E\in(E_0-\delta,\ E_0+\delta)\ \Lambda_{L_0}(0)$ is (m_0', E)-regular$\} \geqq 1 - 1/L_0^p - 1/L_0^q \geqq 1 - 1/L_0^{p'}$ if L_0 is sufficiently large, how large depending only on p, q and p'.

Thus Theorem 2.1 will follow from Theorems 2.2 and 2.3.

Theorem 2.2. *Let $I \subset \mathbf{R}$ be an interval. Suppose that, for some $L_0 > 0$, we have:*

(K1) $\mathbf{P}\{$*for any $E\in I$ either $\Lambda_{L_0}(x)$ or $\Lambda_{L_0}(y)$ is (m_0, E)-regular*$\} \geqq 1 - 1/L_0^{2p}$ *for some $p > d$, $m_0 > 0$, and any $x, y\in \mathbf{Z}^d$ with $\|x-y\| > L_0$.*

(K2) $\mathbf{P}\{d(E, \sigma(H_{\Lambda_L(0)})) < e^{-L^\beta}\} \leqq 1/L^q$ *for some β and q, $0<\beta<1$ and $q > 4p+6d$, all E with $d(E, I) \leqq \frac{1}{2}e^{-L^\beta}$, and all $L \geqq L_0$.*

Then there exists $\alpha = \alpha(p,d)$, $1<\alpha<2$, such that if we set $L_{k+1} = L_k^\alpha$, $k = 0, 1, 2, \ldots$, and pick m, $0 < m < m_0$, we can find $Q = Q(p, d, \beta, q, m_0, \alpha, m) < \infty$, such that if $L_0 > Q$, we have that, for any $k = 0, 1, 2, \ldots$,

$$\mathbf{P}\{\textit{for any } E\in I \textit{ either } \Lambda_{L_k}(x) \textit{ or } \Lambda_{L_k}(y) \textit{ is } (m, E)\textit{-regular}\} \geqq 1 - \frac{1}{L_k^{2p}}$$

for any x, $y\in\mathbf{Z}^d$ with $\|x-y\| > L_k$.

Remarks. 1. To understand Theorem 2.2, which is our basic technical result, it is useful to notice that the von Dreifus–Spencer basic technical result [12] states that, under hypothesis (P1) and (P2),

$$\mathbf{P}\{\Lambda_{L_k}(0) \text{ is } (m, E_0)\text{-regular}\} \geqq 1 - \frac{1}{L_k^p}$$

for all $k = 0, 1, \ldots$, where $L_{k+1} = L_k^\alpha$ for some $\alpha, 1<\alpha<2, 0<m<m_0$, if L_0 is large enough. From this result the original Fröhlich and Spencer results [7] can be derived [12]. Similarly the Fröhlich, Martinelli, Scoppola and Spencer results [4] follow from Theorem 2.2.

2. In Theorem 2.2 we can pick any α such that $1<\alpha<\alpha_0$, where $\alpha_0 = (J+1)p/2p + (J+1)d$, J being the smallest odd integer $> (p + d/p - d)$. Notice $1 < \alpha_0 < 2$. If $p > 2d$, $J = 3$, so $\alpha_0 = 2p/p + 2d$.

3. We will need $m_0 > 8JL_0^{-(1-\beta)}$. If in (P2) and (K2) we had an estimate on $\mathbf{P}\{d(E, \sigma(H_{\Lambda_L(0)})) < 1/L^s\}$ for some $s > 0$, we would need $m_0 \geqq C(\log L_0/L_0)$, where $C = C(s, d, J)$ is some constant. Notice that in many cases we have such an estimate (see the Appendix).

4. Theorem 2.2 is still true if in the definition of (m, E)-regularity for a cube $\Lambda_L(x)$ we had required that $d(E, \sigma(H_{\Lambda_L(x)})) \geqq \frac{1}{2}e^{-L^\beta}$.

Theorem 2.3. *Let $I\subset\mathbf{R}$ be an interval, and let $p > d$, $L_0 > 0$, $1<\alpha<2p/d$, $m > 0$. Set $L_{k+1} = L_k^\alpha$, $k = 0, 1, 2, \ldots$. Suppose that, for any $k = 0, 1, 2, \ldots$,*

$$\mathbf{P}\{\textit{for any } E\in I \textit{ either } \Lambda_{L_k}(x) \textit{ or } \Lambda_{L_k}(y) \textit{ is } (m, E)\textit{-regular}\} \geqq 1 - \frac{1}{L_k^{2p}}$$

for any $x, y\in\mathbf{Z}^d$ with $\|x-y\| > L_k$.

Then, with probability one, the spectrum of H in I is pure point and the *eigenfunctions corresponding to eigenvalues in I decay exponentially fast at infinity with mass m.*

3. Proof of Theorem 2.3.

We follow the strategy of [4, 5].

Definition. *E is a generalized eigenvalue for $H = -\Delta + V$ if there exists a nonzero polynomially bounded function ψ on $\mathbf{Z}^d$ such that $H\psi = E\psi$. In this case ψ is called a generalized eigenfunction.*

We will use the following basic result [15, 16]:

With respect to the spectral measure of H, almost every energy is a generalized eigenvalue.

Thus Theorem 2.3 follows from

Lemma 3.1. *Under the hypothesis of Theorem 2.3, with probability one the generalized eigenfunctions of $H = -\Delta + V$ corresponding to generalized eigenvalues in I decay exponentially fast at infinity with mass m.*

Proof. Let b be a positive integer to be chosen later on. For $x_0 \in \mathbf{Z}^d$ let

$$A_{k+1}(x_0) = \Lambda_{2bL_{k+1}}(x_0) \backslash \Lambda_{2L_k}(x_0)$$

for $k = 0, 1, \ldots$, and let us define the event

$E_k(x_0) = \{\Lambda_{L_k}(x_0)$ and $\Lambda_{L_k}(x)$ are (m, E)-singular for some $E \in I$ and $x \in A_{k+1}(x_0)\}$.

By our hypothesis,

$$\mathbf{P}(E_k(x_0)) \leqq \frac{(2bL_{k+1} + 1)^d}{L_k^{2p}} \leqq \frac{(2b+1)^d}{L_k^{2p-\alpha d}}.$$

Since $\alpha < 2p/d$,

$$\sum_{k=0}^{\infty} \mathbf{P}(E_k(x_0)) < \infty,$$

so it follows from the Borel Cantelli Lemma that for each $x_0 \in \mathbf{Z}^d$,

$$\mathbf{P}\{E_k(x_0) \text{ occurs infinitely often}\} = 0.$$

Thus

$$\mathbf{P}\{E_k(x_0) \text{ occurs infinitely often for some } x_0 \in \mathbf{Z}^d\} = 0.$$

So, if we let $\Omega_0 = \{E_k(x_0)$ occurs only finitely many times for each $x_0 \in \mathbf{Z}^d\}$, we have that $\mathbf{P}(\Omega_0) = 1$.

Now let $V \in \Omega_0$, and let $E \in I$ be a generalized eigenvalue for $H = -\Delta + V$, with the corresponding nonzero polynomially bounded generalized eigenfunction, i.e., $H\psi = E\psi$, $|\psi(x)| \leqq C(1 + \|x\|)^t$ for some $C < \infty$ and positive integer t, and we can find $x_0 \in \mathbf{Z}^d$ such that $\psi(x_0) \neq 0$.

If $E \notin \sigma(H_{\Lambda_{L_k}(x_0)})$, we can recover ψ from its boundary values by

$$\psi(x_0) = \sum_{\langle y, y' \rangle \in \partial\Lambda_{L_k}(x_0)} G_{\Lambda_{L_k}(x_0)}(E; x_0, y)\psi(y').$$

If $\Lambda_{L_k}(x_0)$ is (m, E)-regular, we get

$$|\psi(x_0)| \leqq s_d L_k^{d-1} e^{-mL_k/2} C(1 + \|x_0\| + L_k)^t.$$

Since $\psi(x_0) \neq 0$, it follows that there exists $k_1 = k_1(V, E, x_0)$ such that $\Lambda_{L_k}(x_0)$ is (m, E)-singular for all $k \geqq k_1$. On the other hand, since $V \in \Omega_0$, we can find $k_2 = k_2(V, x_0)$ such that, if $k \geqq k_2$, $E_k(x_0)$ does not occur. Let us take k_3 to be the biggest of k_1 and k_2, $k_3 = k_3(V, E, x_0)$. If $k \geqq k_3$, we conclude that $\Lambda_{L_k}(x)$ is (m, E)-regular for all $x \in A_{k+1}(x_0)$.

Now, let ρ, $0 < \rho < 1$, be given, we pick $b > 1 + \rho/1 - \rho$ and define

$$\tilde{A}_{k+1}(x_0) = \Lambda_{[2b/(1+\rho)]L_{k+1}}(x_0) \backslash \Lambda_{[2/(1-\rho)]L_k}(x_0).$$

Then $\tilde{A}_{k+1}(x_0) \subset A_{k+1}(x_0)$, and, if $x \in \tilde{A}_{k+1}(x_0)$, we have

$$d(x, \partial A_{k+1}(x_0)) \geqq \rho \|x - x_0\|.$$

Moreover, if $\|x - x_0\| > L_0/1 - \rho$, we have that $x \in \tilde{A}_{k+1}(x_0)$ for some k.

Now let $k \geqq k_3$, so $\Lambda_{L_k}(y)$ is (m, E)-regular for any $y \in A_{k+1}(x_0)$. As before,

$$\psi(y) = \sum_{\langle u, u' \rangle \in \partial \Lambda_{L_k}(y)} G_{\Lambda_{L_k}(y)}(E; y, u) \psi(u')$$

or

$$|\psi(y)| \leqq s_d L_k^{d-1} e^{-mL_k/2} |\psi(u_1')|$$

for some $u_1' \in \partial \Lambda_{L_k}^+(y)$.

Thus, if $x \in \tilde{A}_{k+1}(x_0)$ with $k \geqq k_3$, we can repeat this procedure at least $((L_k/2) + 1)^{-1} \rho \|x - x_0\|$ times, and use the polynomial bound on ψ to obtain

$$|\psi(x)| \leqq (s_d L_k^{d-1} e^{-mL_k/2})^{(L_k/2+1)^{-1} \rho \|x - x_0\|} C(1 + \|x_0\| + bL_{k+1})^t.$$

We can conclude that, given ρ', $0 < \rho' < 1$, we can find $k_4 \geqq k_3$ such that if $k \geqq k_4$ we have

$$|\psi(x)| \leqq e^{-\rho' \rho \|x - x_0\|}$$

if $\|x - x_0\| \geqq L_{k_4}/1 - \rho$.

Thus ψ decays exponentially, and

$$\overline{\lim_{\|x\| \to \infty}} \frac{\log |\psi(x)|}{\|x\|} \leqq -\rho' \rho m$$

for any ρ, $\rho' \in (0, 1)$.

Theorem 2.3 is proved.

4. Proof of Theorem 2.2.

Let us fix the interval $I \subset \mathbf{R}$ and $p > d$. For a given L and $m > 0$ we will denote by $R(L, m)$ the statement:

$$\mathbf{P}\{\textit{for any } E \in I \textit{ either } \Lambda_L(x) \textit{ or } \Lambda_L(y) \textit{ is } (m, E)\textit{-regular}\}$$

$$\geqq 1 - \frac{1}{L^{2p}}, \quad \textit{for any } x, y \in \mathbf{Z}^d \textit{ with } \|x - y\| > L.$$

The proof of Theorem 2.1 will proceed by induction. Notice that $R(L_0, m)$ is just (K1).

The induction step is

Lemma 4.1. *Let* $\alpha_0 = (J+1)p/2p + (J+1)d$, *J being the smallest odd integer* $> p + d/p - d$. *Suppose* $R(l, m_l)$ *holds with* $m_l > 8J/l^{1-\beta}$ *and* (K2) *holds for all* $L \geqq l$. *Pick* α, $1 < \alpha < \alpha_0$. *Then there exists* $Q_1 = Q_1(p, d, \beta, q, \alpha) < \infty$, *such that if* $l > Q_1$, *then* $R(L, m_L)$ *holds with* $L = l^\alpha$ *and*

$$m_L \geqq m_l - \left[\frac{5(J+1)}{l^{\alpha-1}} m_l + \frac{3}{l^{\alpha(1-\beta)}}\right] \geqq \frac{8J}{L^{1-\beta}}.$$

The proof of Lemma 4.1 has a deterministic and a probabilistic component. We start by proving the deterministic step; but first we need the following (deterministic) definition.

Definition. *A cube* $\Lambda_L(x)$ *is non-resonant at the energy E if* $d(E, \sigma(H_{\Lambda_L(x)}) \geqq \frac{1}{2} e^{-L^\beta}$, i.e., *if and only if* $\|G_{\Lambda_L(x)}(E)\| \leqq 2e^{L^\beta}$. *In this case we will say that* $\Lambda_L(x)$ *is* $E - NR$.

Lemma 4.2. *Let* $L = l^\alpha$ *with* $1 < \alpha < 2$, $E \in \mathbf{R}$, *J an arbitrary positive integer, and* $m_l > 8J/l^{1-\beta}$. *Suppose*:

(i) $\Lambda_L(x)$ *is* $E - NR$.
(ii) $\Lambda_{j(2l+1)}(y)$ *is* $E - NR$ *for all* $j = 1, 2, \ldots, J$ *and* $y \in \Lambda_L(x)$ *with* $\Lambda_{j(2l+1)}(y) \subset \Lambda_L(x)$.
(iii) *There exist at most J non-overlapping cubes of side l contained in* $\Lambda_l(x)$ *that are* (m_l, E)*-singular.*

Then there exists $Q_2 = Q_2(J, d, \beta, \alpha) < \infty$ *such that if* $l \geqq Q_2$, *we have* $\Lambda_L(x)$ (m_L, E)*-regular with*

$$m_L \geqq m_l - \left[\frac{5(J+1)}{l^{\alpha-1}} m_l + \frac{3}{l^{\alpha(1-\beta)}}\right] \geqq \frac{8J}{L^{1-\beta}}.$$

Proof. By (iii) we have at most J non-overlapping cubes of side l contained in $\Lambda_L(x)$ that are (m, E)-singular. It follows that we can find $u_i \in \Lambda_L(x)$ with $d(u_i, \partial\Lambda_L(x)) \geqq l/2$, $i = 1, \ldots, r$, where $r \leqq J$, such that if $u \in \Lambda_L(x) \Big/ \bigcup_{i=1}^{r} \Lambda_{2l}(u_i)$ with $d(u, \partial\Lambda_L(x)) \geqq l/2$, then $\Lambda_l(u)$ is (m, E)-regular.

An easy geometric argument shows that we can find cubes $\Lambda_{l_i} \subset \Lambda_L(x)$ with side $l_i \in \{j(2l+1),\ j = 1, 2, \ldots, J\}$, $i = 1, 2, \ldots, t$, $t \leqq r$, such that $d(\Lambda_{l_i}, \Lambda_{l_j}) \geqq 1$ if $i \neq j, \ldots,$

$$\bigcup_{i=1}^{r} \Lambda_{2l}(u_i) \subset \bigcup_{j=1}^{t} \Lambda_{l_t}, \quad \text{and} \quad \sum_{i=1}^{t} l_i \leqq J(2l+1).$$

It follows that if $u \in \Lambda_L(x) \Big/ \bigcup_{i=1}^{t} \Lambda_{l_i}$, $d(u, \partial\Lambda_L(x)) \geqq l/2$, we have that $\Lambda_l(u)$ is (m, E)-regular. Also notice that if $u \in \partial\Lambda_{l_j}^+$ for some $j = 1, \ldots, t$ then $u \notin \bigcup_{i=1}^{t} \Lambda_{l_t}$.

The basic tool in the proof is the resolvent identity as used in [7, 4]. If Λ is a cube contained in $\Lambda_L(x)$, let $u\in\Lambda$, $v\in\Lambda_L(x)\backslash\Lambda$. Then it follows from the resolvent identity that

$$G_{\Lambda_L(x)}(E;u,v)=\sum_{\langle w,w'\rangle\in\partial\Lambda} G_\Lambda(E;\,u,\,w)\,G_{\Lambda_L(x)}(E;\,w',\,v).$$

Here $\partial\Lambda$ denotes the boundary of Λ in $\Lambda_L(x)$. Thus

$$|G_{\Lambda_L(x)}(E;u,v)|\leqq\left[\sum_{w\in\partial\Lambda}|G_\Lambda(E;u,w)|\right]|G_{\Lambda_L(x)}(E;w_1,v)| \tag{4.1}$$

for some $w_1\in\partial\Lambda^+$.

Let us now fix $y\in\partial\Lambda_L(x)$ and let $u\in\Lambda_L(x)$, $d(u,\partial\Lambda_L(x))\leqq l/2$. We have two cases:

(a) $\Lambda_l(u)$ is (m_l,E)-regular. In this case

$$\sum_{w\in\partial\Lambda_l(u)}|G_{\Lambda_l(u)}(E;u,w)|\leqq s_d l^{d-1}e^{-m_l l/2}. \tag{4.2}$$

(b) $\Lambda_l(u)$ is (m_l,E)-singular. It follows that $u\in\Lambda_{l_i}$ for some $i=1,\dots,t$. Thus (4.1) gives

$$|G_{\Lambda_L(x)}(E;u,y)|\leqq\left[\sum_{w\in\partial\Lambda_{l_i}}|G_{\Lambda_{l_i}}(E;u,w)|\right]|G_{\Lambda_L(x)}(E;w_1,v)|,$$

where $w_1\in\partial\Lambda_{l_i}^+$. If $d(\Lambda_{l_i},\,\partial\Lambda_L(x))\geqq l/2+1$, then $d(w_1,\partial\Lambda_L(x))\geqq l/2$. In this case we use (ii) to estimate the term in brackets and (4.2) to estimate the other factor getting

$$|G_{\Lambda_L(x)}(E;u,y)|\leqq s_d^2 2^d(l+1)^{2(d-1)}J^{d-1}e^{(J(2l+1))^\beta-m_l l/2}|G_{\Lambda_L(x)}(E;w_2,y)|,$$

where $w_2\in\partial\Lambda_l^+(w_1)$.

Thus

$$|G_{\Lambda_L(x)}(E;u,y))\leqq e^{-m_l' l/2}|G_{\Lambda_L(x)}(E;w_2,y)|, \tag{4.3}$$

where

$$m_l'=m_l-\frac{2}{l}[J(2l+1)^\beta+2(d-1)\log(l+1)+\log(s_d^2 2^d J^{d-1})]\geqq m_l-\frac{8J}{l^{1-\beta}}>0$$

if $l\geqq Q_3$ for some $Q_3=Q_3(J,d,\beta)<\infty$.

If $u\in\Lambda_L(x)$, $d(u,\partial\Lambda_L(x))\geqq l/2$, let

$$Z(u)=\begin{cases} s_d l^{d-1}e^{-m_l l/2} & \text{if } u \text{ is as case (a)}\\ e^{-m' l/2} & \text{if } u \text{ is as case (b)}\end{cases}.$$

Then (4.1) and (4.2) for case (a), and (4.3) for case (b), say that

$$|G_{\Lambda_L(x)}(E;u,y)|\leqq Z(u)|G_{\Lambda_L(x)}(E;w,y)|$$

for some $w\in\Lambda_L(x)$.

To estimate $|G_{\Lambda_L(x)}(E;x,y)|$, we start from x, the center of the cube $\Lambda_L(x)$, and apply the above procedure repeatedly, when possible, getting, after n steps,

$$\begin{aligned}|G_{\Lambda_L(x)}(E;x,y)|&\leqq Z(x)|G_{\Lambda_L(x)}(E;w_1,y)|\leqq Z(x)Z(w_1)|G_{\Lambda_L(x)}(E;w_2,y)|\leqq\cdots\leqq\\ &\leqq Z(x)Z(w_1)\cdots Z(w_{n-1})|G_{\Lambda_L(x)}(E;w_n,y)|.\end{aligned}$$

For this to be possible, we need $w_1, \ldots, w_{n-1}$ to satisfy the conditions of either (a) or (b).

Now let n_1 be the number of times we were in case (a), $n_2 = n - n_1$. We have

$$|G_{\Lambda_L(x)}(E; x, y)| \leqq (s_d l^{d-1} e^{-m_l l/2})^{n_1} (e^{-m_l' l/2})^{n_2} |G_{\Lambda_L(x)}(E; w_{n+1}, y)|.$$

Since $m_l' > 0$, the procedure can always be repeated as long as

$$n_1 \leqq \frac{L/2 - [J(2l+1) + l/2 + 1]}{l/2 + 1}.$$

Thus, since $\Lambda_L(x)$ is $E - NR$ by (i), we can always get

$$|G_{\Lambda_L(x)}(E; x, y)| \leqq (s_d l^{d-1} e^{-m_l l/2})^{n_3} 2e^{L^\beta},$$

where

$$n_3 \leqq \frac{L/2 - [J(2l+1) + l/2 + 1]}{l/2 + 1} - 1.$$

Then we have $|G_{\Lambda_L(x)}(E; x, y)| \leqq e^{-m_L L/2}$, where

$$m_L \geqq m_l - m_l \left[\frac{4l(J+1)}{L} + \frac{2}{l} \right] - \frac{2}{l} \log (s_d l^{d-1}) - \frac{2}{L} \log 2 - \frac{2}{L^{1-\beta}}.$$

Recalling $L = l^\alpha$, $1 < \alpha < 2$, we get

$$\begin{aligned} m_L &\geqq m_l - m_l \left[\frac{4(J+1)}{l^{\alpha-1}} + \frac{2}{l} \right] - \frac{2d \log l}{l} - \frac{\log 4}{l^\alpha} - \frac{2}{l^{\alpha(1-\beta)}} \\ &\geqq m_l - m_l \frac{5(J+1)}{l^{\alpha-1}} - \frac{3}{l^{\alpha(1-\beta)}} \geqq \frac{8J}{L^{1-\beta}} \end{aligned}$$

if $l \geqq Q_4$ for some $Q_4 = Q_4(J, d, \beta, \alpha) < \infty$.

Thus if we take Q_2 to be the biggest of Q_3 and Q_4, we have that $Q_2 = Q_2(J, d, \beta, \alpha) < \infty$ and Lemma 4.2 holds if $l \geqq Q_2$.

We now prove Lemma 4.1. Pick α, $1 < \alpha < \alpha_0$. If $l \geqq Q_2$ of Lemma 4.2, it suffices to show that we can find $Q_5 = Q_5(p, d, \beta, q, \alpha)$ such that, if $l \geqq Q_5$, we have that

$$\mathbf{P}\{\text{for any } E \in I, \text{(i), (ii) and (iii) of Lemma 4.2 hold for either } \Lambda_L(x) \text{or } \Lambda_L(y)\} \geqq 1 - \frac{1}{L^{2p}}$$

for any $x, y \in \mathbf{Z}^d$ with $\|x - y\| > L$. Here $L = l^\alpha$. If Λ_r is a cube with side r, let

$$\sigma'(H_{\Lambda_r}) = \sigma(H_{\Lambda_r}) \cap \{E \in \mathbf{R}; d(E, I) \leqq \tfrac{1}{2} e^{-r^\beta}\}.$$

It follows from (K2) that if Λ_{l_i}, $i = 1, 2$, are non-overlapping cubes with sides $l_i \geqq l$, $i = 1, 2$, respectively, then (see [4, 5])

$$\mathbf{P}\{d(\sigma'(H_{\Lambda_{l_1}}), \sigma'(H_{\Lambda_{l_2}})) < e^{-l_1^\beta}\} \leqq \frac{(l_2+1)^d}{l_1^q}. \tag{4.4}$$

To see that, for any cube Λ let $V_\Lambda = \{V(x), x \in \Lambda\}$, $\mathbf{P}_\Lambda$ and $\mathbf{E}_\Lambda$ being the corresponding probability measure and expectation. Since Λ_{l_1} and Λ_{l_2} are non-overlapping $V_{\Lambda_{l_1}}$ and $V_{\Lambda_{l_2}}$ are independent, so

$$\mathbf{P}\{d(\sigma'(H_{\Lambda_{l_1}}),\sigma'(H_{\Lambda_{l_2}}))<e^{-l_1^\beta}\}=\mathbf{E}_{\Lambda_{l_2}}\mathbf{P}_{\Lambda_{l_1}}\{d(\sigma'(H_{\Lambda_{l_1}}),\sigma'(H_{\Lambda_{l_2}}))<e^{-l_1^\beta}\}.$$

Let us fix $V_{\Lambda_{l_2}}$, we can write $\sigma'(H_{\Lambda_{l_2}})=\{\lambda_1,\dots,\lambda_k\}$, where $k\leqq(l_2+1)^d$. We have

$$\mathbf{P}_{\Lambda_{l_1}}\{d(\sigma'(H_{\Lambda_{l_1}}),\sigma'(H_{\Lambda_{l_1}}))<e^{-l_1^\beta}\}\leqq\sum_{i=1}^{k}\mathbf{P}_{\Lambda_{l_1}}\{d(\lambda_i,\sigma'(H_{\Lambda_{l_1}}))<e^{-l_1^\beta}\}\leqq\frac{(l_2+1)^d}{l_1^q}$$

by (K2), the estimate being uniform in $V_{\Lambda_{l_2}}$. This proves (4.4).

Let us now fix $x,y\in\mathbf{Z}^d$, $\|x-y\|>L$. It follows from (4.4) that, with $l_1\wedge l_2=$ smallest of l_1, l_2, we have

$$\begin{aligned}&\mathbf{P}\{d(\sigma'(H_{\Lambda_{l_1(x')}}),\sigma'(H_{\Lambda_{l_2(y')}}))<e^{-(l_1\wedge l_2)^\beta}\text{ for some }x'\in\Lambda_L(x),y'\in\Lambda_L(y),l_1,l_2\\&\quad=L\text{ or }j(2l+1),j=1,2,\dots,J,\text{ with }\Lambda_{l_1}(x')\subset\Lambda_L(x),\ \Lambda_{l_2}(y')\subset\Lambda_L(y)\}\\&\quad\leqq\frac{(J+1)^2(L+1)^{3d}}{(2l+1)^q}.\end{aligned}\tag{4.5}$$

Now let $E\in I$ and let l' be the largest of L or $j(2l+1)$, $j=1,2,\dots J$, such that there exists $u\in\Lambda_L(z)$ with $\Lambda_{l'}(u)\subset\Lambda_L(z)$, z being either x or y, with $d(E,\sigma'(H_{\Lambda_{l'}(u)}))<\frac{1}{2}e^{-l'^\beta}$ (if such l' exists). For clarity of argument, say $z=x$. If the event whose probability is estimated in (4.5) does not occur, we have that $\Lambda_{l'}(y')$ is $E-NR$ for all $y'\in\Lambda_L(y)$, $l'=L$ or $j(2l+1)$, $j=1,\dots,J$, with $\Lambda_{l'}(y')\subset\Lambda_L(y)$. Thus, for $l\geqq Q_5$,

$$\begin{aligned}&\mathbf{P}\{\text{for any }E\in I,\text{ (i) and (ii) of Lemma 4.2 hold for either }\Lambda_L(x)\text{ or }\Lambda_L(y)\}\\&\quad\geqq1-\frac{(J+1)^2(L+1)^{3d}}{(2l+1)^q}.\end{aligned}\tag{4.6}$$

Now, since J is an odd integer, for any $z\in\mathbf{Z}^d$ we have, using the independence of the potential at different sites, that

$$\begin{aligned}&\mathbf{P}\{\text{for some }E\in I\text{ there are at least }J+1\text{ non-overlapping }(m_l,E)\text{-singular}\\&\quad\text{cubes of side }l\text{ contained in }\Lambda_L(z)\}\leqq\\&\mathbf{P}\{\text{for some }E\in I\text{ there are at least two non-overlapping }(m_l,E)\text{-singular}\\&\quad\text{cubes of side }l\text{ contained in }\Lambda_L(z)\}]^{(J+1)/2}\\&\leqq\left[\frac{(L+1)^{2d}}{l^{2p}}\right]^{(J+1)/2}=\left[\frac{(L+1)^d}{l^p}\right]^{J+1},\end{aligned}\tag{4.7}$$

since $R(l,m_l)$ holds by hypothesis.

Thus, combining (4.6) and (4.7) we get

$$\begin{aligned}&\mathbf{P}\{\text{for any }E\in I,\text{ (i), (ii) and (iii) of Lemma 4.2 hold for either}\\&\quad\Lambda_L(x)\text{ or }\Lambda_L(y)\}\geqq1-\frac{1}{L^{2p}}\text{ for }l\geqq Q_5,\end{aligned}$$

some $Q_5=Q_5(p,d,q,\alpha)<\infty$, since $q>4p+6d$ and $\alpha<\alpha_0<2$.

This proves Lemma 4.1.

We now complete the proof of Theorem 2.2. Let α_0 and J be as in Lemma 4.1, let $1 < \alpha < \alpha_0$, $L_k = L_k^\alpha$, $k = 0, 1, \dots$, and suppose $L_0 > Q_1$ of Lemma 4.1. Let

$$m_{k+1} = m_k - \left[\frac{5(J+1)}{L_k^{\alpha-1}} m_k + \frac{3}{L_k^{\alpha(1-\beta)}} \right],$$

$k = 0, 1, \dots$. Let us pick m, $0 < m < m_0$. Theorem 2.2 follows immediately from Lemma 4.1 by an induction argument if

$$\sum_{k=0}^{\infty} (m_k - m_{k+1}) \leqq m_0 - m.$$

But

$$\sum_{k=0}^{\infty} (m_k - m_{k+1}) \leqq 5(J+1) m_0 \sum_{k=0}^{\infty} \frac{1}{L_k^{\alpha-1}} + 3 \sum_{k=0}^{\infty} \frac{1}{L_k^{\alpha(1-\beta)}}$$

$$\leqq 5(J+1) m_0 \sum_{k=0}^{\infty} \frac{1}{L_0^{(\alpha-1)\alpha^k}} + 3 \sum_{k=0}^{\infty} \frac{1}{L_0^{\alpha(1-\beta)\alpha^k}} \leqq m_0 - m$$

if $L_0 \geqq Q_6$, some $Q_6 = Q_6(p, d, \beta, q, m_0, \alpha, m) < \infty$.
Theorem 2.2 is proved.

Appendix: Validity of (P1) and (P2)

In this Appendix we will discuss conditions on the potential probability distribution μ under which (P1) and (P2) have been proven.

A.1. Arbitrary Dimension. For arbitrary dimension hypotheses like (P1) have only been proven for either "low energy" or "high disorder" (see [7, 4, 5, 12]). For completeness we will sketch the proof of (P1) under those conditions, notice that the proofs yield (P1) directly.

For "low energy" the result is

Proposition A.1.1. *Let μ be any probability measure, and let $L_0 > 0$, $p > d$ and $m_0 > 0$ be given. Then there exists $E_0 = E_0(\mu, L_0, p, d, m_0) < \infty$, such that*

$$\mathbf{P}\{\textit{for any } E \textit{ with } |E| \geqq E_0 \ \Lambda_{L_0}(0) \textit{ is } (m_0, E)\textit{-regular}\} \geqq 1 - \frac{1}{L_0^p}.$$

Proof. Note that $\|\Delta_{\Lambda_{L_0}(0)}\| \leqq \|\Delta\| = 2d$, so if we have $|V(x)| \leqq E_0 - 2d - e^{m_0 L_0/2}$ for all $x \in \Lambda_{L_0}(0)$, we get $\|G_{\Lambda_{L_0}(0)}(E)\| \leqq e^{-m_0 L_0/2}$ for all E with $|E| \geqq E_0$.
Since we have that

$$\mathbf{P}\{|V(x)| \leqq E_0 - 2d - e^{m_0 L_0/2} \text{ for all } x \in \Lambda_{L_0(0)}\}$$
$$= [\mu\{[-E_0 + 2d + e^{m_0 L_0/2}, E_0 - 2d - e^{m_0 L_0/2}\}]^{(L_0+1)^d} \to 1 \text{ as } E_0 \to \infty,$$

the proposition follows.

We will say that a probability measure μ is Holder continuous of order $\rho > 0$ [6] if $\delta_\rho(\mu)^{-1} \equiv \inf_{\tau > 0} \sup_{|b-a| \leqq \tau} |b-a|^{-\rho} \mu([a,b]) < \infty$. In this case we will call $\delta_\rho(\mu)$ the disorder of μ. Notice that if μ is absolutely continuous with a density in L^q, $1 < q \leqq \infty$, then μ is Holder continuous of order $1/q' = 1 - 1/q$, and $\delta_{1/q}(\mu)^{-1} \leqq$

$\| d\mu/dv \|_q$. Thus if μ has a bounded density our notion of disorder coincides with the one in [7,4,5].

Given such μ and $\lambda > 0$, let μ_λ be the probability distribution of λV, i.e.,

$$\mu_\lambda([a,b]) = \mu\left(\left[\frac{a}{\lambda}, \frac{b}{\lambda}\right]\right).$$

Notice that $\delta_\rho(\mu_\lambda) = \lambda^\rho \delta_\rho(\mu)$. For "high disorder" we have

Proposition A.1.2. *Let $\mu = (1-t)\mu_\lambda + tv$, where μ_1 is a Holder continuous probability measure, v is an arbitrary probability measure, $\lambda > 0$, $0 \leqq t < 1$. Let $L_0 > 0$, $p > d$ and $m_0 > 0$ be given. Then, given $\eta > 0$, there exist $\lambda_1 = \lambda_1(\mu_1, L_0, p, d, m_0) < \infty$ and $0 \leqq t_1 = t_1(L_0, p, d, m_0) < 1$, such that*

$$\mathbf{P}\{\textit{for any } E \in [E_0 - \eta, E_0 + \eta]\ \Lambda_{L_0(0)} \textit{ is } (m_0, E)\textit{-regular}\}$$
$$\geqq 1 - \frac{1}{L_0^p} \textit{ for any } E_0 \in \mathbf{R},\ \lambda \geqq \lambda_1 \quad \textit{and} \quad 0 \leqq t \leqq t_1.$$

Proof. Fix $E_0 \in \mathbf{R}$, $\eta > 0$. Notice that if $|V(x) - E| \geqq 2d + \eta + e^{m_0 L_0/2}$ for all $x \in \Lambda_{L_0}(0)$, we get $\| G_{\Lambda_{L_0(0)}}(E) \| \leqq e^{-m_0 L_0/2}$ for all $E \in [E_0 - \eta,\ E_0 + \eta]$. Since

$$\mathbf{P}\{|V(x) - E_0| < 2d + 2\eta + e^{m_0 L_0/2} \text{ for some } x \in \Lambda_{L_0(0)}\}$$
$$\leqq (L_0 + 1)^d \mu[E_0 - 2d - 2\eta - e^{m_0 L_0/2},\ E_0 + 2d + 2\eta + e^{m_0 L_0/2}],$$

the proposition follows.

We will now list conditions under which (P2) has been proven for arbitrary dimension.

Theorem A.1.3.

(i) *If μ is absolutely continuous with a bounded density, then for any $L > 0$, $E \in \mathbf{R}$* and $\varepsilon > 0$,

$$\mathbf{P}\{d(E, \sigma(H_{\Lambda_{L(0)}})) \leqq \varepsilon\} \leqq 2 \left\| \frac{d\mu}{dv} \right\|_\infty^{-1} (L+1)^d \varepsilon.$$

(ii) *If $\mu([a,b]) \leqq \delta^{-1} |b - a|^\rho$ for some $0 < \rho < 1$, $\delta > 0$, and all $a < b \in \mathbf{R}$, we have*

$$\mathbf{P}\{d(E, \sigma(H_{\Lambda_{L(0)}})) \leqq \varepsilon\} \leqq 2(1-\rho)^{-1} \delta^{-1} (L+1)^d \varepsilon^\rho \textit{ for all } E \in \mathbf{R},\ L > 0 \textit{ and } \varepsilon > 0.$$

(iii) *Let μ be Holder continuous of order $\rho > 0$,* i.e., *$\delta_\rho(\mu) > 0$. Then for any $0 < \delta < \delta_\rho(\mu)$, we can find $\eta = \eta(\mu, \delta) > 0$ such that*

$$\mathbf{P}\{d(E, \sigma(H_{\Lambda_{L(0)}})) \leqq \varepsilon\} \leqq \delta^{-1} 2^\rho (L+1)^{d(1+\rho)} \varepsilon^\rho$$

for all $E \in \mathbf{R}$, $\varepsilon > 0$ and $L > 0$ such that $\varepsilon(L+1)^d < \eta$.

(iv) *Let $\mu = (1-t)\mu_1 + t\mu_2$, where $0 \leqq t < 1$ and μ_1, μ_2 are probability measures with compact support such that μ_1 is Holder continuous of order $\rho > 0$ and μ_2 is otherwise arbitrary. Let $0 < \delta < \delta_\rho(\mu_1)$ and suppose $0 < \varepsilon_L^a < (L+1)^{-d}$ for and some $0 < a < \frac{1}{2}$ and all L large enough. Then there exist $t(d)$, where $t(1) = t(2) = 1$*

and $0 < t(d) < 1$ *for* $d \geqq 3$, *such that for all L large enough we have*

$$\mathbf{P}\{d(E, \sigma(H_{\Lambda_L(0)})) \leqq \varepsilon_L\} \leqq \delta^{-1} 2^{\rho}(L+1)^{d(1+\rho)} \varepsilon_L^{\alpha} + \varepsilon_L^{\eta} \text{ for some}$$
$$\alpha = \alpha(\delta, \rho, a) > 0 \text{ and } \eta = \eta(t) > 0 \text{ for } 0 \leqq t < t(d),$$
$$\text{and } \eta(t) \to \infty \text{ as } t \to 0.$$

(i) was proven by Fröhlich and Spencer [7] from Wegner's estimate on the density of states [17]. (ii), (iii) and (iv) were proven by Carmona, Klein and Martinelli [6].

A.2. One Dimension. If μ is not a single delta function and $\int |v|^{\eta} d\mu(v) < \infty$ for some $\eta > 0$, (P1) and (P2) always hold in one dimension. The proof in this generality use the Theory of Random Matrices, namely the circle of ideas associated to Furstenberg's Theorem (e.g., [18]).

(P2) was proved by Carmona, Klein and Martinelli [6]. Their precise result is

Theorem A.2.1. *Let* $d = 1$. *Suppose the support of* μ *is not concentrated in a single point and* $\int |v|^{\eta} d\mu(v) < \infty$ *for some* $\eta > 0$. *Let I be a compact interval. For any* $0 < \beta < 1$ *and* $\sigma > 0$ *there exist* $L_0 = L_0(I, \beta, \sigma) < \infty$ *and* $\tau = \tau(I, \beta, \sigma) > 0$ *such that*

$$\mathbf{P}\{d(E, \sigma(H_{\Lambda_L(0)}) \leqq e^{-\sigma L^{\beta}}\} \leqq e^{-\tau L^{\beta}} \quad \text{for all} \quad E \in I \quad \text{and} \quad L \geqq L_0.$$

We now turn to (P1). Frohlich, Martinelli, Scoppola and Spencer [4, 5] showed how to obtain a related hypothesis from Furstenberg's theorem. For completeness we will sketch a proof of (P1), based on a similar result for the strip by Klein, Lacroix and Speis [19]. We will denote by $\gamma(E)$ the Lyapunov exponent (e.g., [18]).

Theorem A.2.2. *Let* $d = 1$. *Suppose the support of* μ *is not concentrated in a single point and* $\int |v|^{\eta} d\mu(v) < \infty$ *for some* $\eta > 0$. *Let* $\varepsilon > 0$ *and* $0 < \beta < 1$ *be given. Fix* $E_0 \in \mathbf{R}$. *Then there exist* $L_0 = L_0(E_0, \varepsilon, \beta) < \infty$ *and* $k = k(E_0, \varepsilon, \beta) > 0$ *such that*

$$\mathbf{P}\{\Lambda_L(0) \text{ is } (\gamma(E_0) - \varepsilon, E_0)\text{-regular}\} \geqq 1 - e^{-kL^{\beta}} \quad \text{for all} \quad L \geqq L_0.$$

Proof. We will use the notation $H_{[j_1, j_2]}$, $G_{[j_1, j_2]}(E)$, $[j_1, j_2]$ being an interval in $\mathbf{Z}$. Notice that $\Lambda_L(0) = [-L/2, L/2]$ if L in an even positive integer.

It suffices to show that, for given $E_0 \in \mathbf{R}$, $\varepsilon > 0$ and $0 < \beta < 1$, we can find N_0 and $k > 0$ such that

$$\mathbf{P}\{E_0 \notin \sigma(H_{[-n,n]}) \text{ and } |G_{[-n,n]}(E_0, 0, n)| \leqq e^{-(\gamma(E_0) - \varepsilon)n}\} \geqq 1 - e^{-kn^{\beta}}$$

for all integers $n \geqq N_0$.

By the resolvent identity,

$$\begin{aligned} G_{[-n,n]}(E_0, 0, n) &= G_{[0,n]}(E_0, 0, n) + G_{[-n,n]}(E_0; 0, -1) G_{[0,n]}(E_0, 0, n) \\ &= G_{[0,n]}(E_0; 0, n)(1 + G_{[-n,n]}(E_0, 0, -1)). \end{aligned}$$

In view of Theorem A.2.1, it suffices to prove that given $E_0 \in \mathbf{R}$, $\varepsilon > 0$ and $0 < \beta < 1$, there exist N_1 and $\theta > 0$ such that

$$\begin{aligned} \mathbf{P}\{E_0 \notin \sigma(H_{[0,n]}) \text{ and } |G_{[0,n]}(E_0; 0, n)| \leqq e^{-(\gamma(E_0) - \varepsilon)n}\} \\ \geqq 1 - e^{-\theta n^{\beta}} \quad \text{for all} \quad n \geqq N_1. \end{aligned}$$

Let us consider the operator

$$\tilde{H}_{[0,n]} = H_{[0,n]} - iP_n,$$

where P_n is the projection onto the delta function at the lattice site n i.e., $P_n\varphi = \varphi(n)\delta_n$, where $\delta_n(x) = 1$ if $x = n$ and 0 otherwise.

If $E \in \mathbf{R}$, $\tilde{H}_{[0,n]} - E$ is always invertible. To see this notice that if $(\tilde{H}_{[0,n]} - E)\varphi = 0$, then $[(\tilde{H}_{[0,n]} - E)\varphi](x) = 0$ for $x \in [0, n-1]$. From the uniqueness of solutions it follows that $\varphi = \varphi(0)\varphi_1$, where φ_1 is real valued. But then $(\tilde{H}_{[0,n]} - E)\varphi_1 = 0$, so we can conclude that $\varphi_1(n) = \varphi_1(n-1) = 0$, and hence that φ_1 is identically zero.

Let $\tilde{G}_{[0,n]}(E) = (\tilde{H}_{[0,n]} - E)^{-1}$, and let ψ_E be the unique solution of the equation

$$H_{[0,\infty)}\psi_E = E\psi_E \quad \text{with} \quad \psi_E(0) = 1.$$

It follows that

$$\tilde{G}_{[0,n]}(E; 0, n) = \psi_E(n+1) - i\psi_E(n).$$

A careful analysis of Lemma 5.1 in [6] and its proof, followed by the use of Chebychev's inequality, gives that, for given $E_0 \in \mathbf{R}$ and $\varepsilon > 0$,

$$\mathbf{P}\{|\tilde{G}_{[0,n]}(E_0; 0, n)| \geqq e^{-(\gamma(E_0) - \varepsilon)n}\}$$
$$\geqq e^{-\tau n} \text{ for some } \tau > 0 \text{ and all } n \geqq N_2 \text{ for some } N_2 < \infty.$$

By the resolvent equation,

$$G_{[0,n]}(E_0) = \tilde{G}_{[0,n]}(E_0) - i\tilde{G}_{[0,n]}(E_0)P_nG_{[0,n]}(E_0)$$

and hence

$$G_{[0,n]}(E_0; 0, n) = \tilde{G}_{[0,n]}(E_0; 0, n)[1 - iG_{[0,n]}(E_0; n, n)].$$

If

$$|\tilde{G}_{[0,n]}(E_0; 0, n) \leqq e^{-(\gamma(E_0) - \varepsilon)n}, \quad \| G_{[0,n]}(E_0) \| \leqq e^{n\beta},$$

we get

$$|G_{[0,n]}(E_0; 0, n)| \leqq e^{-(\gamma(E_0) - 2\varepsilon)n}$$

for all $n \geqq N_3$ for some $N_3 < \infty$.

Using once again Theorem A.2.1, the Theorem follows.

Theorems A.2.1 and A.2.2 have been extended to a one-dimensional strip by Klein, Lacroix and Speis [9].

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