Commun. Math. Phys. 121, 85–90 (1989)

Communications in Mathematical Physics

Why Instantons are Monopoles*

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Abstract. It is shown that instantons are hyperbolic monopoles for the loop group with non-maximal symmetry breaking at infinity.

1. Introduction

Monopoles in R^3 have been usually considered as time invariant instantons. We wish to turn this idea on its head and show that instantons are really monopoles on hyperbolic three space with structure group the loop group and with reduction at infinity to the subgroup of constant loops. This means that instantons are monopoles with non-maximal symmetry breaking.

This approach to instantons, while giving no new results, does shed light on the result of Atiyah that instantons are equivalent to rational maps from one dimensional complex projective space into the based loops which is the homogeneous space of the loop group divided by the constant loops. Atiyah's result then becomes another example of the general conjecture that monopoles, on R^3 or H^3 for a group K with reduction at infinity to a subgroup H are equivalent to rational maps of the two sphere into the homogeneous space K/H, or as Atiyah has described them, instantons for the corresponding two dimensional sigma model.

The status of this conjecture now is that on R^3 it has been proved for $SU(2)$ by Donaldson (1984) and for the other classical groups by Hurtubise (1988), all in the case of maximal symmetry breaking. For those hyperbolic monopoles which arise from invariant instantons (Atiyah 1984a) Atiyah has shown that the conjecture is also true, and lastly our observation along with Atiyah's results for instantons show that it is true for particular hyperbolic monopoles with loop group structure group and non-maximal symmetry breaking.

The correspondence between instantons and monopoles is explained in Sect. 2 and in Sect. 3 we show how the corresponding "twistor pictures" relate and how the spectral curve of an instanton arises. In the final section we make a conjecture motivated by this correspondence.

* Research supported in part by NSF grant DMS-8506130

2. Instantons and Monopoles

Consider a K bundle $P \to S^4$ with connection $\hat{A}$. Assume that $U(1)$ acts on the bundle so that its action on S^4 has a two sphere S^2_∞ of fixed points and acts freely on the rest (an explicit action is described below). The quotient of $S^4 - S^2_\infty$ by the circle action is (conformally) equivalent to hyperbolic three space H^3. Let $U_0 = S^4 - S^2_\infty = H^3 \times S^1$ and $U_\infty = S^4 - \{0\} \times S^1$, where 0 is the origin in hyperbolic space. Then U_∞ is diffeomorphic to four space minus a line by stereographic projection. The intersection of these two open sets is $(0, \infty) \times S^2_\infty \times S^1$.

Choose sections

$$s_0 : U_0 \to P, \quad s_\infty : U_\infty \to P. \tag{2.1}$$

This is possible as U_0 and U_∞ have no four dimensional topology. The transition function defined by $s_0 = g s_\infty$ is essentially a map $g : S^1 \times S^2 \to K$ and has a degree

$$\deg(g) : Z \simeq H^2(S^2) \otimes H^1(S^1) \to H^3(K) \simeq Z, \tag{2.2}$$

which is the Pontrjagin class of the bundle P.

The manifold $S^4 - S^2$ is conformally equivalent to $H^3 \times S^1$, where H^3 is hyperbolic three space. We can define a connection $A = \sum_{i=1}^{3} (A_i, 0) dx^i$ and Higgs field $\Phi = (A_\theta, i)$ on H^3 with values in $L\Omega K \oplus LU(1)$, the semi-direct product of the loop algebra and the Lie algebra of $U(1)$, by letting

$$s_0^* \hat{A} = \sum_{i=1}^{3} A_i dx^i + A_\theta d\theta. \tag{2.3}$$

It was shown in Garland and Murray (1988), and it is straightforward to check here, that when the.connection on the four sphere is self-dual the connection and Higgs field satisfy the Bogomolny equations. To show that they are the connection and Higgs field of a monopole we just have to check that the Higgs field satisfies the correct boundary conditions. Because the Yang–Mills–Higgs action of the monopole is the Yang–Mills action of the instanton it is finite, and therefore we need only check what kind of symmetry breaking occurs at infinity.

Under the conformal equivalence the length of the unit tangent vector to the circle in $H^3 \times S^1$ approaches zero as we approach the sphere at infinity which is S^2_∞. It follows that $\langle s_\infty^* \hat{A}, \partial/\partial\theta \rangle \to 0$ as we approach infinity. Hence

$$\Phi \to (g^{-1} \partial_\theta g, 1)$$

asymptotically in H^3. If we choose a section s_0 corresponding to parallel transport along radial directions in H^3, then as we approach the sphere at infinity g has a limit

$$g_\infty : S^2_\infty \times S^1 \to K$$

and the Higgs field also has a limit

$$\Phi_\infty : S^2_\infty \to L\Omega K \oplus LU(1), \tag{2.4}$$

whose image lies in an orbit of $\Omega K \times U(1)$ with isotropy subgroup $K \times U(1)$.

This monopole defined by an instanton is therefore not one with maximal symmetry breaking as the maximal torus of $\Omega K \times U(1)$ is the subgroup of constant loops into the torus of K semi-direct product with $U(1)$. The subgroup of all constant loops is the centraliser of the smaller torus $1 \times U(1)$. This is consistent with the results of Atiyah (1984a) as the based loops can be identified with the homogeneous space $\Omega K/K$.

The Higgs field at infinity then is a map

$$\Phi_\infty : S^2_\infty \to \Omega K/K$$

and defines a class in $\pi_2(\Omega K/K) \simeq Z$ which is the magnetic charge and equal to the Pontrjagin class of the original bundle. It is also the chern class of the line bundle obtained by pulling back the line bundle on $\Omega K/K$ corresponding to the central extension of the loop group.

If we change the sections to $s_0' = h_0 s_0$ and $s_\infty' = h_\infty s_\infty$, then

$$A_i' = h_0^{-1} A_i h_0 + h_0^{-1} \frac{\partial h_0}{\partial x^i}, \quad \Phi' = h_0^{-1} \Phi h_0 + h_0^{-1} \frac{\partial h_0}{\partial \theta}.$$

This means that, as expected, the connection and Higgs field gauge transform according to the twisted action of the semi-direct product $U(1) \times \Omega K$.

3. The Twistor Picture

To understand the circle action on CP_3, the twistor space of S^4, consider H the space of quaternions and regard the direct sum $H \oplus H$ as a right quaternionic vector space. The twistor fibering is

$$\begin{aligned} \pi : CP_3 &\to HP_1, \\ [z_0, \ldots, z_3] &\mapsto [z_0 + jz_3, z_1 - jz_2], \end{aligned} \tag{3.1}$$

and we identify HP_1 with S^4.

If $\lambda \in S^1$ then it can act on $H \oplus H$ by left multiplication and this is a quaternionic linear map so it defines an action on $P_1 H$. This circle action lifts and complexifies to an action of $C^\times$ given by

$$\lambda[z_0, z_1, z_2, z_3] = [\lambda^{1/2} z_0, \lambda^{1/2} z_1, \lambda^{-1/2} z_2, \lambda^{-1/2} z_3] \tag{3.2}$$

on CP_3. Because we can multiply through by $\lambda^{1/2}$ or $\lambda^{-1/2}$ the ambiguity in the square roots is only a problem if we try to lift the action from projective space to C^4.

The fixed point set of the circle action on $P_1 H$ is $S^2_\infty = P_1 C \subset P_1 H$ and that of $C^\times$ *on* CP_3 *is the union of two copies of* P_1,

$$P_1^+ = [z_0, z_1, 0, 0] \subset CP_3, \tag{3.3}$$

$$P_1^- = [0, 0, z_2, z_3] \subset CP_3, \tag{3.4}$$

which both project diffeomorphically to S^2_∞. The induced diffeomorphism from P_1^+ to P_1^- is anti-holomorphic.

If we remove P_1^+ and P_1^- from CP_3 there is a $C^\times$ fibering

$$\begin{aligned} CP_3 - P_1^+ \cup P_1^- &\to P_1^+ \times P_1^-, \\ [z_0, z_1, z_2, z_3] &\mapsto ([z_0, z_1, 0, 0], [0, 0, z_2, z_3]), \end{aligned} \tag{3.5}$$

whose fibres are the orbits of the $C^\times$ action. This fibering is therefore a $C^\times$ principal bundle which is, in fact, the frame bundle of the line bundle $\mathcal{O}(-1,1)$. This is the line bundle on $P_1^+ \times P_1^-$ which has chern class -1 times the P_1^+ generator and $+1$ times the P_1^- generator.

Recall the twistor correspondence for calorons in Garland & Murray 1988. The caloron is an instanton on $R^3 \times S^1$ and this has as twistor space $(CP_3 - CP_1)/2\pi Z$ which is a $C^\times$ bundle over TP_1 the minitwistor space of R^3.

If we replace R^3 by H^3 then, following Hitchin 1982, the minitwistor space to consider is the space of oriented geodesics. For the ball model of hyperbolic space the geodesics are arcs of circles intersecting the boundary two sphere "at infinity" orthogonally, so the oriented geodesics are parametrized by ordered pairs of distinct points on the sphere at infinity. Because of the definition of the complex structure on minitwistor space it is more natural to label a geodesic by the point at $+\infty$ and the antipode of the point at $-\infty$. So the minitwistor space of H^3 is $S^2_\infty \times S^2_\infty$ with the antidiagonal removed.

Atiyah (1984b) showed that the hyperbolic monopoles which are circle invariant instantons have the property that their holomorphic bundle on minitwistor space extends across the antidiagonal. Similarly here the twistor space of $H^3 \times S^1$ is $CP_3 - \pi^{-1}(S^2_\infty)$.This is a $C^\times$ bundle over $P_1^+ \times P_1^- - \bar{\Delta}$, where $\bar{\Delta} = \{([z_0, z_1], [\bar{z}_1, -\bar{z}_0]\}$, however as we have seen the instanton bundle is defined on all of $P_1^+ \times P_1^-$.

In the caloron case (Garland & Murray 1988) it was useful to compactify the $C^\times$ bundle over minitwistor space and we can do this for instantons as well. The result is a P_1 bundle $\mathscr{X}$ over $P_1^+ \times P_1^-$ which is the blow up of CP_3 along P_1^+ and P_1^-. This P_1 bundle has two distinguished sections $\mathscr{X}^\infty$ and $\mathscr{X}^0$ and the complement of these is identified with the original $C^\times$ bundle. These subvarieties are identified by the projection map with $P_1^+ \times P_1^-$.

There is another projection map $\mathscr{X} \to CP_3$ which is biholomorphic away from the zero and infinity subvarieties and restricted to them becomes the two natural projections of $P_1^+ \times P_1^-$ to P_1^+ and P_1^-. The instanton bundle on CP_3 pulls back under this map to define a bundle $\mathscr{E}$ on $\mathscr{X}$. In the same way as for calorons (Garland and Murray 1988) we can push this bundle down to $P_1^+ \times P_1^-$ and obtain an infinite dimensional bundle which is the bundle that would be obtained by applying the twistor correspondence for hyperbolic monopoles (Atiyah 1984b) directly to the loop group bundle on hyperbolic space.

The points on $P_1^+ \times P_1^-$ for which the restriction of the bundle $\mathscr{E}$ to the corresponding fibre in $\mathscr{X}$ is not trivial define the spectral curve of the instanton. (See also Hurtubise 1986). There will be some additional spectral data in the form of a section of a sheaf over this curve which will determine the instanton as is shown in Hurtubise and Murray 1988. Rather than describe that when we all ready have a good description of instantons as rational maps, let us consider the well understood case of charge 1 instantons.

The charge 1 instantons have a moduli space which is the five ball. There is the rotationally symmetric instanton which is the centre of the ball and for every x in S^4, the boundary of the 5 ball, the instantons with maximum field strength at x lie along the ray from the centre to x. The point x itself can be thought of as an instanton with a delta function field strength at x.

The possible spectral curves are in one to one correspondence with the points of H^3 as in the flat case (see Atiyah 1984b or Hitchin 1982). The explicit description in Atiyah 1979 gives us the following description of the spectral curves of an instanton. We should think of the 5 ball as analogous to the three ball with the vertical axis replaced by a copy of H^3 intersecting the boundary S^4 in a copy of S^2_∞. The circle acts by rotations in the "lines of latitude" around this H^3. The H^3 parametrizes the spectral curves and the instantons with the same spectral curve form rotationally invariant surfaces cutting this H^3 once.

4. Invariant Instantons

In Atiyah 1984b it was shown that instantons invariant under a circle action are the same as certain hyperbolic monopoles. In this section we show how our results reduce to that case.

To discuss invariant instantons you have to choose an action of the circle on the principal bundle P covering the action of S^4. Then the section s_0 is chosen to be invariant under this action. If the connection is invariant it follows that its pullback under s_0 is invariant and that the connection and Higgs field of the monopole are invariant. As we approach S^2_∞ the invariant section has to have as limit over each point at infinity a circle orbit. If we identify the fibre of P over the point with K then this circle in K is (conjugate to) a subgroup, and hence the Higgs field at infinity is of the form shown by Atiyah; it is conjugate to an element of the Lie algebra of K whose exponential generates a loop.

Notice that this does not contradict the fact that the Higgs field was shown in Sect. 2 to be in the orbit of 0. It still is if you allow the action of the loop group. However if you want to consider invariant loop group monopoles as monopoles for K then you must only allow the action of K.

5. Conjecture

It is believed that the Yang–Mills action has no higher order critical points, although this has not been proved. Taubes (1982) showed that there are higher order critical points of the YMH action in R^3 and he also observed that his method of proof did not work for the Yang–Mills action. It is tempting to conjecture that the difference between these two cases is the difference between the flat and hyperbolic metrics. Are there any higher order critical points of the YMH action on hyperbolic space?

References

Atiyah, M. F.: Geometry of Yang–Mills Fields. Accademia Nazionale dei Lincei Scuola Normale Superiore.Ležioni Fermiane Pisa-1979. Also available reprinted by Oxford University Press (1988) as part of the collected works of M.F. Atiyah

Atiyah, M. F.: Instantons in two and four dimensions. Commun. Math. Phys. **93**, 437–451 (1984a)
Atiyah, M. F.: Magnetic monopoles in hyperbolic space. In: Proceedings of the International Colloqium on Vector Bundles. Tata Institute, Bombay (1984b), pp 1–34
Atiyah, M. F., Hitchin, N. J., Singer, I. M.: Self-duality in four-dimensional Riemannian geometry. Proc. R. Soc. Lond. **A362**, 425–461 (1978)
Donaldson, S. K.: Nahm's equations and the Classification of Monopoles. Commun. Math. Phys. **96**, 387–407 (1984)
Garland, H., Murray, M. K.: Kac-Moody monopoles and periodic instantons. Commun. Math. Phys. **120**, 335–351 (1988)
Hitchin, N. J.: Monopoles and geodesics. Commun. Math. Phys. **83**, 579–602 (1982)
Hurtubise, J. C.: Instantons and jumping lines. Commun. Math. Phys. **105**, 107–122 (1986)
Hurtubise, J. C.: The classification of monopoles for the classical groups. Commun. Math. Phys. (1988)
Murray, M. K.: Non-abelian magnetic monopoles. Commun. Math. Phys. **96**, 539–565 (1984)
Taubes, C. H.: The existence of a non-minimal solution to the Yang–Mills–Higgs equations on R^3. Parts I and II. Commun. Math. Phys. **86**, 257–298 and 299–320 (1982)

Communicated by A. Jaffe

Received August 16, 1988