

Canonical Partition Functions of Hamiltonian Systems and the Stationary Phase Formula

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Abstract. We show that there is a Symplectic Action of a Torus associated to Harmonic flows on the Cotangent Bundle of a semi-simple Lie Algebra. This allows to obtain a completely classical proof of the Gallavotti–Marchioro Formula by the method of the Stationary Phase.

We compute the Canonical Partition Function,

$$Z(\beta) = \int_{V^{2m}} \exp(-\beta H) \Omega \quad (\beta > 0)$$

for a class of integrable Hamiltonian systems. H is the Hamiltonian function, $\Omega = \omega^m/m!$ is the volume form associated to a symplectic form ω of a symplectic manifold V^{2m} . We compute this integral by a singularity analysis and by an adaptation of the Duistermaat–Heckman theorem [D–H] on the exactness of the stationary phase formula. The proof that Berline–Vergne [B–V] gave of this theorem can be adapted to our situation where V^{2m} is not compact and where the parameter β is positive.

We obtain, for instance, the expression of the Canonical Partition Function for integrable systems which are reductions of the Harmonic flow on the cotangent bundle of a semi-simple Lie algebra. To this purpose, we need to check that these systems are associated to an Hamiltonian action of the torus. We show that the Hamiltonian flows of the eigenvalues of the Lax matrix are all periodic.

Our result extends a formula which was precedingly obtained by Gallavotti and Marchioro [G–M] for the Calogero–Moser system with an external quadratic potential which corresponds to the root system of type A_{m-1} . Gallavotti and Marchioro got their result by considering the quantum system and then by taking a limit ($\hbar \rightarrow 0$).

On the contrary, our proof is purely classical; it illustrates how the Duistermaat–Heckman formula can be used also to compute thermodynamical sums.

1 Adaptation of the Stationary Phase Method

We consider the proof of [B. V.] of the theorem of the stationary phase of Duistermaat and Heckman.

Let V be an arbitrary n -dimensional manifold and let ξ be a vector field on V . Let $\mathcal{Q} = \mathcal{Q}^0 + \dots + \mathcal{Q}^n$ be the De Rham complex on V , with the differential $d: \mathcal{Q}^i \rightarrow \mathcal{Q}^{i+1}$. Let δ_ξ be the operator $d + i(\xi)$. By the Cartan formula, we get $\delta_\xi^2 = L_\xi$ (Lie derivative along the vector field ξ).

We assume furthermore that $n = 2m$, V is symplectic, that there is an Hamiltonian action of the torus \mathbb{T}^m on V . Let J be the moment map, we take as ξ a vector field so that $\xi = J^*(X)$ for an element X of the Lie algebra of the torus and that the zeroes of ξ are fixed points for the action of the torus.

Let us put on V an invariant Riemannian metric $\langle \cdot, \cdot \rangle$. We define then a 1-form θ on the open set $V_0 = (x \in V / \xi(x) \neq 0)$ as follows:

$$\theta_x(v) = \langle \xi(x), v \rangle / \langle \xi(x), \xi(x) \rangle. \quad (1)$$

This 1-form has the properties:

- i) $L_\xi \theta = 0$,
- ii) $\theta(\xi) = 1$.

Lemma 1-1. (Berline-Vergne). *Let $\mu = \mu_{2m} + \dots + \mu_0$ be an even element of \mathcal{G} such that $\delta_\xi \mu = 0$. Then μ_{2m} is d -exact on V_0 .*

Proof. We introduce

$$v = \theta \wedge (1 + d\theta)^{-1} \wedge \mu. \quad (2)$$

We get from i) and ii) $i(\xi)dv = i(\xi)\mu$, and since on V_0 , $i(\xi): \mathcal{Q}^{2m} \rightarrow \mathcal{Q}^{2m-1}$ is injective, $\mu_{2m} = dv_{2m-1}$. \square

If we set $\alpha = H - \omega$, where H is the Hamiltonian of ξ , then, we have, $\delta_\xi \alpha = 0$. We apply the lemma to

$$\mu = \exp - \beta \alpha = \exp(-\beta H)(1 + B\omega + \dots + (\beta^m/m!) \omega^m). \quad (3)$$

We get:

$$\exp(-\beta H)\Omega = d(\beta^{-m} v_{2m-1}).$$

We assume furthermore that:

—(c₁) H is a rational (respectively meromorphic) positive function on \mathbb{R}^{2m} ; it blows up along finitely many algebraic (respectively analytic) subsets of codimension 1 denoted $\Delta_i (i = 1, \dots, m)$.

—(c₂) $V = T^*W$, $W = \mathbb{R}^m / U_{i=1}^m \Delta_i$, $\omega = \Sigma dx_1 \wedge dy_i$

—(c₃) The only critical points of H are minima. This implies that the critical points of H must be fixed points for the action of the torus \mathbb{T}^m . (otherwise they could not be isolated)

—(c₄) $H(x, y) \rightarrow +\infty$ when $(x, y) \rightarrow \infty$.

Remark. In our examples, H is convex, so that the condition (c₃) is naturally fulfilled.

Let $d(p, \Delta)$ be the distance in \mathbb{R}^{2m} from a point p to a closed set Δ . We introduce the set:

$$D_R = \{(x, y) / \|(x, y)\| < R, d((x, y), \Delta_i) > R^{-1} \text{ for all } (i = 1, \dots, s)\}$$

We denote by C , the set of the Critical points of the function H and let $B_p, p \in C$, be small balls centered about the critical points of H . From our assumptions, the Canonical Partition Function of H :

$$Z(\beta) = \int_{\mathbb{R}^{2m}} \exp(-\beta H) \Omega \tag{4}$$

exists and is equal to

$$Z(\beta) = \lim_{R \rightarrow \infty} \int_{D_R} \exp(-\beta H) \Omega.$$

By Lemma 1–1, there is a $2m - 1$ form v_{2m-1} defined by (2) so that by Stokes' theorem, we have,

$$Z(\beta) = \sum_{p \in C} \int_{B_p} \exp(-\beta H) \Omega - \int_{\delta B_p} \beta^{-m} v_{2m-1} + \lim_{R \rightarrow \infty} \int_{\delta D_R} \beta^{-m} v_{2m-1}. \tag{5}$$

Our situation differs from the Duistermaat–Heckman case as β is a positive parameter (versus a purely imaginary one) and as the manifold V is not compact. So we have to show that the limit in (5) is equal to zero. This can be done with the Lojasiewicz inequality [L].

Lemma 1–2. *The contribution of the boundary vanishes:*

$$\lim_{R \rightarrow \infty} \int_{\delta D_R} \beta^{-m} v_{2m-1} = 0.$$

Proof. H is a rational (respectively meromorphic) function on \mathbb{R}^{2m} , since it blows up either when (x, y) tends to infinity or along the algebraic (respectively analytic) subsets $\Delta_i (i = 1, \dots, s)$, the Lojasiewicz inequality [L] shows the existence of a positive constant 1 such that

$$\exp(-\beta H(x, y)) \ll \exp(-R^1) \quad \text{for } (x, y) \in \delta D_R, R > 0.$$

From (1), the 1-form θ blows up eventually when (x, y) approaches the subsets Δ_i ; but, again by the Lojasiewicz inequality, it has at most a polynomial growth there.

So that by (2) the form v_{2m-1} satisfies

$$\|v_{2m-1}\| \ll cR^\tau \exp - R^1 \quad \text{for some constants } c, \tau > 0. \tag{6}$$

Also, we see easily that the volume of δD_R increases as an algebraic function of R as R tends to infinity. This ends the proof. \square

We finish now as in [B–V]. By the canonical form theorem [G–S] (from the assumption that the critical points of H are of Morse type and that H is associated to the action of a torus), we can find symplectic coordinates $(x_1, \dots, x_m; y_1, \dots, y_m)$, given by a local diffeomorphism Φ , on the ball B_p such that:

$$\Phi^* H = H(p) + \sum_{k=1}^m \lambda_k(p) (x_k^2 + y_k^2) = H_0. \tag{7}$$

If we compute the terms on the right-hand side of (5) by making a second application of Stokes' theorem, we get:

$$\int_{\Phi^{-1}(B_p)} \exp - \beta H_0 \Omega - \int_{\delta \Phi^{-1}(B_p)} v_{2m-1}^0 = \int_{\mathbb{R}^{2m}} \exp - \beta H_0 \Omega,$$

where $v_{2m-1}^0 = \Phi^*(v_{2m-1})$.

The right-hand side of Eq. (5) can now be easily computed. We see that $Z(\beta)$ is determined by the critical values of H and by the eigenvalues of the Hessian of H at its critical points. We have now proved the

Theorem 1–3. *Under the conditions (c), and with the notations of (7), the Canonical Partition Function of H is equal to:*

$$Z(\beta) = \sum_{p \in \mathcal{C}} \pi^m \beta^{-m} \left(\prod_{k=1}^m \lambda_k(p) \right)^{-1} \exp - \beta H(p). \quad (8)$$

2 Hamiltonian Torus Action

Let \mathcal{G} be a semi-simple Lie algebra of dimension m . The cotangent bundle $T^*\mathcal{G}$ may be identified with $\mathcal{G} \times \mathcal{G}^* \simeq \mathcal{G} \times \mathcal{G}$. As a cotangent bundle, it has a symplectic structure which can be written [K. K. S.]:

$$\omega = \text{Tr } dX \wedge dL, \quad (X, L) \in \mathcal{G} \times \mathcal{G}.$$

We define on $T^*\mathcal{G}$ the Harmonic Flow by the Hamiltonian Function

$$H = (1/2) \text{Tr}(\lambda^2 X^2 + L^2). \quad (9)$$

The group G acts on \mathcal{G} by the adjoint action; this action lifts into an Hamiltonian action on $T^*\mathcal{G}$. The Harmonic flow has m commuting independent integrals

$$H_k = \text{Tr}(\lambda^2 X^2 + L^2)^k,$$

which are G -invariant functions.

Following [O. P], a projection of the Hamiltonian system (9) on the cotangent bundle of the Weyl chamber T^*W , defines the system

$$H = (1/2) \sum_{i=1}^m (y_i^2 + \lambda^2 x_i^2) + (1/2) \sum_{\alpha \in A^+} g_\alpha^2 |\alpha|^2 / \langle x, \alpha \rangle^2, \quad (10)$$

$$\omega = \sum_{i=1}^m dx_i \wedge dy_i.$$

In (10) A^+ means the set of the positive roots of the Lie algebra \mathcal{G} and the g_α are arbitrary positive constants which are equal for two elements α, α' which are conjugated under the Weyl group.

Examples:

1—If \mathcal{G} is of A_{m-1} type, the walls of the Weyl chamber are given by the hyperplanes

$$(x_i = x_j, i \neq j),$$

the Weyl group is the permutation group G_m , it acts transitively on the set A^+ , so all the constants g_α must be equal. In that case, we get

$$H = (1/2) \sum_{i=1}^m (y_i^2 + \lambda^2 x_i^2) + g^2 \sum_{i < j} (x_i - x_j)^{-2}, \quad (11)$$

which is the Calogero–Moser system with an external quadratic potential.

2—In the $B_m = C_m$ case, the walls of the Weyl chambers are given by the hyperplanes

$$(x_i = x_j, i \neq j), \quad (x_i = -x_j, i \neq j), \quad (x_i = 0).$$

The set $A +$ has two orbits under the action of the Weyl group. So that there are two constants g_1 and g_2 and that we get

$$H = (1/2) \sum_{i=1}^m (y_i^2 + \lambda^2 x_i^2) + g_1^2 \left(\sum_{i < j} (x_i - x_j)^{-2} + \sum_{i < j} (x_i + x_j)^{-2} \right) + g_2^2 \sum_{i=1}^m x_i^{-2}. \quad (12)$$

3—In the D_m case, we obtain the Hamiltonian system which corresponds to (12) with $g_2 = 0$.

We explain now in details why we can associate an Hamiltonian torus action to the Calogero–Moser system with an external quadratic potential. We can proceed in quite a similar way for the more general system (10) (type V in the Olshanetsky–Perelomov classification [O. P]) with slightly different notations.

We use the matrices (X, L) defined by:

$$\begin{aligned} X_{ij} &= x_i \delta_{ij}, \\ L_{ij} &= y_i \delta_{ij} + \sqrt{-1} g / (x_i - x_j) (1 - \delta_{ij}). \end{aligned} \quad (13)$$

We note that we have the relation $[X, L] = \sqrt{-1} g C$, where the matrix C is defined by $C_{ij} = 1 - \delta_{ij}$. Under the Hamiltonian flow of the function $H = (1/2) \text{Tr}(\lambda^2 X^2 + L^2)$, the matrices X, L have the following evolution:

$$\dot{X} = [A, X] + L, \quad \dot{L} = [A, L] - \lambda^2 X, \quad (14)$$

where A is given by:

$$A_{ij} = \sqrt{-1} \delta_{ij} \sum_{k \neq i} g / (x_i - x_k)^2 + (1 - \delta_{ij}) (g \sqrt{-1}) / (x_i - x_j)^2. \quad (15)$$

We introduce the matrices $Z = \sqrt{-1} \lambda X + L$ and $Z^* = -\sqrt{-1} \lambda X + L$; the matrix $P = ZZ^*$ undergoes an isospectral deformation

$$\dot{P} = [A, P], \quad (16)$$

and we remark that $Z^* = \overline{Z}$ (transposed of the complex conjugated) and accordingly that $P^* = P$. The eigenvalues of P are algebraic functions on T^*W . Let $\Lambda (= \Lambda_p)$ be one of these eigenvalues.

Theorem. *The Hamiltonian flow defined by Λ on T^*W is periodic. All the periodic orbits have the same period (not necessarily primitive) $2\pi/\lambda$.*

Let us introduce the eigenvector Ψ of P for the eigenvalue Λ . The Hamiltonian flow of Λ is defined by the equations:

$$\{\Lambda, x_i\} = \partial \Lambda / \partial y_i, \quad (17)$$

$$\{\Lambda, y_i\} = -\partial \Lambda / \partial x_i. \quad (18)$$

The matrix P can be diagonalized by an element V of the unitary group. If we denote by \langle , \rangle the Hermitian product, we get

$$\begin{aligned}\partial\Lambda/\partial x_i &= \partial\langle P\Psi, \Psi\rangle/\partial x_i = \langle \partial P/\partial x_i \Psi, \Psi\rangle, \\ \partial\Lambda/\partial y_i &= \partial\langle P\Psi, \Psi\rangle/\partial y_i = \langle \partial P/\partial y_i \Psi, \Psi\rangle.\end{aligned}\quad (19)$$

Let us introduce the matrix T :

$$T_{ij} = \Psi_i \Psi_j^*$$

(* for the complex conjugation).

Lemma 2–1. *The matrix $V^{-1}TV$ is diagonal such that:*

$$V^{-1}TV_{ij} = \Delta_{ij} = \delta_{ip}\delta_{ij}.$$

Proof. We remind the reader that the index p refers to the eigenvalue Λ . We have

$$T_{ij} = V\Delta V_{ij}^{-1} = \sum_{1,s} V_{i1}\Delta_{1s}V_{sj}^{-1}$$

because

$$T_{ij} = \Psi_i \Psi_j^* = V_{ip}V_{jp}. \quad \square$$

As a corollary, we get: $[P, T] = 0$. We prove now the

Proposition 2–1. *Under the Hamiltonian flow of Λ , the time evolution of X and L is given by the equations:*

$$\dot{X} = [A^{(p)}, X] + \{\{L, T\}\}, \quad (20)$$

$$\dot{L} = [A^{(p)}, L] - \lambda^2\{\{X, T\}\}, \quad (21)$$

where the elements of the matrix $A^{(p)}$ are:

$$\begin{aligned}(i \neq j)A_{ij} &= (y_i + y_j)/(x_i - x_j)T_{ij} + \sqrt{-1g(T_{ii} + T_{jj})}/(x_i - x_j)^2 \\ &+ \sum_{k \neq i,j} \sqrt{-1gT_{kj}}/(x_i - x_j)(x_i - x_k) + \sum_{k \neq i,j} \sqrt{-1gT_{ik}}/(x_i - x_j)(x_k - x_j),\end{aligned}\quad (22)$$

$$A_{ii} = - \sum_{s \neq i} (A_{is} - \lambda\sqrt{-1T_{is}}) + \lambda\sqrt{-1T_{ii}} = - \sum_{s \neq i} (A_{si} - \lambda\sqrt{-1T_{si}}) + \lambda\sqrt{-1T_{ii}}, \quad (23)$$

and where the symbol $\{\{ , \}\}$ means the anticommutator of two matrices.

Proof. We write the elements of the matrix P :

$$\begin{aligned}P_{ii} &= \lambda^2 x_i^2 + y_i^2 + \sum_{k \neq i} g^2/(x_i - x_k)^2, \\ (i \neq j)P_{ij} &= \sum_{k \neq i,j} -g^2/(x_i - x_k)(x_k - x_j) \\ &+ \sqrt{-1g}(y_i + y_j)/(x_i - x_j) - \lambda g.\end{aligned}\quad (24)$$

To check (20), we have first to show that:

$$\dot{x}_i = \{\{L, T\}\}_{ii} = \sum_j L_{ij}T_{ji} + L_{ji}T_{ij}. \quad (25)$$

The sum of the right-hand side is:

$$2y_i T_{ii} + \sum_{k \neq i} \sqrt{-1g(T_{ki} - T_{ik})/(x_i - x_k)},$$

which is \dot{x}_i by (18) and (24).

Then for $i \neq j$, we have to prove that

$$0 = [A^{(p)}, X]_{ij} + \{\{L, T\}\}_{ij}.$$

We get:

$$A_{ij}(x_i - x_j) = \{\{L, T\}\}_{ij},$$

which is precisely (22).

Let us see now the diagonal term of (21), we have to check that

$$\dot{y}_i = - \sum_{j \neq i} \sqrt{-1g(A_{ij} + A_{ji})/(x_i - x_j)} - 2\lambda^2 x_i T_{ii},$$

which is precisely given by (19), (22) and (24).

The off-diagonal terms of (21) deserve the most part of our attention; we get:

$$\begin{aligned} & - \sqrt{-1g(\dot{x}_i - \dot{x}_j)/(x_i - x_j)^2} \\ & = \sqrt{-1g(A_{ii} - A_{jj})/(x_i - x_j)} + A_{ij}(y_j - y_i) + \sum_{k \neq i, j} \sqrt{-1gA_{ik}/(x_k - x_j)} \\ & \quad - \sqrt{-1gA_{kj}/(x_i - x_k)} - \lambda^2(x_i + x_j)T_{ij}. \end{aligned}$$

If we multiply by $(x_i - x_j)/\sqrt{-1g}$, this gives:

$$\begin{aligned} & - \sum_{k \neq i, j} \sqrt{-1g(T_{ik} - T_{ki})/(x_i - x_k)(x_i - x_j)} \\ & + \sum_{k \neq i, j} \sqrt{-1g(T_{jk} - T_{kj})/(x_j - x_k)(x_i - x_j)} + \dots \end{aligned} \quad (26)$$

$$(y_i + y_j)(T_{ii} - T_{jj})/(x_i - x_j) + \dots \quad (27)$$

$$\sqrt{-1g^{-1}((\lambda^2 x_i^2 + y_i^2) - (\lambda^2 x_j^2 + y_j^2))} T_{ij} + \dots \quad (28)$$

$$\sum_{k \neq i, j} T_{kj}(y_j - y_i)/(x_i - x_k) + \sum_{k \neq i, j} T_{ik}(y_j - y_i)/(x_k - x_j) \quad (29)$$

$$\sum_{s \neq i, j} (A_{is}(x_i - x_j)/(x_s - x_j) - A_{sj}(x_i - x_j)/(x_i - x_s)) + \dots \quad (30)$$

$$A_{ii} - A_{jj} = 0. \quad (31)$$

To show this equality, we begin to remark that:

$$\sum_{s \neq i, j} (A_{is}(x_i - x_s)/(x_s - x_j) - A_{sj}(x_s - x_j)/(x_i - x_s)) \quad (32)$$

$$\sum_{s \neq i} A_{is} - \sum_{s \neq j} A_{sj}, \quad (33)$$

$$\sum_{s \neq i, j} T_{is}(y_j + y_s)/(x_s - x_j) - \sum_{s \neq i, j} T_{sj}(y_i + y_s)/(x_i - x_s) \quad (34)$$

$$\sum_{s \neq i, j} T_{is}(y_i + y_s)/(x_s - x_j) + \sum_{s \neq i, j} T_{sj}(y_s + y_j)/(x_i - x_s) \quad (35)$$

$$\sum_{s \neq i, j} T_{is}(y_i + y_s)/(x_s - x_j) - \sum_{s \neq i, j} T_{sj}(y_s + y_j)/(x_i - x_s) + \dots \quad (36)$$

$$\sum_{s \neq i, j} \sqrt{-1g(T_{ii} - T_{jj})/(x_i - x_s)(x_s - x_j)} + \dots \quad (37)$$

$$\sum_{s \neq i, j; k \neq i, j, s} \sqrt{-1gT_{ik}/(x_k - x_s)(x_s - x_j)} - \sum_{s \neq i, j; k \neq i, j, s} \sqrt{-1gT_{kj}/(x_s - x_k)(x_i - x_s)} \quad (38)$$

$$+ \dots \sum_{s \neq i} \sqrt{-1gT_{ij}/(x_i - x_s)^2} - \sum_{s \neq j} \sqrt{-1gT_{ij}/(x_j - x_s)^2} + \dots \quad (39)$$

$$\sum_{s \neq i, j} \sqrt{-1gT_{js}/(x_s - x_j)(x_i - x_j)} - \sqrt{-1gT_{si}/(x_i - x_s)(x_i - x_j)}. \quad (40)$$

We have that:

$$(29) + (36) = (34)$$

$$(27) + (37) = -\sqrt{-1g^{-1}P_{ij}(T_{ii} - T_{jj})} - \lambda\sqrt{-1(T_{ii} - T_{jj})},$$

$$(28) + (39) = -\sqrt{-1g^{-1}(-P_{ii} + P_{jj})T_{ij}},$$

$$(26) + (40) = -\sum_{k \neq i, j} \sqrt{-1gT_{ik}/(x_i - x_j)(x_i - x_k)} - \sum_{k \neq i, j} \sqrt{-1gT_{kj}/(x_i - x_j)(x_j - x_k)}, \quad (41)$$

$$(38) + (41) = -\sum_{s \neq i, k; k \neq i, j} \sqrt{-1gT_{kj}/(x_s - x_k)(x_i - x_s)} + \sum_{s \neq j, k; k \neq i, j} \sqrt{-1gT_{ik}/(x_k - x_s)(x_s - x_j)}, \quad (42)$$

$$(34) + (42) = -\sqrt{-1g^{-1} \sum_{s \neq i, j} (P_{sj}T_{is} - P_{is}T_{sj})} - \lambda\sqrt{-1 \sum_{s \neq i} T_{is}} + \lambda\sqrt{-1 \sum_{s \neq j} T_{sj}},$$

and so finally:

$$(27) + (28) + (31) + (32) + (30) + (34) + (37) + (39) + (42) = \sqrt{-1g^{-1}[P, T]_{ij}} + \sum_{s \neq i} A_{is} - \lambda\sqrt{-1 \sum_{s \neq i} T_{is}} - \sum_{s \neq j} A_{sj} + \lambda\sqrt{-1 \sum_{s \neq j} T_{sj}} - \lambda\sqrt{-1 T_{ii}} + \lambda\sqrt{-1 T_{jj}} + A_{ii} - A_{jj} = 0.$$

Note that to end the proof of the proposition, we have to check the coherence of (23), that is:

$$\sum_{s \neq i} A_{is} - \lambda\sqrt{-1 T_{is}} = \sum_{s \neq i} A_{si} - \lambda\sqrt{-1 T_{si}},$$

but this is a direct consequence of $\sqrt{-1g^{-1}[P, T]_{ii}} = 0$. \square

From (20) and (21), and again the fact that $[P, T] = 0$, we deduce that the matrix P is isospectral for the flows of its eigenvalues. We find by this way that the eigenvalues of P are in involution.

At this point, we introduce the matrix U , solution of the differential equation:

$$\dot{U} = AU$$

so that $U(\infty) = 1$ (identity matrix).

For imaginary values of λ , we have unbounded orbits for the flow of the Hamiltonian, and so Z, Z^* and $P = ZZ^*$ tend to diagonal matrices. We have,

$$dUPU^{-1}/dt = 0 \quad \text{and so,}$$

$UPU^{-1} = U(\infty)U^{-1}(\infty)$ is a diagonal matrix also for real values of λ . Since T commutes with P , UTU^{-1} is diagonal, and by Lemma 2–1, we have necessarily, $UTU^{-1} = \Delta$.

We introduce now the matrices:

$$\xi = UZU^{-1}, \quad \xi^* = UZ^*U^{-1}.$$

to write that (20) and (21) imply

$$\dot{\xi} = \sqrt{-1\lambda}\{\{\xi, \Delta\}\}, \quad \dot{\xi}^* = -\sqrt{-1\lambda}\{\{\xi^*, \Delta\}\}.$$

These equations can be integrated into:

$$\dot{\xi}(t) = e^{\sqrt{-1}\lambda t\Delta}\xi(0)e^{\sqrt{-1}\lambda t\Delta}, \quad \dot{\xi}^*(t) = e^{-\sqrt{-1}\lambda t\Delta}\xi^*(0)e^{-\sqrt{-1}\lambda t\Delta}.$$

By taking the difference, we get that $X(t) = UXU^{-1}$ is a periodic matrix of period $2\pi/\lambda$. The traces of $X(t)^j$ for all integers j are periodic functions of t of period $2\pi/\lambda$. We obtain that all the functions $x_i(t)$ are periodic of period $2\pi/\lambda$ (of course, it is not necessarily the primitive period.)

We conclude now this paragraph. The Hamiltonian flows generated by the functions $\Lambda_1, \dots, \Lambda_m$ commute to each other and are periodic.

They define an Hamiltonian Action of the torus \mathbb{T}^m on T^*W , the moment map J being given by

$$J: T^*W \rightarrow \mathbb{R}^m(x, y) \rightarrow (t_1, \dots, t_m) = (\Lambda_1(x, y), \dots, \Lambda_m(x, y)).$$

The Hamiltonian system which defines the Calogero–Moser system with an external quadratic potential, or more generally the Olshanetsky–Perelomov systems of type V , is given by

$$H = \Lambda_1 + \dots + \Lambda_m,$$

so that for the element $X = (1, \dots, 1)$ of $(\mathbb{T}^m)^*$, the Hamiltonian vector field ξ of H satisfies $\xi = J^*(X)$. We can apply the method of the paragraph (1) to compute its Canonical Partition Function.

3 Generalizations of the Gallavotti–Marchioro Formula

From [C] and [P], we can deduce the following analysis of the set of critical points of H given by (10). Since H is convex, it has only one critical point in each connected component of the Weyl chamber and it is a minimum; the value of this minimum is equal to $H = \lambda \sum_{\alpha \in R^+} g_\alpha$. At each of these minima, the Hessian of H is equal to:

$$(1/2) \sum_{i=1}^m (y_i^2 + \lambda^2 v_i^2 x_i^2),$$

where $v_i (i = 1, \dots, m)$ denote the degrees of the basic invariants of the Weyl group [Ch]. We can now prove the

Theorem 3–1. *Let H be the Hamiltonian system given by (10), the Canonical Partition Function $Z(\beta)$ of H is given by:*

$$Z(\beta) = (2\pi/\beta\lambda)^m \exp - \left(\beta\lambda \sum_{\alpha \in A^+} g_\alpha \right).$$

Proof. There is a symplectic system of coordinates (p_i, q_i) so that

$$y_i = (\lambda v_i)^{1/2} p_i, \quad q_i = (\lambda v_i)^{1/2} x_i, \quad H = \lambda \sum_{\alpha \in A^+} g_\alpha + (\lambda/2) \sum_{i=1}^m v_i (p_i^2 + q_i^2).$$

The formula (8) gives for each connected component of the Weyl chamber, a contribution

$$(2\pi/\beta\lambda)^m (v_1 \cdots v_m)^{-1} \exp - \left(\beta\lambda \sum_{\alpha \in R^+} g_\alpha \right)$$

to the integral $Z(\beta)$. The order of the Weyl group is equal to the product $v_1 \cdots v_m$ [Ch], and so we obtain the announced formula. \square

Examples:

1—In the A_{m-1} case, we obtain

$$Z(\beta) = (2\pi/\beta\lambda)^m \exp - (\beta\lambda g m(m-1)/2)$$

which is the Gallavotti–Marchioro formula [G. M.].

2—For the $B_m = C_m$ case, we have:

$$Z(\beta) = (2\pi/\beta\lambda)^m \exp - \beta(g_1 \lambda m(m-1) + g_2 \lambda m).$$

3—The case D_m corresponds to $g_2 = 0$.

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