Commun. Math. Phys. 115, 167–175 (1988)

Communications in
Mathematical
Physics

Holomorphic Coordinates for Supermoduli Space

Philip Nelson

Department of Physics, Boston University, 590 Commonwealth Ave., Boston, MA 02215, USA

Abstract. Two-dimensional supergravity provides complex coordinates for supermoduli space compatible with its natural complex structure. Such coordinates are useful for investigating questions of holomorphic factorization of superdeterminants. They also demonstrate explicitly that the complex structure on supermoduli space is indeed integrable.

1. Introduction

In the first quantized theory of bosonic strings, string amplitudes are expressed as integrals over the space of all Riemann surfaces. The integrands of such integrals satisfy a remarkable holomorphy property, which serves as the basis for applications of algebraic geometry to strings.[1] Namely, the moduli space $\mathscr{M}$ of all Riemann surfaces has a natural complex structure, so that it makes sense to ask whether functions, forms, and so on are holomorphic on this space. The regulated determinants of $2d$ quantum field theory, while a priori only smooth functions of the moduli, occur in a combination with a special relationship to the complex structure [2, 3]. We can state this property precisely as follows. If we take the string measure μ and divide by

$$\mu_0 \equiv (\det \operatorname{Im} \tau)^{-13} |\psi^1 \wedge \ldots \wedge \psi^{3g-3}|^2 , \tag{1.1}$$

then the result is locally the absolute value squared of a holomorphic function on $\mathscr{M}$. Here τ is the period matrix of the Riemann surface in some marking and $\{\psi^i\}$ are a basis of quadratic differentials chosen to vary holomorphically. Given another such basis, or a different marking, (1.1) changes by the absolute square of some holomorphic function, so our statement of the holomorphy property is well-defined.

To establish that $\mathscr{M}$ has an integrable complex structure one first singles out certain tangent directions to $\mathscr{M}$ as holomorphic, and then shows that the tangent

[1] For a review see [1]

subspaces thus defined are natural and satisfy a differential condition, the vanishing of the Nijenhuis tensor [4]. In complex geometry this condition suffices to guarantee the existence of good complex coordinates on patches of $\mathcal{M}$, by an integrability theorem [4]. More directly, one can simply find a set of good complex coordinates near any (nonsingular) point of $\mathcal{M}$, compatible with the given choice of holomorphic directions.

In this paper I will work through the corresponding situation in the case of super Riemann surfaces (SRS). Here we get a moduli superspace $\hat{\mathcal{M}}$, which again has a natural choice of holomorphic directions satisfying a differential condition [5]. To show that they define a good complex structure on $\hat{\mathcal{M}}$ (again away from its singular orbifold points) one must either generalize the integrability theorem or else construct complex coordinates $\mathbf{t}$, $\boldsymbol{\zeta}$. Since the latter are useful in their own right, I will follow this route. For example, a function on $\hat{\mathcal{M}}$ is holomorphic exactly when it is an analytic function of $\mathbf{t}$ and $\boldsymbol{\zeta}$ in the usual sense; the nonlinearity of the holomorphy condition in [5] is already subsumed in the definition of these coordinates, which are holomorphic to all orders.

The strategy is to use the solution to the torsion constraints of $2d$ supergravity given in [6], suitably modified for compact euclidean 2-surfaces. One must then show that these coordinates are compatible with the almost complex structure of $\hat{\mathcal{M}}$, a fact which is not obvious from inspection. Throughout I will take the view of SRS advocated in [8, 9]. In particular, a superconformal structure is defined globally, not on patches, by a framing defined up to a certain group. No connection on the SRS is introduced. This fact simplifies some of the formulas and focuses attention on the geometry of the SRS.

Instead of the Wess-Zumino gauge solution which we will use, one could introduce the general solution to the torsion constraints given by Gates and Nishino [10], fixing the gauge only at the end. Such an approach may simplify the intermediate stages of the derivations given here.

Another approach to finding coordinates on supermoduli space is the generalization of the Bers embedding studied in [11]. One can also divide the super half-plane by the action of a Fuchsian supergroup [12–14]. The latter coordinates do not show the complex structure, however, so we will not use them here.

The considerations here can all be straightforwardly transcribed from nonchiral superfields to the case of heterotic geometry [15]. Again one obtains good holomorphic coordinates for chiral supermoduli space using a solution to the torsion constraints [16].

It is possible that the special family of local coordinate systems for $\hat{\mathcal{M}}$ defined in the sequel induces extra structure on $\hat{\mathcal{M}}$ besides a complex structure. Thus for example one can think of the class of inertial frames in special relativity as the physical construction which *determines* the metric, and not the other way round. I will mention this possibility at the end.

2. The Classical Case

To describe the complex structure on ordinary moduli space $\mathcal{M}$ we take its complex tangent $T_c\mathcal{M}$ at each point and split it into the sum of two conjugate subspaces: the holomorphic tangent space $T^{1,0}\mathscr{M}$ and its complement $T^{0,1}\mathscr{M} \equiv \overline{T^{1,0}\mathscr{M}}$. To accomplish the splitting, represent a given complex structure by a family of local real frames $e^a_{0,\alpha}$, $a=1,2$ on a fixed covering $\{U_\alpha\}$ of the fixed real 2-surface X. We will always look at just one patch and so drop the subscript α, keeping in mind that our constructions should not change if $e_0{}^a$ is replaced by $e'^a = e_0{}^b R^a_b$ for some function R from X to $GL(2,R)$.

Let $e_0{}^z \equiv e_0{}^1 + ie_0{}^2$ and $e_0{}^{\bar z} \equiv \overline{e_0{}^z}$. Then we can rephrase the above remark by saying that $e_0{}^z$ represents the same complex structure as $e'^z = fe_0{}^z$, where f is a nowhere-vanishing complex function. Let $\mathbf{C}^\times$ denote the group $\mathbf{C}-\{0\}$, so that $f: X \to \mathbf{C}^\times$.

A small fluctuation δe^a represents a tangent to moduli space. Define δe^z, $\delta e^{\bar z}$ as before. For a complex tangent vector these will not in general be complex conjugates of each other, so we can define holomorphic tangents as variations with $\delta e^z \neq 0$, $\delta e^{\bar z} = 0$. More precisely, let

$$h_a{}^b = (\delta e_a{}^n) e_n{}^b .$$

Then fluctuations of the form

$$h_z{}^z ; \quad h_{\bar z}{}^{\bar z} \tag{2.1}$$

produce no change at all on $\mathscr{M}$, being purely Weyl scalings or $U(1)$ rotations of $e_0{}^a$. Also

$$h_z{}^{\bar z} = D_z V^{\bar z} ; \quad h_{\bar z}{}^z = D_{\bar z} V^z \tag{2.2}$$

are pure gauge, where D_z is the Riemannian covariant derivative associated to $e_0{}^a$. Of the remaining tangent directions we declare that $h_{\bar z}{}^z$ spans the holomorphic tangents while $h_z{}^{\bar z}$ spans the antiholomorphic ones. Thus the complex structures on $\mathscr{M}$ and on X itself are entwined: the division between z and $\bar z$ tensor indices defined by e_0 itself is used to produce a division between $h_{\bar z}{}^z$ and $h_z{}^{\bar z}$ variations away from e_0.

Actually, so far we have only defined an "almost-complex" structure on moduli space [4]. To show that there exist good local complex coordinates we must establish the integrability of the almost-complex structure. For this it is sufficient to verify that the torsion of the almost-complex structure [4] (its "Nijenhuis tensor") vanish. For the case in question, however, it is just as easy to find explicit complex coordinates. Consider the family of frames

$$e^z(\mathbf{t}) = \mathrm{d}u + \mu^{iu}_{\bar u} t^i \, \mathrm{d}\bar u . \tag{2.3}$$

Here t^i are coordinates for a neighborhood of zero in[2] $\mathbf{C}^{3g-3}$, and u is any complex coordinate in which

$$e_0{}^z = \phi \, \mathrm{d}u \tag{2.4}$$

for some nonvanishing complex function ϕ. Clearly, if we replace $e_0{}^z$ by $f \cdot e_0{}^z$ for some $\mathbf{C}^\times$ function f the same coordinate u will do. If we change u to some new $\tilde u$ satisfying the same condition then (2.3) suffers only an overall $\mathbf{C}^\times$ transformation by $\dfrac{\mathrm{d}u'}{\mathrm{d}u}$. Hence the complex structure defined by some starting point $e_0 \in \mathscr{M}$ and a point $t \in \mathbf{C}^{3g-3}$ is globally well defined on the surface X.

[2] For simplicity we will always assume that the genus $g > 1$ in the sequel

We must now check three things. First, the coordinate system defined by (2.3) does not degenerate in a small neighborhood of e_0 if the Beltrami differentials μ^i are chosen appropriately. For example, if a metric is given and e^a are taken to be orthonormal in every patch, then one can take $\mu^i = |\phi|^2 \bar{\psi}^i$, where $\{\psi^i\}$ is a basis of quadratic differentials for the complex structure given by e_0. With this choice $\delta e|_{e_0}$ is in fact orthogonal to the gauge directions (2.2) [17].[3] Second, our splitting of the tangents into $h_z{}^z$; $h_{\bar{z}}{}^{\bar{z}}$ can be seen to be gauge-invariant, so it really does descend to $\mathcal{M}$.

Third, we must show that (2.3) is holomorphic, not just at e_0 but throughout a finite neighborhood. This follows at once since if $\delta t^i \neq 0$ but $\delta \bar{t}^i = 0$ we have $\delta e^z \neq 0$ but $\delta e^{\bar{z}} = 0$, and so $h_z{}^{\bar{z}} = 0$.

We can if we like substitute (2.3) into the bosonic string action, $S = \int (\det e_0) d^2 u \, \partial_z X \cdot \partial_{\bar{z}} X$. The result to first order is, for a holomorphic variation about some nonzero $\mathbf{t}$,

$$\delta S = \int (\det e) d^2 u (\partial_z X)^2 \mu_{\bar{z}\bar{z}} ,$$

where

$$\mu_{\bar{z}\bar{z}} \equiv \phi \bar{\phi}^{-1} \mu_{\bar{u}}^{iu} \delta t^i . \tag{2.5}$$

Thus we have the familiar result that a holomorphic variation of the coordinate t^i leads to the insertion of a certain mode of the $2d$ stress tensor.

3. The Super Case

In the superconformal case e^a gets replaced by a frame for a real superspace $\hat{X}$ of $2|2$ dimensions. The frame is $E^A \equiv dy^N E_N{}^A$, where A runs over $z, \bar{z}, +, -$ and y^N runs over the four coordinates of $\hat{X}$. It must satisfy certain integrability conditions, as explained in [6, 8, 9]. E^A is also defined only up to a group of frame rotations which this time is not $\mathbf{C}^\times$ but the group G of invertible complex matrices of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ [8, 9]. Such a frame defines a super Riemann surface, or SRS.

The general solution to the torsion constraints can be written succinctly with the following conventions. Begin with a Riemann surface X with frame $e_0{}^a$. Let e denote the family of one-forms (2.3), so $e \equiv e^z(\mathbf{t})$. Let u be a local conformal coordinate for e as in (2.4). Over the fixed smooth 2-surface X choose a fixed spin structure. Given the initial conformal structure represented by $e_0{}^a$, we now construct a real "square root" $\sqrt{TX}$ of the 2-dimensional real tangent TX:[4] $\sqrt{TX}$ is the bundle whose complexification is $\sqrt{TX}_c \equiv (T^{1,0}X)^{1/2} \oplus (T^{0,1}X)^{1/2}$. Let θ denote the local section of $\sqrt{TX}_c$ whose square is e. Construct $\hat{X}$, the supermanifold $(X, \wedge \sqrt{TX})$. The notation means that the body of $\hat{X}$ is X, while the anticommuting coordinates transform as spinors. Thus we can take as coordinates for $\hat{X}$ u, $\bar{u}$, θ, and $\bar{\theta}$, where now θ, $\bar{\theta}$ are viewed as anticommuting generators of $\wedge \sqrt{TX} \otimes \mathbf{C}$. Let $|\theta|^2$ denote $\bar{\theta}\theta$, and so on.

[3] Strictly speaking, what we have are coordinates for Teichmüller space, since we have neglected the possibility that e_0 may be a fixed point of a large diffeomorphism. Since the almost-complex structure is defined by equations invariant under both large and small diffeomorphisms, however, it is well-defined even at such a singular point. A similar remark applies to the discussion of the next section

[4] I thank V. Dellapietra for helping clarify this construction

Using e we can define a connection $\omega \equiv \omega_z$ on the base manifold X. We have

$$[\partial_z, \partial_{\bar{z}}] \equiv -i(\omega\partial_{\bar{z}} + \bar{\omega}\partial_z),$$

where $\partial_z = \phi^{-1}\frac{\partial}{\partial u}$ is a vector field on X and ϕ is the factor in (2.4). In turn ω defines a covariant derivative $D \equiv D_z$. Finally, choose a set of $2g-2$ sections χ^i of $\bar{K} \otimes K^{-1/2}$, where K is the holomorphic tangent of X in the complex structure e and $K^{1/2}$ its holomorphic square root determined by the choice $\sqrt{TX}$. Let ζ^i be a set of anticommuting generators for $\wedge\, \mathbf{C}^{2g-2}$, and

$$\chi = \chi_{\bar{z}}^{i+} \zeta^i, \tag{3.1}$$

where the function $\chi_{\bar{z}}^{i+}$ is χ^i expressed in terms of the frame. We then have (see [6])

$$\begin{aligned}
E^z &= e - \mathrm{d}\theta \cdot \theta + \bar{e}\theta\chi, \\
E^+ &= \mathrm{d}\theta + \frac{1}{2}e(i\theta\omega + |\theta|^2\overline{D\chi}) + \frac{1}{2}\bar{e}(i\theta\bar{\omega} + \chi), \\
E_z &= \left(1 - \frac{1}{4}|\theta|^2|\chi|^2\right)\partial_z - \frac{1}{2}\bar{\theta}\bar{\chi}\partial_{\bar{z}} + \tfrac{1}{2}(-\bar{\chi} + i\bar{\theta}\omega)\partial_{\bar{\theta}} \\
&\quad + \left(\frac{1}{4}\bar{\theta}|\chi|^2 - \frac{i}{2}\theta\omega - \frac{1}{2}|\theta|^2\left(\overline{D\chi} + \frac{i}{2}\bar{\chi}\bar{\omega}\right)\right)\partial_\theta, \\
E_+ &= \theta\partial_z + \frac{1}{2}\bar{\chi}|\theta|^2\partial_{\bar{z}} + \left(1 - \frac{1}{4}|\theta|^2|\chi|^2\right)\partial_\theta + \frac{1}{2}(\bar{\chi}\theta - i|\theta|^2\omega)\partial_{\bar{\theta}}.
\end{aligned} \tag{3.2}$$

The normalization chosen is convenient in that in the flat case we get the usual $E_z = \frac{\partial}{\partial u}$, $E_+ = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial u}$. We define $E^{\bar{z}}$ and E^- to be conjugates of these, with the rule that $\overline{\theta\chi} = \bar{\theta}\bar{\chi}$, $\overline{|\theta|^2} = -|\theta|^2$, etc. One can show that E_A and E^A are dual, and that they satisfy the integrability condition for a superconformal structure [8]:

$$\begin{aligned}
t_{++}{}^z &= 2, \\
t_{+-}{}^z = t_{++}{}^{\bar{z}} &= t_{++}{}^- = 0, \\
t_{++}{}^+ + 2t_{+-}{}^- &= 0,
\end{aligned} \tag{3.3}$$

and the conjugate equations, where $[E_A, E_B] = t_{AB}{}^C E_C$ defines $t_{AB}{}^C$. These constraints include the three "essential" ones defining a superconformal structure as well as the two which fix the gauge group from G to superweyl and local $U(1)$ [8]. Note that in (3.3) and below we never introduce a connection V on $\hat{X}$.

We interpret (3.2) as follows.[5] Let V be a neighborhood of 0 in $\mathbf{C}^{3g-3}$. We can use (2.3) to make $V \times X$ into a complex manifold of dimension $3g-2$, a holomorphic *family* of Riemann surfaces. Letting $\hat{V} = (V, \wedge\, \mathbf{C}^{2g-2})$ we can similarly regard (3.2) as making $\hat{V} \times \hat{X}$ into a complex supermanifold of dimension

[5] Another construction of families of SRS appears at the end of [15]. The two approaches appear to be equivalent. Also, as mentioned before a generalization of the Bers embedding is given in [11]

$3g-2|2g-1$, a family of SRS. This family depends on the original family of ordinary frames e and on the choice of the χ^i.

One can now ask whether (3.2) varies holomorphically with t and ζ. This is not so obvious. Indeed, the various appearances of $\bar{\theta}$, $\bar{e}$, $\bar{D}$, $\bar{\chi}$, and $\bar{\omega}$ in E_z, E_+ make it clear that $\partial E_z/\partial \bar{t}^i$, etc., will not vanish. Nevertheless, it is possible that these variations vanish *modulo gauge*, i.e. they may induce holomorphic tangents to $\hat{\mathcal{M}}$. This is what we will show in the next section.

Before proceeding, however, we should check that (3.2) is globally defined on $\hat{X}$. Suppose that we replace e by $\tilde{e}=f^2\cdot e$, where f is a map to $U(1)$. Then θ becomes $\tilde{\theta}=f\theta$ and $\mathrm{d}\tilde{\theta}=f\cdot\mathrm{d}\theta+(\mathrm{d}f)\cdot\theta$, while $\tilde{\omega}=f^{-2}(\omega+2if^{-1}\partial_z f)$. It is easy to see that the transformations of $\mathrm{d}\theta$ and ω combine to make the new frame $\tilde{E}^A$ defined by (3.2) a phase rotation of E^A. Thus $\tilde{E}^A$ defines the same superconformal structure as E^A; in particular, this structure is continuous across patch boundaries.

Similarly, if $\tilde{e}=f^2 e$, where f is a *real* nonvanishing function, then E^A suffers only a superweyl transformation. Thus our family of SRS depends only on the given family of Riemann surfaces, not on the slice chosen to represent it.[6]

4. Complex Structure

Following the classical case we can define a holomorphic complex variation of a frame E^A to be one for which $\delta E^{\bar{z}}=\delta E^{-}=0$. This is essentially the prescription of [5] since supermoduli space sits inside the space of all complex supermanifolds [8, 9]. More precisely, we begin with a given SRS.[7] There are many representations of its superconformal structure as frames on $\hat{X}$, and we choose any one. Let $H_A{}^B=(\delta E_A{}^N)E_N{}^B$ represent a small fluctuation. We call it holomorphic if $H_A{}^{\bar{z}}$ and $H_A{}^{-}$ both vanish up to gauge fluctuations. The latter include the generators of local G-transformations, $H: X\to G$, as well as small diffeomorphisms:

$$\delta E_A=[V, E_A]; \qquad H_A{}^B=V^C t_{CA}{}^B-\partial_A V^B, \tag{4.1}$$

where V is some vector field on X.

One must show that the above definition factors through the gauge equivalences to define a splitting of the tangent spaces to $\mathcal{M}$. For the this follows simply because by definition G does not mix E^z, E^+ with $E^{\bar{z}}, E^-$. For diffeomorphisms it follows because the vanishing of $H_A{}^{\bar{z}}, H_A{}^{-}$ is a set of scalar equations.

Of course not just any $H_A{}^B$ is permitted. The frame $E_A+\delta E_A$ must satisfy the same torsion constraints as E_A itself. To write these out we must first derive certain relations among the components of $t_{AB}{}^C$, similar to the Bianchi identities in [6].

If a frame E satisfies the constraints (3.3), then we have

$$[E_+, E_-]=-\tfrac{1}{2}(\bar{\tau}E_+ +\tau E_-), \tag{4.2}$$

$$[E_+, E_+]=2E_z+\tau E_+, \tag{4.3}$$

[6] For this to work it is important that the super-slice be determined by tensors χ in $\bar{K}\otimes K^{-1/2}$, not $\bar{K}^{3/2}$, so that $\chi_{\bar{u}}^{\theta}$ and not $\chi_{\bar{u}\theta}$ is Weyl invariant. For this and other reasons the alternate choice made in [20] is not useful

[7] In fact, we are always implicitly considering *families* of SRS, as in [19]

where $\tau \equiv t_{++}{}^{+}$ is an odd function. For the particular solution (3.2) we have

$$\tau = t_{++}{}^{+} = i\theta\omega + |\theta|^2\left(\frac{i}{2}\bar{\chi}\bar{\omega} + \overline{D\chi}\right), \tag{4.4}$$

but for now we proceed in a general gauge. Using (4.3) and the Jacobi identities we can rewrite $[E_z, E_+]$ in terms of τ, to get

$$t_{z+}{}^{+} = \tfrac{1}{2}\partial_+\tau; \quad t_{z+}{}^{z} = -\tau; \quad t_{z+}{}^{-} = t_{z+}{}^{\bar{z}} = 0, \tag{4.5}$$

where ∂_+ is E_+ regarded as a differential operator. Similarly,

$$t_{z-}{}^{z} = \bar{\tau}; \quad t_{z-}{}^{\bar{z}} = 0; \quad t_{z-}{}^{-} = -\tfrac{1}{2}\partial_+\tau. \tag{4.6}$$

Now consider the distorted frame $E_A + H_A{}^B E_B$. It has torsion $t_{AB}{}^C + \delta t_{AB}{}^C$, where

$$\delta t_{AB}{}^C = \partial_A H_B{}^C + H_A{}^E t_{EB}{}^C - (-)^{AB}(A \leftrightarrow B) - t_{AB}{}^D H_D{}^C.$$

(Compare the formula with connections in [21, 7].) Setting the variation of the constraints (3.3) equal to zero we find that $H_A{}^B$ must obey five equations and their conjugates:

$$\begin{aligned}
H_z{}^z &= \tfrac{1}{2}(2\partial_+ H_+{}^z - H_+{}^z\tau),\\
H_z{}^{\bar{z}} &= \tfrac{1}{2}(2\partial_+ H_+{}^{\bar{z}} - 3\tau H_+{}^{\bar{z}}),\\
H_-{}^{+} &= -\tfrac{1}{2}(\partial_+ H_-{}^z + \partial_- H_+{}^z + \tfrac{1}{2}H_+{}^z\bar{\tau} - \tfrac{3}{2}H_-{}^z\tau),\\
H_{\bar{z}}{}^{+} &= \partial_- H_-{}^{+} - H_-{}^{+}\bar{\tau} + H_-{}^z t_{z-}{}^{+},\\
H_{\bar{z}}{}^{-} &= \partial_+ H_-{}^{+} + H_+{}^z t_{z-}{}^{+} + H_-{}^{+}\tau.
\end{aligned} \tag{4.7}$$

Of the six remaining complex degrees of freedom in $H_A{}^B$, two are Weyl and $U(1)$ and can be removed by setting

$$H_+{}^{+} = H_-{}^{-} = 0, \tag{4.8}$$

analogously to our earlier elimination of $h_z{}^z$ and $h_{\bar{z}}{}^{\bar{z}}$.

We can now give a solution to (4.7) and (4.8) which is holomorphic in the above sense: let $H_-{}^z$ be an arbitrary superfield and

$$\begin{aligned}
H_{\bar{z}}{}^z &= \partial_- H_-{}^z + \tfrac{3}{2}H_-{}^z\bar{\tau},\\
H_-{}^{+} &= -\tfrac{1}{2}(\partial_+ H_-{}^z + \tfrac{3}{2}\tau H_-{}^z),\\
H_{\bar{z}}{}^{+} &= \partial_- H_-{}^{+} - H_-{}^{+}\bar{\tau} + H_-{}^z t_{z-}{}^{+},\\
H_{\bar{z}}{}^{-} &= \partial_+ H_-{}^{+} + H_-{}^{+}\tau.
\end{aligned} \tag{4.9}$$

All eleven of the other components vanish. Furthermore, one can choose

$$H_-{}^z = \bar{\theta}\bar{m}_{\bar{z}\bar{z}} - |\theta|^2\bar{\kappa}_{\bar{z}-}, \tag{4.10}$$

where $\bar{m}$ is a quadratic differential and $\bar{\kappa}$ is a differential of spin $\frac{3}{2}$, times an anticommuting parameter.[8] Equation (4.10) is essentially the super-Beltrami differential considered in [11] and elsewhere.

---

[8] This differs from the prescription in [5]. One can show that the lowest term in $H_-{}^z$ can be gauged away and hence cannot contain the supermoduli

To show that the family of SRS defined by (3.2) is holomorphic we could choose δt^i, $\delta\zeta^i \neq 0$; $\delta\bar{t}^i$, $\delta\bar{\zeta}^i = 0$ and show that the resulting δE_A is gauge-equivalent to a fluctuation of the form (4.9). This procedure requires that we find the explicit gauge transformation taking (4.9) into the Wess-Zumino gauge of (3.2). Instead we can simply substitute both (3.2) and (4.9) into the string action and verify that the fluctuation (4.9) has the same effect as a certain μ [see (2.5)] and a certain $\delta\chi$ [see (3.1)].

Begin with the action [6]

$$S = \int (\det E_N{}^A)(\mathrm{d}^2 u \mathrm{d}^2\theta)\, \partial_+ X \cdot \partial_- X\,,$$

where det is the Berezin determinant and $\mathrm{d}u\mathrm{d}\theta$ is the Berezin integral form. One then shows that $\det E_N{}^A = |\phi|^2(1 + \frac{1}{4}|\theta|^2|\chi|^2)$. Expand the superfield X as $y + \theta\psi + \bar{\theta}\bar{\psi}$; as usual we can drop the auxiliary field in the last term. Substituting (3.2) gives [see (2.4)]

$$S_{WZ} = \int |\phi|^2 \mathrm{d}^2 u\{-\partial_z y \partial_{\bar{z}} y + \psi\partial_{\bar{z}}\psi + \bar{\psi}\partial_z\bar{\psi} + \bar{\chi}(\partial_{\bar{z}} y)\bar{\psi} + \chi(\partial_z y)\psi + \tfrac{1}{2}|\chi|^2|\psi|^2\}\,.$$

The variation (4.9) gives to first order

$$\delta S_{WZ} = \int |\phi|^2 \mathrm{d}^2 u\{m(\psi\partial_z\psi - (\partial_z y)^2 + (\partial_z y)\bar{\chi}\bar{\psi}) + \kappa((\partial_z y)\psi - \tfrac{1}{2}\bar{\chi}|\psi|^2)\}\,.$$

Thus the holomorphic fluctuation (4.9) is the same as the variation $\mu = m$, $\delta\chi = \kappa$, $\bar{\mu} = \delta\bar{\chi} = 0$ [see (2.5)], so that our coordinates are holomorphic to first order about every $(\mathbf{t}, \boldsymbol{\zeta})$. Therefore, they are holomorphic coordinates to all orders. In particular, any other local coordinate system $(\tilde{t}, \tilde{\zeta})$ for $\hat{\mathscr{M}}$ built in this way using a different holomorphic family of Riemann surfaces and a different family of $\{\chi^i\}$ will differ from (t, ζ) by an analytic map of $\mathbf{C}^{3g-3|2g-2}$ to itself, and so $\hat{\mathscr{M}}$ is a complex manifold away from its orbifold points.

Finally, we also learn that the coordinates (t, ζ) are nondegenerate for an appropriate choice of $\{\chi^i\}$. This follows since once can show that (4.10) cannot be gauged away, using (4.1), (3.3), (4.5), (4.6), and (4.4).

5. Conclusion

The Wess-Zumino gauge solution to the constraints of $2d$ supergravity supplies a class of special coordinate systems (t, ζ) for the superspace $\hat{\mathscr{M}}$ of all super Riemann surfaces. These coordinate systems all have something in common: each singles out a subspace of $T_c\hat{\mathscr{M}}$ defined by $\delta\bar{t} = 0$, $\delta\bar{\zeta} = 0$, and the holomorphic tangent space defined in this way is independent of the choices made in defining the coordinates. Moreover, it agrees with the natural almost-complex structure on $\hat{\mathscr{M}}$. It is possible that the family of coordinate systems defined by (3.2) contain more invariant information than just a complex structure. For example, they may reduce the structure group of $\hat{\mathscr{M}}$, just as a superconformal structure on $\hat{X}$ reduces its structure group [8]. Such additional structure on $\hat{\mathscr{M}}$ (or a generalization of this space such as the one described in [22]) may prove to be essential to specify fully the integration prescription on $\hat{\mathscr{M}}$ needed in the fermionic string.

Notes added. (a) G. Moore as pointed out that the complex coordinates constructed here are not the most general ones, and that other choices may be important. One obtains the most general coordinates by composing the choice here with a holomorphic map from $\mathbf{C}^{3g-3|2g-2}$ to itself.

Equivalently one generalizes (2.3) and (3.1) from functions linear in t, ζ, respectively to general analytic functions of both t and ζ.

b) The frame (3.2) can be simplified if we require only that it satisfy the essential torsion constraints. A G-transformation which is neither superweyl nor local $U(1)$ suffices to remove the middle term of E^+, though it modifies the last constraint in (3.3).

Acknowledgements. I would like to thank V. Dellapietra, S. Giddings, A. Morozov, and M. Rothstein for many discussions and especially G. Moore for collaboration and insight. Also I thank the Aspen Center for Physics for hospitality while this work was being done. This work was supported in part by DOE contract DE-AC02-88ER40284.

References

1. Nelson, P.: Lectures on strings and moduli space. Phys. Rep. **149**, 304 (1987)
2. Belavin, A., Knizhnik, V.: Algebraic geometry and the geometry of quantum strings. Phys. Lett. B **168**, 201 (1986); Complex geometry and theory of quantum strings. ZETF **91**, 364 (1986) [=Sov. Phys. JETP **64**, 214 (1987)]
3. Bost, J., Jolicœur, J.: A holomorphy property and critical dimension in string theory from an index theorem. Phys. Lett. B **174**, 273 (1986)
4. Chern, S.S.: Complex manifolds without potential theory, 2nd ed. Berlin, Heidelberg, New York: Springer 1979
5. D'Hoker, E., Phong, D.: Superholomorphic anomalies and supermoduli space. Nucl. Phys. B **292**, 317 (1987)
6. Howe, P.: Super Weyl transformations in two dimensions. J. Phys. A **12**, 393 (1979); see also the appendix to [7]
7. Nelson, P., Moore, G.: Heterotic geometry. Nucl. Phys. B **274**, 509 (1986)
8. Giddings, S., Nelson, P.: Torsion constraints and super Riemann surfaces. Phys. Rev. Lett. **59**, 2619 (1987)
9. Giddings, S., Nelson, P.: The geometry of super Riemann surfaces. Commun. Math. Phys. (to appear)
10. Gates, S., Nishino, H.: $D=2$ superfield supergravity, ... Class. Quant. Grav. **3**, 443 (1986)
11. Crane, L., Rabin, J.: Super Riemann surfaces. Commun. Math. Phys. **113**, 601–623 (1988)
    Rabin, J.: Teichmüller deformations of super Riemann surfaces. Phys. Lett. B **190**, 40 (1987)
12. See e.g. Baranov, M., Shvarts, A.: Multiloop contribution in string theory. Pisma ZETF **42**, 340 (1985) [=JETP Letters **42**, 419 (1986)];
    Baranov, M., Manin, Yu., Frolov, I., Shvarts, A.: Multiloop contribution to fermion string. Yad. Phys. **43**, 1053 (1986) [=Sov. J. Nucl. Phys. **43**, 670 (1986)]
13. Baranov, M., Frolov, I., Schwarz, A.: Geometry of two-dimensional superconformal field theories. Teor. Mat. Fiz. **70**, 92 (1987) [=Theor. Math. Phys. **70**, 64 (1987)]
14. Batchelor, M., Bryant, P.: Graded Riemann surfaces. Preprint 1987
15. Brooks, R., Muhammad, F., Gates, S.: Unidexterous $D=2$ supersymmetry in superspace. Nucl. Phys. B **268**, 47 (1986);
    Moore, G., Nelson, P., Polchinski, J.: Strings and supermoduli. Phys. Lett. B **169**, 47 (1986);
    Evans, M., Ovrut, B.: The world sheet supergravity of the heterotic string. Phys. Lett. **171** B, 177 (1986)
16. Moore, G., Nelson, P.: (unpublished)
17. Alvarez, O.: Theory of strings with boundary. Nucl. Phys. B **216**, 125 (1983)
18. Gates, S.J.: Harvard Preprint HUTP-78/A028 (1978) (unpublished);
    Brown, M., Gates, S.J.: Ann. Phys. **122**, 443 (1979)
19. Rothstein, M., Le Brun, C.: Moduli of super Riemann surfaces. IAS preprint 1987
20. Bonini, M., Iengo, R.: The measure for the supermoduli. Phys. Lett. **191** B, 56 (1987)
21. Martinec, E.: Superspace geometry of fermionic strings. Phys. Rev. D **28**, 2604 (1983)
22. Baranov, M., Schwarz, A.: On the multiloop contribution to the strong theory. Int. J. Mod. Phys. A **2**, 1773 (1987)

Communicated by L. Alvarez-Gaumé

Received September 21, 1987