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# Block Spin Approach to the Singularity Properties of the Continued Fractions

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**Abstract.** The massless singularity of a ferromagnetic Gaussian measure on $\mathbb{Z}_+$ is studied by means of the coarse graining renormalization group method. The result gives information about a singularity behavior of a continued fraction and a time decay rate of a diffusion (random walk) on $\mathbb{Z}_+$.

## 1. Introduction: Problem and Results

We regard $\mathbb{R}^{\mathbb{Z}_+}$ as a measurable space with the $\sigma$-algebra generated by the cylinder subsets of $\mathbb{R}^{\mathbb{Z}_+}$. Let us introduce the notion of ferromagnetic Gaussian measures on $\mathbb{R}^{\mathbb{Z}_+}$. For bounded positive sequences $J=(J_n)_{n\in\mathbb{Z}_+}$ and $g=(g_n)_{n\in\mathbb{Z}_+}$ satisfying

$$\inf_{n\geqq 0} g_n > 0, \tag{1.1}$$

the pair $(J,g)$ is called a *ferromagnetic pair*. We define, for a ferromagnetic pair $(J,g)$, matrices $H(J)$ and $D(g)$ by putting, for $n,m\in\mathbb{Z}_+$,

$$\begin{aligned} H_{nm}(J) &= 0, \quad |n-m|>2, \\ &= J_{n\wedge m}, \quad |n-m|=1, \\ &= -J_{n-1}-J_n, \quad n=m, \end{aligned} \tag{1.2}$$

and

$$D_{nm}(g)=\delta_{nm}g_n, \tag{1.3}$$

where $n\wedge m=\min(n,m)$ and $J_{-1}=0$. The matrix $D(g)-H(J)$ induces a bounded linear operator on $l^\infty(\mathbb{Z}_+)=\{(\phi_n)_{n\in\mathbb{Z}_+} \mid \sup_{n\in\mathbb{Z}_+}|\phi_n|<\infty\}$ and it has a symmetric positive definite inverse (see Lemma 2.1 and 2.2). Then there exists a unique Gaussian probability measure $\mu_{Jg}$ on $\mathbb{R}^{\mathbb{Z}_+}$ with mean 0 and covariance $(D(g)-H(J))^{-1}$. We refer to the probability measure $\mu_{Jg}$ as the *ferromagnetic Gaussian measure* characterized by $(J,g)$ and write

$$\langle F(\phi)\rangle(J,g)=\int F(\phi)\mu_{Jg}(d\phi)$$

for any integrable function $F(\phi)$ on $\mathbb{R}^{\mathbb{Z}_+}$. In particular,

$$\langle \phi_n \phi_m \rangle (J, g) = (D(g) - H(J))^{-1}_{nm}, \quad n, m \in \mathbb{Z}_+, \tag{1.4}$$

holds. Our concern in this paper is the quantity:

$$f(J, g) = \langle \phi_0^2 \rangle (J, g).$$

Ferromagnetic Gaussian measures appear in the statistical mechanical theory of spin systems under the name "Gaussian model" or "free field" (for example, see [1]). In such a literature, $J_{nm}$ is called a ferromagnetic (nearest-neighbor) interaction, and $g_n$ corresponds to the square of mass. As is shown in Corollary 2.12, the function $f(J, tg)$ diverges as $t \downarrow 0$, where $(tg)_n = tg_n, n \in \mathbb{Z}_+$. The aim of this paper is to study the "massless singularity" of $f(J, tg)$ as $t \downarrow 0$. We show the following:

**Theorem 1.1.** *If a ferromagnetic pair $(J, g)$ satisfies*

$$C_1 n^{-\gamma} \leqq J_n \leqq C_2 n^{-\gamma}, \quad n > 0, \tag{1.5}$$

*for some constants $C_1, C_2 > 0$ and $\gamma \geqq 0$, then it holds that*

$$\lim_{t \downarrow 0} \frac{\log f(J, tg)}{\log t} = -\frac{\gamma + 1}{\gamma + 2}.$$

The theorem is restated in several ways. First we note that $f(J, tg)$ has an expression in the form of the continued fraction (see Appendix):

$$f(J, tg) = \frac{1}{tg_0 +} \frac{1}{J_0^{-1} +} \frac{1}{tg_1 +} \frac{1}{J_1^{-1} +} \cdots. \tag{1.6}$$

Then we have:

**Corollary 1.2.** *Let $(a_n)_{n \in \mathbb{Z}_+}$ and $(b_n)_{n \in \mathbb{Z}_+}$ be positive sequences such that*

$$C_3 < a_n < C_4, \quad n \geqq 1,$$
$$C_3 n^\gamma < b_n < C_4 n^\gamma, \quad n \geqq 1,$$

*for some constants $C_3, C_4 > 0$ and $\gamma \geqq 0$. Then, the continued fraction*

$$L(t) = \frac{1}{ta_0 +} \frac{1}{b_0 +} \frac{1}{ta_1 +} \frac{1}{b_1 +} \cdots, \quad t > 0, \tag{1.7}$$

*satisfies*

$$\lim_{t \downarrow 0} \frac{\log L(t)}{\log t} = -\frac{\gamma + 1}{\gamma + 2}. \tag{1.8}$$

Secondly a ferromagnetic Gaussian measure can be related to the diffusion (random walk) problem on $\mathbb{Z}_+$. Consider the diffusion equation:

$$\frac{d}{d\tau} u(\tau) = H(J) u(\tau), \quad \tau > 0, \tag{1.9}$$

$$u_n(0) = \delta_{0n}, \quad n \in \mathbb{Z}_+. \tag{1.10}$$

It is easily seen that the Laplace transform of $u_n(\tau)$ is given by the correlation function of a ferromagnetic Gaussian measure:

$$\int_0^\infty u_n(\tau)e^{-\tau t}d\tau = \langle \phi_0 \phi_n \rangle (J, t\mathbf{1}), \quad n \in \mathbb{Z}_+,$$

where $\mathbf{1} = (1, 1, \ldots) \in l^\infty(\mathbb{Z}_+)$. Then, employing the Abelian theorem, we obtain the following corollary.

**Corollary 1.3.** *Assume that the condition of the theorem is satisfied and that the solution of (1.9), (1.10) has the estimate*

$$C_5 \tau^{-\tilde{D}/2} < u_0(\tau) < C_6 \tau^{-\tilde{D}/2}, \quad \tau > 1, \tag{1.11}$$

*for some $C_5, C_6 > 0$ and $\tilde{D} > 0$. Then the exponent $\tilde{D}$ is given by*

$$\tilde{D} = \frac{2}{\gamma + 2}. \tag{1.12}$$

The exponent $\tilde{D}$ is called the spectral dimension [3] (see also Definition of $\tilde{d}(J, g)$ in Chap. 2.2).

The one dimensional diffusion problem has been extensively investigated by several authors in a general situation [4]. In particular, the fact stated in Theorem 1.1 may be obtained as a special case of the results of [5], where Krein's theory was used. We shall show the theorem by a quite different method, i.e. the coarse-graining renormalization group (*block spin*) method. Our analysis is an application of the renormalization group method for free fields on fractals studied in [6].

Our program is as follows. In Chap. 2, we shall show the well-definedness of the ferromagnetic Gaussian measure and prove some basic estimates. In Chap. 3, the coarse-graining renormalization for the Gaussian measure will be introduced. This plays the central role in Chap. 4 which is devoted to the proof of the main theorem.

Related problems are considered in [7, 8].

## 2. Ferromagnetic Gaussian Measure

In this chapter we show the well-definedness of the ferromagnetic Gaussian measure on one dimensional chain introduced in Chap. 1 and list basic properties that we use in the proof of Theorem 1.1.

### 2.1. *Well-Definedness of $\mu_{Jg}$.*

Let $\mathbf{M}$ be the set of all real matrices $A = (A_{nm})_{n,m \in \mathbb{Z}_+}$ satisfying

$$\| A \| = \sup_{n \in \mathbb{Z}_+} \sum_{m \in \mathbb{Z}_+} |A_{nm}| < \infty. \tag{2.1}$$

Then $(\mathbf{M}, \| \quad \|)$ turns out to be a Banach algebra with the identity $I$ and acts on $l^\infty(\mathbb{Z}_+)$ in the canonical sense.

**Lemma 2.1.** *For a ferromagnetic pair $(J, g)$, we define $H(J)$ and $D(g)$ by (1.2) and (1.3), respectively. Then $H(J)$ and $D(g)$ are in $\mathbf{M}$ and there exists a symmetric matrix*

$R(J,g)\in\mathbf{M}$ *such that*

$$(D(g)-H(J))R(J,g)=I, \tag{2.2}$$

$$R(J,g)(D(g)-H(J))=I \tag{2.3}$$

*with the estimate*

$$\|R(J,g)\| \leqq \left(\sup_{n\in\mathbb{Z}_+} g_n\right)^{-1}. \tag{2.4}$$

*Proof.* It is easily seen that $H(J), D(g)\in\mathbf{M}$. Let us show the existence of $R(J,g)$. Put

$$\mu_n = g_n + J_{n-1} + J_n, \quad n\in\mathbb{Z}_+.$$

We decompose $D(g)-H(J)$ as a sum of its diagonal part $D(\mu)$ and off-diagonal part $E(J)$:

$$D(g)-H(J)=D(\mu)-E(J),$$

where

$$\begin{aligned} D(\mu)_{nm} &= 0, \quad n\neq m,\\ &= \mu_n = g_n + J_{n-1} + J_n, \quad n=m,\\ E(J)_{nm} &= 0, \quad |n-m|\neq 1,\\ &= J_{n\wedge m}, \quad |n-m|=1. \end{aligned}$$

Then the Neumann series $\sum_{N=0}^{\infty}(D(\mu)^{-1}E(J))^N$ converges in $\mathbf{M}$, and hence $D(g)-H(J)=D(\mu)(I-D(\mu)^{-1}E(J))$ has the inverse $R(J,g)$:

$$R(J,g)=\sum_{N=0}^{\infty}(D(\mu)^{-1}E(J))^N D(\mu)^{-1}. \tag{2.5}$$

The symmetry of $R(J,g)$ is trivial. In order to show (2.4), we rewrite (2.3) as

$$R(J,g)D(g)=I+R(J,g)H(J).$$

Put $\mathbf{1}={}^t(1,1,\ldots)\in l^\infty(\mathbb{Z}_+)$. If we note that $H(J)\mathbf{1}=0$, we have

$$R(J,g)D(g)\mathbf{1}=\mathbf{1}. \tag{2.6}$$

This implies (2.4). □

We now need a positive definiteness of $R(J,g)$.

**Lemma 2.2.** *For* $\xi\in l^\infty(\mathbb{Z}_+)$ *such that* $\xi\neq 0$ *and* $\xi_n\neq 0$ *only for finite* $n$'s, *it holds that*

$$\langle \xi, R(J,g)\xi\rangle > 0. \tag{2.7}$$

*Proof.* Put $\eta = R(J,g)\xi$. Then:

$$\sum_{n\in\mathbb{Z}_+}\eta_n^2<\infty, \quad \langle \xi, R(J,g)\xi\rangle = \langle (D(g)-H(J))\eta, \eta\rangle.$$

If we note that

$$\langle (D(g)-H(J))\eta,\eta\rangle = \sum_{n\in\mathbb{Z}_+}[g_n\eta_n^2 + J_n(\eta_n-\eta_{n+1})^2],$$

we have the lemma. □

By the help of the above lemmas, the standard method of the probability theory ensures the existence of the ferromagnetic Gaussian measure $\mu_{Jg}$ with mean 0 and covariance $R(J,g)$.

We now pick up some convenient formulas from the above argument. Let us prepare some notations. For $i,j\in\mathbb{Z}_+$, we say that a sequence $w=(w_0,w_1,\dots,w_N)\subset\mathbb{Z}_+$ is a *walk* from $i$ to $j$ if $w_0=i$, $w_N=j$ and $|w_k-w_{k+1}|=1$, $0\leqq k\leqq N-1$. The set of all walks from $i$ to $j$ is denoted by $W(i,j)$. For a walk $w=(w_0,w_1,\dots,w_N)\in W(i,j)$ and a ferromagnetic pair $(J,g)$, we put

$$J_w=\prod_{k=0}^{N-1}J_{w_k\wedge w_{k+1}},$$

$$\mu_n=g_n+J_n+J_{n-1},$$

$$\mu_w=\prod_{k=0}^{N}\mu_{w_k}.$$

**Proposition 2.3.** *The correlation function of a ferromagnetic Gaussian measure satisfies the following equalities*:

$$\langle\phi_i\phi_j\rangle(J,g)=\sum_{w\in W(i,j)}J_w/\mu_w, \tag{2.8}$$

$$\sum_{j\in\mathbb{Z}_+}\langle\phi_i\phi_j\rangle(J,g)g_j=1 \tag{2.9}$$

$$\langle\phi_i\phi_j\rangle(cJ,cg)=c^{-1}\langle\phi_i\phi_j\rangle(J,g), \tag{2.10}$$

*where* $i,j\in\mathbb{Z}_+$ *and* $c>0$.

*Proof.* The "random walk representation" (2.8) is equivalent to (2.5) and the formula (2.9) is nothing but (2.6). The last equality is trivial. □

*2.2. Basic Properties.*

In the following, unless otherwise stated, $(J,g)$ and $(J',g')$ are arbitrary ferromagnetic pairs.

*Definition.* Consider the quantity $f(J,g)=\langle\phi_0^2\rangle(J,g)$. If the limit

$$\lim_{t\downarrow 0}\frac{\log f(J,tg)}{\log t}$$

exists, we say that the ferromagnetic pair $(J,g)$ has the spectral dimension, and we define $\tilde{d}(J,g)$, the spectral dimension of $(J,g)$, by,

$$\tilde{d}(J,g)/2-1=\lim_{t\downarrow 0}\frac{\log f(J,tg)}{\log t}.$$

In the remainder of this chapter, we shall study the behavior of $f(J,g)$ and $\tilde{d}(J,g)$ under the change of the parameters $J$ and $g$.

**Lemma 2.4.** *For a ferromagnetic pair* $(J, g)$,

$$0 < \langle \phi_0 \phi_n \rangle (J, g) \leqq \frac{1}{(n+1) \inf\limits_{m \in \mathbb{Z}_+} g_m}, \quad n \in \mathbb{Z}_+. \tag{2.11}$$

*Proof.* The positivity of $\langle \phi_0 \phi_n \rangle (J, g)$ is trivial from the random walk representation (2.8):

$$\langle \phi_0 \phi_n \rangle (J, g) = \sum_{w \in W(0,n)} \frac{J_w}{\mu_w}.$$

Since each term of the random walk representation is positive, we can make resummations and throw away terms to obtain lower bounds. We follow the method of stopping time arguments:

$$\begin{aligned}\langle \phi_0 \phi_n \rangle (J, g) &= \sum_{w \in W(0,n)} \frac{J_w}{\mu_w} = \sum_{w'}{}' \frac{J_{w'}}{\mu_{w'}} \mu_{n+1} \sum_{w'' \in W(n+1,0)} \frac{J_{w''}}{\mu_{w''}} + \sum_{w}{}'' \frac{J_w}{\mu_w} \\ &> \frac{J_n}{\mu_n} \sum_{w'' \in W(n+1,0)} \frac{J_{w''}}{\mu_{w''}} > \sum_{w'' \in W(n+1,0)} \frac{J_{w''}}{\mu_{w''}} \\ &= \langle \phi_0 \phi_{n+1} \rangle (J, g). \end{aligned} \tag{2.12}$$

Here the summation $\sum'$ is over all walks $w' = (n, n+1)$ or $(n, j_1, j_2, \ldots, j_m, n+1)$ $(m = 1, 2, \ldots)$ starting from $n$ and ending at $n+1$ with the property that $j_k \neq n+1$ for all $k \in \{1, 2, \ldots, m\}$, and the summation $\sum''$ is over all walks $w = (n, j_1, j_2, \ldots, j_m, 0)$ (or $(n, 0)$ if $n = 1$) starting from $n$ and ending at 0 with the property that $j_k \neq n+1$ for all $k \in \{1, 2, \ldots, m\}$. In the calculation, we have also used the fact that

$$\mu_n = g_n + J_n + J_{n-1} > J_n.$$

Combining (2.12) with (2.9), we have

$$\begin{aligned} 1 &= \sum_{m=0}^{\infty} g_m \langle \phi_0 \phi_m \rangle (J, g) > \left( \inf_{m \in \mathbb{Z}_+} g_m \right) \sum_{m=0}^{n} \langle \phi_0 \phi_m \rangle (J, g) \\ &> \left( \inf_{m \in \mathbb{Z}_+} g_m \right) (n+1) \langle \phi_0 \phi_n \rangle (J, g). \quad \square \end{aligned}$$

**Lemma 2.5.** *Let* $(J, g)$ *and* $(J, g')$ *be two ferromagnetic pairs. Define* $g(s) = (g(s)_n)_{n \in \mathbb{Z}_+}$, $(0 \leqq s \leqq 1)$ *by,*

$$g(s)_n = g_n s + g'_n (1 - s). \tag{2.13}$$

*Then,*

$$\frac{d}{ds} f(J, g(s)) = -\sum_n (g_n - g'_n) \langle \phi_0 \phi_n \rangle (J, g(s))^2. \tag{2.14}$$

*Proof.* From (2.2) and (2.3), we have, for $s, s' > 0$;

$$\sum_{k \in \mathbb{Z}_+} (D(g(s)) - H(J))_{nk} \langle \phi_k \phi_0 \rangle (J, g(s)) = \delta_{n,0}, \tag{2.15}$$

and

$$\sum\sum_{n,k\in\mathbb{Z}_+} \langle \phi_m\phi_n\rangle(J,g(s'))(D(g(s'))-H(J))_{nk}\langle \phi_k\phi_0\rangle(J,g(s))$$
$$=\langle \phi_m\phi_0\rangle(J,g(s)). \tag{2.16}$$

Multiply (2.15) by $\langle \phi_m\phi_n\rangle(J,g(s'))$, sum over $n\in\mathbb{Z}_+$, and subtract (2.16) to obtain,

$$\begin{aligned}\langle \phi_m\phi_0\rangle(J,g(s')) &- \langle \phi_m\phi_0\rangle(J,g(s))\\ &= \sum_{n\in\mathbb{Z}_+}(g(s)-g(s'))_n\langle \phi_m\phi_n\rangle(J,g(s'))\langle \phi_n\phi_0\rangle(J,g(s))\\ &= \sum_{n\in\mathbb{Z}_+}(g_n-g'_n)(s-s')\langle \phi_m\phi_n\rangle(J,g(s'))\langle \phi_n\phi_0\rangle(J,g(s)).\end{aligned} \tag{2.17}$$

From (2.17), we have the continuity of $\langle \phi_m\phi_0\rangle(J,g(s))$ with respect to $s$. If we put $m=0$ in (2.17), divide by $(s'-s)$, and use the continuity of $\langle \phi_m\phi_0\rangle(J,g(s))$, we obtain the desired result. □

**Corollary 2.6.** (i) *If*

$$g_n \leqq g'_n, \quad n\in\mathbb{Z}_+,$$

*then*

$$f(J,g')\leqq f(J,g). \tag{2.18}$$

(ii) *If*

$$g_n \leqq g'_n, \quad n\in\mathbb{Z}_+,$$

*then*

$$|\log f(J,g)-\log f(J,g')| \leqq \left(\inf_n g_n\right)^{-2} f(J,g')^{-1}\sum_{n\in\mathbb{Z}_+}(n+1)^{-2}(g'_n-g_n). \tag{2.19}$$

*Proof.* (i). From the assumption and Lemma 2.5, the statement follows directly. (ii). From Lemma 2.5, Lemma 2.4, and the assumption, we have,

$$\left|\frac{d}{ds}f(J,g(s))\right| \leqq \sum_n (g'_n-g_n)(n+1)^{-2}\left(\inf_n g(s)_n\right)^{-2}.$$

Therefore, using the assumption and (i),

$$\begin{aligned}|\log f(J,g)-\log f(J,g')| &= \left|\int_0^1 \frac{d}{ds}\log f(J,g(s))ds\right|\\ &\leqq \int_0^1 \frac{1}{f(J,g(s))}\sum_n (g'_n-g_n)(n+1)^{-2}\left(\inf_n g(s)_n\right)^{-2} ds\\ &\leqq \left(\inf_n g_n\right)^{-2} f(J,g')^{-1}\sum_{n\in\mathbb{Z}_+}(n+1)^{-2}(g'_n-g_n). \quad \square\end{aligned}$$

**Corollory 2.7.** *If $\tilde{d}(J,g)$ exists, then $\tilde{d}(J,g')$ also exists, and $\tilde{d}(J,g)=\tilde{d}(J,g')$.*

*Proof.* Since $g=(g_n)_{n\in\mathbb{Z}_+}$ and $g'=(g'_n)_{n\in\mathbb{Z}_+}$ are bounded positive sequences satisfying

(1.1), there exist positive constants $M$ and $M'$ such that

$$Mg_n < g'_n < M'g_n, \quad \text{for all} \quad n \in \mathbb{Z}_+.$$

From (2.18),

$$f(J, tM'g) \leqq f(J, tg') \leqq f(J, tMg), \quad \text{for} \quad t > 0.$$

Therefore if $0 < t < 1$,

$$\frac{\log f(J, tM'g)}{\log t} \geqq \frac{\log f(J, tg')}{\log t} \geqq \frac{\log f(J, tMg)}{\log t}.$$

Clearly,

$$\lim_{t \downarrow 0} \frac{\log f(J, tMg)}{\log t} = \lim_{t \downarrow 0} \frac{\log f(J, tg)}{\log t - \log M} = \lim_{t \downarrow 0} \frac{\log f(J, tg)}{\log t} = \frac{\tilde{d}(J, g)}{2} - 1,$$

from which the statement follows. □

Corollary 2.7 shows that $\tilde{d}(J, g)$ is *independent* of the choice of $g = (g_n)$. Henceforth we shall write

$$\tilde{d}(J) \equiv \tilde{d}(J, g).$$

**Lemma 2.8.** *Let $(J, g)$ and $(J', g)$ be two ferromagnetic pairs. Define $J(s) = (J(s)_n)_{n \in \mathbb{Z}_+}$, $(0 \leqq s \leqq 1)$ by,*

$$J(s)_n = J_n s + J'_n(1 - s). \tag{2.20}$$

*Then,*

$$\frac{d}{ds} f(J(s), g) = -\sum_n (J_n - J'_n) \tfrac{1}{2} (\langle \phi_0 \phi_n \rangle (J(s), g) - \langle \phi_0 \phi_{n+1} \rangle (J(s), g))^2. \tag{2.21}$$

*Proof.* Direct application of the method used in the proof of Lemma 2.5 proves this lemma. □

**Corollary 2.9.** (i) *If*

$$J_n \leqq J'_n, \quad n \in \mathbb{Z}_+,$$

*then*

$$f(J', g) \leqq f(J, g). \tag{2.22}$$

(ii) *If*

$$J_n \leqq J'_n, \quad n \in \mathbb{Z}_+,$$

*then*

$$|\log f(J, g) - \log f(J', g)| \leqq 2^{-1} \left( \inf_n g_n \right)^{-2} f(J', g)^{-1} \sum_{n \in \mathbb{Z}_+} (n+1)^{-2} (J'_n - J_n). \tag{2.23}$$

*Proof.* (i) From the assumption and Lemma 2.8, the statement follows directly. (ii). From Lemma 2.8, Lemma 2.4, and the assumption, we have,

$$\left| \frac{d}{ds} f(J(s), g) \right| \leqq \sum_n (J'_n - J_n) \tfrac{1}{2} (\langle \phi_0 \phi_n \rangle (J(s), g) - \langle \phi_0 \phi_{n+1} \rangle (J(s), g))^2$$

$$\leqq \sum_n (J'_n - J_n)\tfrac{1}{2}\langle \phi_0 \phi_n \rangle (J(s), g)^2$$

$$\leqq \frac{1}{2} \sum_n (J'_n - J_n)(n+1)^{-2} \left( \inf_n g(s)_n \right)^{-2}.$$

Therefore, using the assumption and (i),

$$|\log f(J,g) - \log f(J',g)| = \left| \int_0^1 \frac{d}{ds} \log f(J(s), g) ds \right|$$

$$\leqq 2^{-1} \left( \inf_n g_n \right)^{-2} f(J',g)^{-1} \sum_{n \in \mathbb{Z}_+} (n+1)^{-2} (J'_n - J_n). \quad \square$$

**Corollary 2.10.** *Assume that $\tilde{d}(J)$ exists. If there exist positive constants $C$ and $C'$ which are independent of $n \in \mathbb{Z}_+$ such that,*

$$C J_n < J'_n < C' J_n, \quad n \in \mathbb{Z}_+,$$

*then $\tilde{d}(J')$ also exists, and*

$$\tilde{d}(J) = \tilde{d}(J'). \tag{2.24}$$

*Proof.* From (2.22),

$$f(CJ, tg) \leqq f(J', tg) \leqq f(C'J, tg), \quad \text{for} \quad t > 0.$$

Using (2.10), we have

$$f(J, tC^{-1} g)/C \leqq f(J', tg) \leqq f(J, tC'^{-1} g)/C'.$$

The statement is now reduced to Corollary 2.7. $\square$

**Lemma 2.11.** *Define a ferromagnetic pair $(J, g)$ by*

$$\begin{aligned} g_0 &= g^*/2, & \\ g_n &= g^*, & n = 1, 2, 3, \ldots \\ J_n &= J^*, & n \in \mathbb{Z}_+, \end{aligned}$$

*where $g^*$ and $J^*$ are positive constants. Then*

$$f(J, g) = (g^* J^* + g^{*2}/4)^{-1/2}.$$

*Proof.* From (1.6) we see that $f(J, g)$ must satisfy,

$$f(J, g) = \frac{1}{g^*/2 + X}, \tag{2.25}$$

where

$$X = \cfrac{1}{J^{*-1} + \cfrac{1}{g^* + X}},$$

from which we obtain

$$X = -g^*/2 \pm (J^* g^* + g^{*2}/4)^{1/2}.$$

If we put this into (2.25), we see that $f(J,g)$ must satisfy:

$$f(J,g) = \pm (g^* J^* + g^{*2}/4)^{-1/2}.$$

Since we already know that $f(J,g)R(J,g)_{00}$ exists and is positive, we have the statement. □

**Corollary 2.12.**

$$\lim_{t \downarrow 0} f(J, tg) = \infty. \tag{2.26}$$

*Proof.* Since $J = (J_n)_{n \in \mathbb{Z}_+}$ and $g = (g_n)_{n \in \mathbb{Z}_+}$ are bounded sequences, there exists a constant $M(>0)$ such that

$$J_n < M, \quad \text{and} \quad g_n < M, \quad \text{for all} \quad n \in \mathbb{Z}_+.$$

Using (2.18), (2.22), and Lemma 2.11, we have

$$f(J, tg) \geqq f(M, tM) = (tM^2 + t^2M^2/4)^{-1/2} \uparrow \infty, \quad \text{as} \quad t \downarrow 0. \quad \square$$

## 3. The Coarse Graining Method

In Chap. 2, we defined the spectral dimension $\tilde{d}(J)$ which describes the "massless singularity" of the measure $\mu_{Jg}$, and derived some properties of $\tilde{d}(J)$, assuming its existence. In this chapter, we prove a simple lemma which gives us a sufficient condition for the existence of $\tilde{d}(J)$. In this lemma, we assume that $f(J,g) = \langle \phi_0^2 \rangle (J,g)$ satisfies an identity (in the massless limit) under the scale change of parameters $(J,g)$. To obtain the identity, we then consider a marginal distribution of $\mu_{Jg}$, by "integrating" the variables $\phi_{2n+1}, n \in \mathbb{Z}_+$. The intuition of this procedure came from the coarse graining renormalization group method, which appears in statistical mechanics.

**Lemma 3.1.** *Consider a ferromagnetic pair* $(J,g)$*. If, there exist positive constants* $\alpha$ *and* $\beta$ *such that* $\beta > \alpha$ *and*

$$\lim_{t \downarrow 0} \frac{f(J, tg)}{f(\alpha J, \beta t g)} = 1,$$

*then* $\tilde{d}(J)$ *exists, and*

$$\tilde{d}(J) = \frac{2 \log \beta}{\log(\beta/\alpha)}. \tag{3.1}$$

*Proof.* Put

$$x = -(\log t)/\log(\beta/\alpha)$$

and define

$$F(x) = \log\{f(J, (\alpha/\beta)^x g)\} + (\log \alpha)x.$$

From the assumption and (2.10),

$$\lim_{x \to \infty} \{F(x) - F(x-1)\} = 0,$$

from which we have

$$\lim_{x\to\infty}\frac{F(x)}{x}=0.$$

From the definition of $x$ and $F(x)$ we have

$$\lim_{t\downarrow 0}\frac{\log f(J,tg)}{\log t}=\frac{\log\alpha}{\log(\beta/\alpha)}.\quad\square$$

**Proposition 3.2.** *Consider a ferromagnetic pair* $(J,g)$. *Define another ferromagnetic pair* $(\tilde{J},\tilde{g})$ *by*

$$\tilde{J}_n=\frac{J_{2n}J_{2n+1}}{g_{2n+1}+J_{2n}+J_{2n+1}},\tag{3.2}$$

$$\tilde{g}_n=g_{2n}+\frac{J_{2n+1}}{g_{2n-1}+J_{2n-2}+J_{2n-1}}g_{2n-1}+\frac{J_{2n}}{g_{2n+1}+J_{2n}+J_{2n+1}}g_{2n+1},\quad n\in\mathbb{Z}_+.$$

*Then*

$$f(J,g)=f(\tilde{J},\tilde{g}).\tag{3.3}$$

*Proof.* The Gaussian probability measure $\mu_{Jg}$ has mean 0 and covariance $R(J,g)=(D(g)-H(J))^{-1}$. Consider a measurable map

$$p:\mathbb{R}^{\mathbb{Z}_+}\to\mathbb{R}^{\mathbb{Z}_+}$$

defined by;

$$p:(\phi_0,\phi_1,\phi_2,\dots)\mapsto(\phi_0,\phi_2,\phi_4,\dots).$$

The image measure

$$\tilde{\mu}_{Jg}=\mu_{Jg}p^{-1}$$

is again a Gaussian probability measure with mean 0 and covariance $\tilde{R}(J,g)=(\tilde{R}(J,g)_{nm})_{n,m\in\mathbb{Z}_+}$, where

$$\tilde{R}(J,g)_{nm}=R(J,g)_{2n,2m}.\tag{3.4}$$

As in Lemma 2.1, we decompose $D(g)-H(J)$ into a sum of diagonal part $D(\mu)$ and off-diagonal part $E(J)$:

$$D(g)-H(J)=D(\mu)-E(J),$$

where

$$\begin{aligned}D(\mu)_{nm}&=0,\quad n\neq m,\\&=\mu_n=g_n+J_{n-1}+J_n,\quad n=m,\\E(J)_{nm}&=0,\quad |n-m|\neq 1,\\&=J_{n\wedge m},\quad |n-m|=1.\end{aligned}$$

As have been proved in Lemma 2.1,

$$\tilde{R}(J,g)_{nm}=R(J,g)_{2n,2m}=\sum_{N=0}^{\infty}\{(D(\mu)^{-1}E(J))^N D(\mu)^{-1}\}_{2n,2m}.$$

On the other hand, from the definition of $D(\mu)$ and $E(J)$, we have, for odd $N$,

$$\{(D(\mu)^{-1}E(J))^N D(\mu)^{-1}\}_{2n,2m}=0.$$

Therefore,

$$\begin{aligned}\tilde{R}(J,g)_{nm}&=\sum_{N=0}^{\infty}\{(D(\mu)^{-1}E(J))^{2N}D(\mu)^{-1}\}_{2n,2m}\\&=\sum_{N=0}^{\infty}\{(D(\mu)^{-1}E(J)D(\mu)^{-1}E(J))^N D(\mu)^{-1}\}_{2n,2m}.\end{aligned}$$

If we define $D_0, D_1 \in \mathbf{M}$ by

$$\begin{aligned}D_{0,nm}&=0, \quad n\neq m,\\&=D(\mu)_{2n}, \quad n=m,\\D_{1,nm}&=0, \quad |n-m|\neq 1,\\&=(E(J)D(\mu)^{-1}E(J))_{2n,2m}, \quad |n-m|=1,\end{aligned}$$

we have

$$\begin{aligned}\sum_{N=0}^{\infty}\{(D(\mu)^{-1}E(J)D(\mu)^{-1}E(J))^N D(\mu)^{-1}\}_{2n,2m}&=\sum_{N=0}^{\infty}\{(D_0^{-1}D_1)^N D_0^{-1}\}_{nm}\\&=(D_0-D_1)^{-1}_{nm},\end{aligned}$$

where the last equality can be proved in the same way as Lemma 2.1. If we write down the last expression explicitly, we find that it is equal to $R(\tilde{J},\tilde{g})$. If we use (3.4) we obtain, in particular,

$$f(J,g)=R(J,g)_{00}=\tilde{R}(J,g)_{00}=R(\tilde{J},\tilde{g})_{00}=f(\tilde{J},\tilde{g}). \quad \square$$

## 4. Proof of the Main Theorem

*Proof of Theorem 1.1.* For $\gamma=0$, the theorem is a direct consequence of Lemma 2.11 and Corollary 2.10. Let us assume $\gamma>0$. We first consider the following nonlinear eigenvalue problems:

$$\alpha J_n=\frac{J_{2n}J_{2n+1}}{J_{2n}+J_{2n+1}}, \quad n\in\mathbb{Z}_+. \tag{4.1}$$

The above set of equations has an explicit solution.

**Lemma 4.1.** *For any fixed $\gamma>0$, (4.1) with $\alpha=2^{-\gamma-1}$ has a solution:*

$$J_n=J_n^*, \quad n\in\mathbb{Z}_+,$$

*where,*

$$\begin{aligned}J_n^*&=\{\alpha/(1-\alpha)\}(2\alpha)^{[\log n/\log 2]}, \quad n\geqq 1,\\&=1, \quad n=0,\end{aligned} \tag{4.2}$$

*and for $x\in\mathbb{R}$, $[x]$ is the largest integer $k$ satisfying $k\leqq x$.*

*Proof.* Straightforward calculation proves the statement. $\square$

Next we consider the following set of equations:

$$\beta g_n = g_{2n} + \frac{J^*_{2n-1}}{J^*_{2n-2} + J^*_{2n-1}} g_{2n-1} + \frac{J^*_{2n}}{J^*_{2n} + J^*_{2n+1}} g_{2n+1}, \quad n \in \mathbb{Z}_+, \tag{4.3}$$

where $g_{-1} \equiv 0$.

**Lemma 4.2.** *Equation (4.3) with $\beta = 2$ has a solution:*

$$g_n = g^*_n, \quad n \in \mathbb{Z}_+,$$

*where,*

$$\begin{aligned} g^*_n &= 1, \quad n \geqq 2, \\ &= 3/\{2(2-\alpha)\}, \quad n = 1, \\ &= 3(1-\alpha)/\{2(2-\alpha)\}, \quad n = 0. \end{aligned} \tag{4.4}$$

*Proof.* Straightforward calculation proves that (4.4) satisfies (4.3). □

We put $\alpha = 2^{-\gamma-1}$ and $\beta = 2$ in the following. We define the family $(J^*(t), tg^*(t))$, $t > 0$, of ferromagnetic pairs on $\mathbb{Z}_+$ by

$$J^*(t)_n = \frac{1}{\alpha} \frac{J^*_{2n} J^*_{2n+1}}{tg^*_{2n+1} + J^*_{2n} + J^*_{2n+1}}, \quad n \in \mathbb{Z}_+, \tag{4.5}$$

$$g^*(t)_n = \frac{1}{\beta} \left\{ g^*_{2n} + \frac{J^*_{2n-1}}{tg^*_{2n-1} + J^*_{2n-2} + J^*_{2n-1}} g^*_{2n-1} + \frac{J^*_{2n}}{tg^*_{2n+1} + J^*_{2n} + J^*_{2n+1}} g^*_{2n+1} \right\}, \quad n \in \mathbb{Z}_+. \tag{4.6}$$

Then (3.3) implies

$$f(J^*, tg^*) = f(\alpha J^*(t), \beta t g^*(t)), \quad \text{for} \quad t > 0. \tag{4.7}$$

The following uniform estimates are easily derived by explicit calculations: For $n \in \mathbb{Z}_+$ and $t > 0$,

$$C_4 n^{-\gamma} < J^*_n < C_5 n^{-\gamma}, \quad n \geqq 1, \tag{4.8}$$

$$2\alpha J^*_n \leqq J^*_{2n}, \tag{4.9}$$

$$0 < J^*_n - J^*(t)_n < C_1 t, \tag{4.10}$$

$$0 < g^*_n - g^*(t)_n < C_2, \tag{4.11}$$

$$\beta g^*(t)_n > g^*_{2n}, \tag{4.12}$$

where $C_i, (i = 1, 2, 4, 5)$ are positive constants independent of $n \in \mathbb{Z}_+$ and $t > 0$.

Next we define, for $t > 0$,

$$h_1(t) = \frac{f(\alpha J^*(t), \beta t g^*(t))}{f(\alpha J^*, \beta t g^*(t))},$$

and

$$h_2(t) = \frac{f(\alpha J^*, \beta t g^*(t))}{f(\alpha J^*, \beta t g^*)}.$$

If we can show $h_1(t)$, $h_2(t) \to 1$ as $t \downarrow 0$, then (3.1) with $\alpha = 2^{-\gamma-1}$ and $\beta = 2$ follows, and hence we have the value of $\tilde{d}(J^*)$.

**Lemma 4.3.** *Suppose that we have the estimate:*

$$\overline{\lim_{t \downarrow 0}} \frac{\log f(J^*, tg^*)}{\log t} \leqq -\delta \tag{4.13}$$

*for some $\delta > 1 - 1/\gamma$. Then it holds that*

$$\lim_{t \downarrow 0} h_i(t) = 1, \quad i = 1, 2,$$

*which, (as we remarked above,) implies*

$$\tilde{d}(J^*) = 2/(\gamma + 2). \tag{4.14}$$

*Proof.* First we note that

$$J_n^*/J^*(t)_n - 1 = t/(4\alpha J_n^*), \quad \text{if} \quad n > 0. \tag{4.15}$$

Let $\varepsilon$ be an arbitrary constant satisfying $0 < \varepsilon < 1$. For sufficiently small $t > 0$, we can define,

$$N(t, \varepsilon) = \max\{N \in \mathbb{Z}_+ \mid J^*(t)_n/J_n^* > 1 - \varepsilon \quad \text{if} \quad 0 \leqq n \leqq N\}.$$

(Since we assumed $\gamma > 0$, we have $N(t, \varepsilon) < \infty$.)

Let us show that if $t > 0$ is sufficiently small,

$$N(t, \varepsilon)^\gamma t > C_3 \varepsilon, \tag{4.16}$$

where $C_3$ is a positive constant independent of $t$ and $\varepsilon$. We have, with $N = N(t, \varepsilon)$:

$$\begin{aligned} N(t, \varepsilon)^\gamma t &= N(t, \varepsilon)^\gamma 4\alpha J_{N+1}^* (J_{N+1}^*/J^*(t)_{N+1} - 1) > N(t, \varepsilon)^\gamma 4\alpha J_{N+1}^* \varepsilon \\ &> 4\alpha C_4 N(t, \varepsilon)^\gamma (N(t, \varepsilon) + 1)^{-\gamma} \varepsilon > C_3 \varepsilon, \end{aligned}$$

where we used (4.15), $J^*(t)_{N+1}/J_{N+1}^* \leqq 1 - \varepsilon$, and (4.8). Thus we obtain (4.16).

We also see from the definition of $N(t, \varepsilon)$ that if $0 \leqq n \leqq N(t, \varepsilon)$,

$$0 < 1 - J^*(t)_n/J_n^* < \varepsilon, \tag{4.17}$$

and from (4.3), (4.6), (4.1), (4.5), $J_{2n}^* \geqq J_{2n+1}^*$, (4.9), and (4.17):

$$\begin{aligned} 0 < g_n^* - g^*(t)_n &\leqq \frac{\alpha}{\beta} \left\{ \frac{J_{n-1}^* - J^*(t)_{n-1}}{J_{2n-2}^*} g_{2n-1}^* + \frac{J_n^* - J^*(t)_n}{J_{2n}^*} g_{2n+1}^* \right\} \\ &< \frac{1}{\beta} \varepsilon \frac{1}{2} (g_{2n-1}^* + g_{2n+1}^*) < \varepsilon/2. \end{aligned} \tag{4.18}$$

Next we decompose $h_1(t)$:

$$\begin{aligned} h_1(t) &= \frac{f(\alpha J^*(t), \beta t g^*(t))}{f(\alpha \tilde{J}(t), \beta t g^*(t))} \frac{f(\alpha \tilde{J}(t), \beta t g^*(t))}{f(\alpha J^*, \beta t g^*(t))} \\ &\equiv P_1(t) Q_1(t), \end{aligned}$$

where

$$\tilde{J}(t)_n = J^*(t)_n, \quad n \leqq N(t,\varepsilon),$$
$$= J_n^*, \quad n > N(t,\varepsilon). \tag{4.19}$$

Let us estimate $P_1(t)$. From (2.10) and (4.13) we have

$$f(\alpha J^*, \beta t g^*) \geqq \alpha^{\delta-\varepsilon-1}\beta^{-\delta+\varepsilon}t^{-\delta+\varepsilon} \tag{4.20}$$

if $t(>0)$ is small enough. Using (2.23), (2.22), and (4.10), we have:

$$|\log P_1(t)| \leqq \frac{C_7}{N(t,\varepsilon)t f(\alpha J^*, t\beta g^*)},$$

where $C_7$ is a positive constant independent of $t$ and $\varepsilon$. Furthermore, by the help of (4.20) and (4.16), we obtain

$$|\log P_1(t)| \leqq C_8(\varepsilon)t^{1/\gamma-1+\delta-\varepsilon},$$

where $C_8(\varepsilon)$ is a positive constant independent of $t$. Therefore, if $\delta > 1 - 1/\gamma$, we have $P_1(t) \to 1$ as $t \downarrow 0$ by choosing sufficiently small $\varepsilon$. Let us estimate $Q_1(t)$. Since (4.17) implies

$$(1-\varepsilon)J_n^* < \tilde{J}(t)_n, \quad n \in \mathbb{Z}_+,$$

we have

$$1 \leqq Q_1(t) < \frac{f(\alpha(1-\varepsilon)J^*, \beta t g^*(t))}{f(\alpha J^*, \beta t g^*(t))} = \frac{1}{1-\varepsilon}\,\frac{f(\alpha J^*, (1-\varepsilon)^{-1}\beta t g^*(t))}{f(\alpha J^*, \beta t g^*(t))} \leqq \frac{1}{1-\varepsilon},$$

where we also used (2.22), (2.10) and (2.18). Thus we see that

$$1 \leqq \varliminf_{t\downarrow 0} h_1(t) \leqq \varlimsup_{t\downarrow 0} h_1(t) \leqq (1-\varepsilon)^{-1}$$

holds for any $\varepsilon < 0$ sufficiently small. This proves $h_1(t) \to 1$ as $t \downarrow 0$.

The proof of $h_2(t) \to 1$ as $t \downarrow 0$ goes along the same line. We put

$$\tilde{g}(t)_n = g^*(t)_n, \quad n \leqq N(t,\varepsilon),$$
$$= g_n^*, \quad n > N(t,\varepsilon),$$

and decompose $h_2(t)$ as

$$h_2(t) = \frac{f(\alpha J^*, \beta t g^*(t))}{f(\alpha J^*, \beta t \tilde{g}(t))}\,\frac{f(\alpha J^*, \beta t \tilde{g}(t))}{f(\alpha J^*, \beta t g^*)}$$
$$\equiv P_2(t) Q_2(t).$$

Equations (2.19) and (2.18) together with (4.11) and (4.12) imply

$$|\log P_2(t)| \leqq \frac{C_{10}}{N(t,\varepsilon)t f(\alpha J^*, t\beta g^*)},$$

which, together with (4.20) and (4.16), yields

$$|\log P_2(t)| \leqq C_{11}(\varepsilon)t^{1/\gamma-1+\delta-\varepsilon},$$

where $C_{11}(\varepsilon)$ is a positive constant independent of $t$. On the other hand, since (4.18) and (4.4) imply

$$(1-\varepsilon)g_n^* \leqq \tilde{g}(t)_n, \quad n \in \mathbb{Z}_+,$$

we have

$$1 \leqq Q_2(t) \leqq \frac{f(\alpha J^*, \beta t(1-\varepsilon)g^*)}{f(\alpha J^*, \beta t g^*)} = \frac{1}{1-\varepsilon} \frac{f((1-\varepsilon)^{-1}\alpha J^*, \beta t g^*)}{f(\alpha J^*, \beta t g^*)} \leqq \frac{1}{1-\varepsilon},$$

where we used (2.19), (2.10) and (2.22). This proves $h_2(t) \to 1$ as $t \downarrow 0$. □

Let us continue the proof of the theorem. For each $\gamma > 0$, we denote by $J_\gamma$, the interaction $J^*$ defined by (4.2) with $\alpha = 2^{-\gamma-1}$, and similarly, we define $g_\gamma$. We have proved that, if there exists a constant $\delta > 1 - 1/\gamma$ that satisfies

$$\overline{\lim_{t \downarrow 0}} \frac{\log f(J_\gamma, t g_\gamma)}{\log t} \leqq -\delta, \tag{4.21}$$

then we have

$$\tilde{d}(J_\gamma) = 2/(\gamma + 2). \tag{4.22}$$

Since, from (2.26), we have the trivial estimate (4.21) with $\delta = 0$, (4.22) holds for $0 < \gamma < 1$, in particular, for $0 < \gamma \leqq 1/2$. Suppose that we have proved (4.22) for $\gamma \leqq \tilde{\gamma}$. Then for any $\gamma$ satisfying $\tilde{\gamma} < \gamma$, we have:

$$\overline{\lim_{t \downarrow 0}} \frac{\log f(J_\gamma, t g_\gamma)}{\log t} \leqq \overline{\lim_{t \downarrow 0}} \frac{\log f(J_{\tilde{\gamma}}, t g_\gamma)}{\log t} = \tilde{d}(J_{\tilde{\gamma}})/2 - 1 = -(\tilde{\gamma}+1)/(\tilde{\gamma}+2).$$

Therefore (4.21) with $\delta = (\tilde{\gamma}+1)/(\tilde{\gamma}+2)$ holds. Since, for $\gamma \leqq \tilde{\gamma} + 1$,

$$1 - 1/\gamma \leqq 1 - 1/(\tilde{\gamma}+1) = \tilde{\gamma}/(\tilde{\gamma}+1) < (\tilde{\gamma}+1)/(\tilde{\gamma}+2) = \delta,$$

(4.22) holds for $\tilde{\gamma} < \gamma \leqq \tilde{\gamma} + 1$. Therefore we have (4.22) for all $\gamma > 0$ by induction.

For any $J$ satisfying (1.5), applying (2.24), we have $\tilde{d}(J) = \tilde{d}(J_\gamma)$. This completes the proof. □

## Appendix

For a ferromagnetic pair $(J, g)$, we prove

$$\langle \phi_0^2 \rangle (J, g) = \frac{1}{g_0 +} \frac{1}{J_0^{-1} +} \frac{1}{g_1 +} \frac{1}{J_1^{-1} +} \cdots, \tag{A.1}$$

that is,

$$\langle \phi_0^2 \rangle (J, g) = \lim_{n \to \infty} f_n, \tag{A.2}$$

where

$$f_{2n-1} = \frac{1}{g_0 +} \frac{1}{J_0^{-1} +} \frac{1}{g_1 +} \frac{1}{J_1^{-1} +} \cdots \frac{1}{+ g_n}, \quad n > 0,$$

$$f_{2n} = \frac{1}{g_0 +} \frac{1}{J_0^{-1} +} \frac{1}{g_1 +} \frac{1}{J_1^{-1} +} \cdots \frac{1}{+ g_n +} \frac{1}{J_n^{-1}}, \quad n > 0.$$

Since Seidel–Stern's theorem ([2] p. 87) implies $\left(\text{because of } \sum_{n\in\mathbb{Z}_+} (g_n + J_n^{-1}) = \infty\right)$

$$\lim_{n\to\infty} f_{2n} = \lim_{n\to\infty} f_{2n-1},$$

it suffices to show that

$$\langle \phi_0^2 \rangle (J, g) = \lim_{N\to\infty} f_{2N}. \tag{A.3}$$

Put $\mu_0 = g_0 + J_0$ and $\mu_n = g_n + J_{n-1} + J_n$, $n > 0$.
From

$$\langle \phi_0 \phi_n \rangle (J, g) = (D(g) - H(J))_{0n}^{-1}, \tag{A.4}$$

we have

$$\mu_0 \langle \phi_0^2 \rangle (J, g) - J_0 \langle \phi_0 \phi_1 \rangle (J, g) = 1, \tag{A.5}$$

$$\mu_n \langle \phi_0 \phi_n \rangle (J, g) - J_{n-1} \langle \phi_0 \phi_{n-1} \rangle (J, g) - J_n \langle \phi_0 \phi_{n+1} \rangle (J, g) = 0, \quad n \geqq 1. \tag{A.6}$$

We modify the original ferromagnetic pair $(J, g)$. Fix $N > 0$. Put $J_N = 0$ and increase $g_N$ so that the value of $\mu_N$ does not change. Write the resulting ferromagnetic pair as $(\tilde{J}, \tilde{g})$. Put

$$c_n = J_{n-1}[1 - \langle \phi_0 \phi_n \rangle (\tilde{J}, \tilde{g}) / \langle \phi_0 \phi_{n-1} \rangle (\tilde{J}, \tilde{g})], \quad n > 0.$$

Then the analog of (A.5) and (A.6) imply

$$f(\tilde{J}, \tilde{g}) = 1/(g_0 + c_1), \tag{A.7}$$

$$c_n = 1/\{1/J_{n-1} + 1/(g_n + c_{n+1})\}, \quad N > n > 0, \tag{A.8}$$

$$c_N = 1/\{1/J_{N-1} + 1/(g_N + J_N)\}. \tag{A.9}$$

As is easily seen from (A.7), (A.8), (A.9), and (A.1), it holds that

$$f(\tilde{J}, \tilde{g}) = f_{2N}.$$

On the other hand, from the definition of $(\tilde{J}, \tilde{g})$, $f(\tilde{J}, \tilde{g})$ can be written in the form of (2.8) that is a finite volume approximation of the original expression (2.8): i.e., the summation is now taken over all walks in $W(0, 0)$ not passing through the point $N + 1$. Since the finite volume approximation of (2.8) converges to $f(J, g) = \langle \phi_0^2 \rangle (J, g)$ in the limit $N \to \infty$, we have (A.3).

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