

Formal Group Laws for Affine Kac-Moody Groups and Group Quantization*

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Abstract. We describe a method for obtaining Formal Group Laws from the structure constants of Affine Kac-Moody groups and then apply a group manifold quantization procedure which permits construction of physical representations by using only canonical structures on the group. As an intermediate step we get an explicit expression for two-cocycles on Loop Groups. The programme is applied to the Affine $SU(2)$ group.

1. Introduction

In recent years Kac-Moody algebras, and the Affine ones in particular, have acquired more and more relevance in Theoretical Physics. Thus, for example, these algebras appear as algebras of hidden symmetries of two-dimensional chiral models [1–3], in self-dual Yang-Mills theory [3], in completely integrable dynamical systems [4, 5], Bose-Fermi correspondence [1, 6], string theory [7–9], conformally invariant field theories [10], etc.

As is well known [11], there exists an isomorphism between the Affine Kac-Moody algebra $L(\tilde{A})$ and the unique one-dimensional non-trivial central extension, $C[t, t^{-1}] \otimes \tilde{L}(A)$, of the gauge-like algebra $C[t, t^{-1}] \otimes L(A)$. The cocycle Σ that fulfills the extension is

$$\Sigma(X \otimes t^n, Y \otimes t^m) = \alpha n \delta_{n, -m} \tau(X, Y), \quad \forall X, Y \in L(A), \quad (1.1)$$

where τ is the killing form of the classical algebra $L(A)$. We have $\dim H^2(C[t, t^{-1}] \otimes L(A), C) = 1$.

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The above structure for $L(\tilde{A})$ or, rather, the corresponding one for groups, is shared with many basic physical symmetries, the simplest example of which is the central extension of the Galilei group, $\tilde{G}_{(m)}$, associated with the quantum free particle. This kind of group, and more generally the groups which also have the structure of a principal bundle with $U(1)$ as the structure group (Quantum Groups), has been used to a great extent to obtain the wave functions of physical systems [12–15] through a Group Quantization Formalism (GQF) [12] which only makes use of the group manifold and of the canonical structures defined on it.

In this paper we apply GQF to the principal bundle $\Omega\tilde{G}$, the central extension of the space of loops in the simple group G , and work out explicitly the wave functions carrying the representations of $\Omega SU(2)$ as functions on the group manifold. To start with, the formalism requires a local group law in local coordinates at the identity. We calculate this group law and, in particular, we give an explicit expression for the two-cocycle of the group using the intrinsic group coordinates. In this respect, our procedure diverges from that of reference [16, 17]. As a general feature the GQF recuperates what would be the Kostant-Souriau Geometric Quantization [18, 19] of the Kirillov-Kostant co-adjoint orbits [19] and solves the problem of the ambiguity of the polarizations in the finite dimensional case as well as that of finding an invariant integration volume. The wave functions that we obtain arise naturally accompanied by a weight function for the invariant scalar product [20].

The fact that we have to resort to a power expansion to find the group law can be seen as a drawback of the procedure. We must note, nevertheless, that the description of a physical system in this way would look like a perturbative one but with the additional advantage of being finite.

We finally remark that obviously GQF provides Hamiltonian (Lagrangian) models in which Affine Kac-Moody algebras are realized not only as infinitesimal actions but also as conserved Noether invariants whose Poisson bracket closes. It must also be noted that “normal ordering” is automatically incorporated into the formalism (through the choice of a polarization) in such a way that central charges already appear at the classical level (in the Hamiltonian-Jacobi version).

II. “Perturbative” Expansion of the Group Law

Formal Group Theory [21] can be converted into an approximate method for computing local group laws of Kac-Moody groups (although the process can of course be applied to any group). We start with formal power series in two variables of “ Z ” \times $\dim G + 1$ components:

$$\begin{aligned}
 F^z(X', X) = & X'^z + X^z + B_{i,j}^z X'^i X^j \\
 & + \frac{1}{2!} B_{ij,\ell}^z X'^i X'^j X^\ell + \frac{1}{2!} B_{i,j\ell}^z X'^i X^j X^\ell \\
 & + \frac{1}{3!} B_{ij\ell,m}^z X'^i X'^j X'^\ell X^m \\
 & + \frac{1}{2!} \frac{1}{2!} B_{ij,\ell m}^z X'^i X'^j X^\ell X^m + \dots,
 \end{aligned} \tag{2.1}$$

where $i, j, \ell, m \in Z \times \{1, 2, \dots, \dim G\}$ and $\alpha = i, j, \dots$ or ϕ , the $U(1)$ index. For this formal series to be a formal group law we only have to impose the associativity conditions [22],

$$F(X'', F(X', X)) = F(F(X'', X'), X), \quad \forall X'', X', X. \tag{2.2}$$

The structure constants of the Lie algebra are obtained from the expressions

$$B_{i,j}^\alpha - B_{j,i}^\alpha = C_{ij}^\alpha, \tag{2.3}$$

to be referred to in the sequel as the second order conditions.

As is well known [21] the Lie algebra determines the formal group structure and, accordingly, the constants $B_{i_1 \dots i_n, j_{n+1} \dots j_m}^\alpha$ ($m > 2$) are determined modulo a group isomorphism by the structure constants C_{ij}^α . From condition (2.2) a set of constraint equations follow for $B_{i_1 \dots i_n, j_{n+1} \dots j_m}^\alpha$ order by order. Thus, for example, the lower orders look like:

$$3^{\text{rd}} \text{ order} \quad B_{i,j\ell}^\alpha - B_{ij,\ell}^\alpha = B_{p,\ell}^\alpha B_{i,j}^\alpha - B_{i,p}^\alpha B_{j,\ell}^\alpha, \tag{2.4}$$

$$4^{\text{th}} \text{ order} \quad \begin{cases} B_{i,j\ell m}^\alpha - B_{ij,\ell m}^\alpha \\ = B_{p,\ell m}^\alpha B_{i,j}^\alpha - B_{i,p}^\alpha B_{j,\ell m}^\alpha - B_{i,m p}^\alpha B_{j,\ell}^\alpha - B_{i,\ell p}^\alpha B_{j,m}^\alpha, \\ B_{ij,\ell m}^\alpha - B_{ij\ell,m}^\alpha \\ = B_{p,m}^\alpha B_{ij,\ell}^\alpha + B_{p j,m}^\alpha B_{i,\ell}^\alpha + B_{p i,m}^\alpha B_{j,\ell}^\alpha - B_{ij,p}^\alpha B_{\ell,m}^\alpha. \end{cases} \tag{2.5}$$

In general, for $n \geq 3$ there exist $n-2$ independent equations (corresponding to $a = 1, 2, \dots, n-2$):

$$n^{\text{th}} \text{ order} \quad \begin{cases} B_{i_1 \dots i_a, j_{m_1 \dots m_{n-a-1}}}^\alpha - B_{i_1 \dots i_a, m_1 \dots m_{n-a-1}}^\alpha \\ = \sum_{\substack{b=1, \dots, a \\ \mu_1 \dots \mu_b \in \{1, \dots, a\}}} \\ \times B_{p i_1 \dots \hat{i}_{\mu_1} \dots \hat{i}_{\mu_2} \dots \hat{i}_{\mu_b} \dots i_a, m_1 \dots m_{n-a-1}}^\alpha B_{i_{\mu_1} i_{\mu_2} \dots i_{\mu_b}, j}^\alpha \\ - \sum_{\substack{c=1, \dots, n-a-1 \\ \nu_1 \dots \nu_c \in \{1, \dots, n-a-1\}}} \\ \times B_{i_1 \dots i_a, m_1 \dots \hat{m}_{\nu_1} \dots \hat{m}_{\nu_2} \dots \hat{m}_{\nu_c} \dots \hat{m}_{n-a-1}, p}^\alpha B_{j, m_{\nu_1} \dots m_{\nu_c}}^\alpha. \end{cases} \tag{2.6}$$

The system (2.6) has the Jacobi identity as the compatibility condition. By successively solving the different sets of equations the formal group law is “perturbatively” derived. It is important to realize that this perturbative expansion is not uniquely determined; there is a certain freedom in the way we choose the B 's in the left-hand side of (2.3)–(2.6) to each order [23]. This “gauge” freedom, which in general corresponds to a group isomorphism, corresponds in particular to the coboundary ambiguity of the central extensions in the case of Affine Kac-Moody groups. Then a rule is needed in working out any formal group law. We point out, however, that QGF is insensitive to the special choice of the rule so that we may adopt the more advantageous one in each case.

III. Formal Group Law for $\widetilde{\Omega SU}(2)$

In the concrete case of the group $\widetilde{\Omega SU}(2)$ the above procedure can be simplified due to the fact that we can write down a formal group law for the $SU(2)$ subgroup. A

formal Laurent expansion of the parameters of the $SU(2)$ group then provides the solution for all the B 's but for those with upper index ϕ . We write the local $SU(2)$ group law in a fibre bundle chart of the Hopf fibration $SU(2) \rightarrow S^2 : \eta = e^{i\theta/2}$ parametrizes the fibre and $2z, 2z^*$ the north pole neighbourhood [24]

$$\begin{aligned} z'' &= z'\eta^{-2} + x'_3 z - \frac{2z}{1+x_3} [z^* z' \eta^{-2} + z'^* z \eta^2], \\ z''^* &= z'^* \eta^2 + x'_3 z^* - \frac{2z^*}{1+x_3} [z z'^* \eta^2 + z' z^* \eta^{-2}], \\ \eta'' &= \sqrt{\frac{2}{1+x'_3}} \left\{ \sqrt{\frac{1+x_3}{2}} \sqrt{\frac{1+x'_3}{2}} \eta' \eta - \sqrt{\frac{2}{1+x'_3}} \sqrt{\frac{2}{1+x_3}} z^* z' \eta^* \eta' \right\}, \\ x_3 &\equiv \sqrt{1-4zz^*}, \quad x'_3 = x'_3 x_3 - 2(z^* z' \eta^{-2} + z'^* z \eta^2). \end{aligned} \tag{3.1}$$

In these coordinates a formal group law for $\Omega SU(2)$ is given by:

$$\begin{aligned} z''_n &= z'_n + z_n - iz'_{n-m} \theta_m - z'_{n-m-p} z_m z_p^* - z'^*_{n-m-p} z_m z_p \\ &\quad - 2z'_{n-m-p} z_m^* z_p - \frac{1}{2} z'_{n-m-p} \theta_m \theta_p + \dots, \\ z''^*_n &= z'^*_n + z^*_n + iz'^*_{n-m} \theta_m - z'^*_{n-m-p} z_m^* z_p - z'_{n-m-p} z_m^* z_p^* \\ &\quad - 2z'^*_{n-m-p} z_m^* z_p - \frac{1}{2} z'^*_{n-m-p} \theta_m \theta_p + \dots, \\ \theta''_n &= \theta'_n + \theta_n + i(z'_{n-m} z_m^* - z'^*_{n-m} z_m) + (z'_{n-m-p} z_m^* + z'^*_{n-m-p} z_m) \theta_p + \dots, \end{aligned} \tag{3.2}$$

where summation of repeated indices is understood.

The procedure of Sect. II is now applied to extend the group law (3.2) to that of $\widehat{\Omega SU}(2)$, i.e., to compute explicitly the group two-cocycle ζ . Using the ‘‘gauge’’

$$\begin{aligned} B^{\phi}_{i,j} &= -B^{\phi}_{j,i}, \quad i, j = z_n, z_m^*, \theta_p, \\ B^{\phi}_{z_n z_m^*, \theta_p} &= 0 = B^{\phi}_{\theta_p, z_n z_m^*}, \quad \forall n, m, p \in \mathbb{Z}, \end{aligned} \tag{3.3}$$

we obtain for ϕ'' ,

$$\phi'' = \phi' + \phi + \alpha \zeta(g', g), \quad g', g \in \Omega SU(2),$$

$$\begin{aligned} \zeta(g', g) &= \frac{1}{2} [n(2z'_n z_{-n} + 2z'_n z_{-n}^* + \theta'_n \theta_{-n}) + i(m-n)z'_n z_m^* \theta_{n-m} \\ &\quad - i(m-n)z'_n z_m \theta_{-n-m} + in\theta'_n z_m z_{-n-m}^* - i\theta'_n z_m^* z_{-n-m} + \dots]. \end{aligned} \tag{3.4}$$

Had we taken another ‘‘gauge’’ for computing the B^{ϕ} 's we would have obtained an equivalent (i.e., differing in a two-coboundary) two-cocycle $\hat{\zeta}$.

IV. Group Quantization and Representations: The $\widehat{\Omega SU}(2)$ case

The basic structure of the GQF [12] is a Lie group \tilde{G} , the ‘‘quantum’’ group, which in turn is a (say) right $U(1)$ -principal bundle (or a R -principal bundle then leading to a classical theory) with a 1-form Θ naturally selected among the left-invariant 1-forms which is a connection form. In the simplest case, and roughly speaking, \tilde{G}

might be either a (global) central extension of a group G by $U(1)$, characterized by a cocycle $\xi \in H^2(G, U(1))$, or a finite dimensional semisimple group with $U(1)$ in the Cartan subgroup. In both cases a vertical left-invariant 1-form is globally defined either by cohomological criteria [20], in the first case, or by means of the Killing form from the fundamental (vertical) left-invariant generator of \tilde{G} in the second one. However, a semisimple quantum group \tilde{G} can also be (equivalently) handled as if it were a central extension by making use of a “pseudo-extension” [25], i.e., a trivial extension by $U(1)$ which modifies the structure constants of the algebra. This can only happen if the cocycle, “pseudo-cocycle,” is locally generated by a linear function on the parameter of the structure subgroup [$U(1)$ or R] of the principal bundle \tilde{G} . This set of coboundaries is a group strongly related to the cohomology group of a contraction. The connection form Θ for pseudo-extended groups is obtained as the left-invariant 1-form dual to the central $U(1)$ generator. We must remark, nevertheless, that in spite of the local considerations just made about the pseudo-cocycle, the associated form is globally defined [see later in this section the comments concerning the $SU(2)$ pseudo-extensions]. This construction was explicitly done in the case of the Poincaré group considered as a principal bundle having the time translation subgroup as the structure group [15].

Going from finite to infinite dimension opens new possibilities as the Whitehead lemma no longer holds. Thus, for example, central extensions are not forbidden for semisimple Lie groups.

The *Full Polarization* \mathcal{P} , which generalizes that of ordinary Geometric Quantization [18, 19] is defined as a *maximal horizontal* (with respect to the connection Θ) *subalgebra of left invariant vector fields* containing the *characteristic subalgebra* of Θ [12, 15], i.e., the algebra generated by the left invariant vector fields that annihilate Θ and $d\Theta$ [26]:

$$\mathcal{G}_\Theta = \langle \tilde{X}^L \in \mathfrak{X}^L(\tilde{G}) / i_{\tilde{X}^L} \Theta = 0 = i_{\tilde{X}^L} d\Theta \rangle. \quad (4.1)$$

Thus, the full set of conditions on $\Psi \in \mathcal{F}(\tilde{G}, \mathbb{C})$ characterizing the wave functions are

$$\Xi \cdot \Psi = i\Psi, \quad \tilde{X}^L \cdot \Psi = 0, \quad \forall \tilde{X}^L \in \mathcal{P}, \quad (4.2)$$

where Ξ is the fundamental vector field (generator) of the principal bundle \tilde{G} . On these functions Ψ , which are constrained by means of left invariant vector fields, the *right invariant vector fields act as ordinary derivations defining the physical operators* of the theory. The fact that (in general) left and right vector fields commute is important for the stability of the wave functions subspace under the operator actions.

To apply the GQF to the special case of $\widehat{\Omega SU}(2)$ we start writing a (formal) group law which exhibits a specific principal bundle structure. To this end we note that besides the principal bundle structure associated with the local central extension characterized by (say) the cocycle ξ in (3.4), the Hopf fibration of the $SU(2)$ subgroup itself defines a Z -parametrized family of pseudo-extensions to be taken into account in writing the group law for the ϕ parameter. In fact, these pseudo-extensions precisely account for the irreducible representations of the $SU(2)$ subgroup; otherwise $SU(2)$ would be trivially represented. As above mentioned, these pseudo-extensions are realized by means of coboundaries locally generated by *linear functions* on the local parameter θ_0 of the $U(1)$ subgroup of

$SU(2)$ [27] ($\xi_{\text{cob}}(g', g) = \delta(g' * g) - \delta(g') - \delta(g)$). Taking into account the global structure of the involved subgroups, the only possible group laws for the $U(1)$ central subgroup of the pseudo-extended group are ($\zeta \equiv e^{i\phi}$, $\eta_0 \equiv e^{i\theta_0/2} = e^{i\delta(g)}$),

$$\zeta'' = \zeta' \zeta (\eta_0'' \eta_0'^{-1} \eta_0^{-1})^N, \quad N \in \mathbb{Z}. \tag{4.3}$$

Thus, an integer constant (the winding number) appears in the characterization of the representations.

Equation (4.3) along with (3.1) constitute the local group law adapted to the globally defined $U(1)$ pseudo-extended $SU(2)$ group, $SU(2) \otimes U(1)$. As a principal bundle $SU(2) \otimes U(1) \rightarrow SU(2)$ is characterized by the Čech-coboundary $\tilde{g}_{\alpha\beta}(\xi) = \pi^* g_{\alpha\beta}(\xi) \equiv g_{\alpha\beta}(\xi + \sigma\xi)$, where π is the $SU(2) \rightarrow S^2$ projection, $g_{\alpha\beta}$ is a Čech-cocycle associated with the $SU(2) \rightarrow S^2$ fibration, $\xi \in S^3 \subset C^2$ and σ are the Pauli matrices.

We now propose as the final group law for $\widetilde{\Omega SU}(2)$ the law given by (3.2) plus

$$\begin{aligned} \phi'' &= \phi' + \phi + \alpha \xi(g', g) + N \xi_{\text{cob}}(g', g), \\ \xi_{\text{cob}}(g', g) &= -i \ln(\eta_0'' \eta_0'^{-1} \eta_0^{-1}) \\ &= \frac{1}{2} \{ i(z'_{-m} z_m^* - z'^*_m z_m) + (z'_{-m-p} z_m^* + z'^*_{-m-p} z_m) \theta_p + \dots \}. \end{aligned}$$

From the group law (3.2), (4.4) we only have to derive the left-invariant fields, the Θ form (dual to the ϕ generator) by means of the duality relations and to obtain the Full Polarization subalgebra. We shall proceed, of course, order by order.

Left-Invariant Vector Fields:

$$\begin{aligned} \tilde{X}_{z_q}^L &= \frac{\partial}{\partial z_q} - iz_{n-q}^* \frac{\partial}{\partial \theta_n} - 2z_m z_{n-m-q}^* \frac{\partial}{\partial z_n} \\ &+ \dots - \left[\left(\alpha q + i \frac{N}{2} \right) z_{-q}^* + \frac{i\alpha}{2} n \theta_n z_{-n-q}^* + \dots \right] \Xi, \\ \tilde{X}_{z_q}^{L*} &= \frac{\partial}{\partial z_q^*} + iz_{n-q} \frac{\partial}{\partial \theta_n} - 2z_m^* z_{n-m-q} \frac{\partial}{\partial z_n^*} \\ &+ \dots - \left[\left(\alpha q - i \frac{N}{2} \right) z_{-q} - i \frac{\alpha}{2} n \theta_n z_{-n-q} + \dots \right] \Xi, \\ \tilde{X}_{\theta_q}^L &= \frac{\partial}{\partial \theta_q} - iz_{n-q} \frac{\partial}{\partial z_n} + iz_{n-q}^* \frac{\partial}{\partial z_n^*} + \left[-\frac{\alpha}{2} q \theta_q + \dots \right] \Xi, \\ \tilde{X}_\phi^L &\equiv \Xi = \frac{\partial}{\partial \phi}. \end{aligned}$$

Lie Algebra Commutators:

$$\begin{aligned} [\tilde{X}_{z_q}^L, \tilde{X}_{z_p}^{L*}] &= 2i \tilde{X}_{\theta_{q+p}}^L + 2 \left(\alpha q + i \frac{N}{2} \right) \delta_{q, -p} \Xi, \\ [\tilde{X}_{\theta_q}^L, \tilde{X}_{z_p}^L] &= i \tilde{X}_{z_{q+p}}^L, \quad [\tilde{X}_{\theta_q}^L, \tilde{X}_{z_p}^{L*}] = -i \tilde{X}_{z_{q+p}^*}^{L*}, \\ [\tilde{X}_{\theta_q}^L, \tilde{X}_{\theta_p}^L] &= \alpha q \delta_{q, -p} \Xi. \end{aligned} \tag{4.6}$$

Quantization Form Θ :

$$\Theta = \left(\alpha q + i \frac{N}{2} \right) [z_q^* dz_q - z_q dz_q^*] + \frac{\alpha}{2} q \theta_{-q} d\theta_q + i \left[\left(\alpha n + i \frac{N}{2} \right) z_{n-q} z_{-n}^* + \left(-\alpha n + i \frac{N}{2} \right) z_{n-q}^* z_{-n} \right] d\theta_q + \dots + d\phi. \quad (4.7)$$

Characteristic Subalgebra. Both solving the equations $i_{\tilde{X}}\Theta = 0 = i_{\tilde{X}}d\Theta$, we find the following possibilities (depending on the values of α and N):

$$\mathcal{G}_\Theta \text{ generated by } \begin{cases} \tilde{X}_{\theta_0}^L & \text{for } \forall \alpha \neq 0, N \neq 0 \end{cases} \quad (4.8a)$$

$$\begin{cases} \tilde{X}_{\theta_0}^L, \tilde{X}_{z_0}^L, \tilde{X}_{z_0^*}^{L*} & \text{for } \forall \alpha \neq 0, N = 0. \end{cases} \quad (4.8b)$$

Equation (4.8b) corresponds to a $SU(2)$ subalgebra [indeed (4.8b) is the original $SU(2)$ subalgebra] while (4.8a) can be seen as the Cartan subalgebra of $SU(2)$.

Full Polarization Subalgebra. Unlike the finite dimensional case there may exist more than one non-equivalent polarization. Indeed we are faced with a “no conjugacy” theorem similar to that of Borel subalgebras [28] of infinite dimensional semisimple algebras [29] to which polarizations parallel. This means that non-equivalent dynamics are associated with different unequivalent polarizations. To each one of the possible choices of \mathcal{G}_Θ we find the following essentially different polarizations (any other is obtained by conjugacy):

$$\mathcal{G}_\Theta = \langle \tilde{X}_{\theta_0}^L \rangle \rightarrow \begin{cases} \mathcal{P}_a = \langle \tilde{X}_{\theta_n \leq 0}^L, \tilde{X}_{z_q}^L \rangle & (4.9a) \\ \mathcal{P}'_a = \langle \tilde{X}_{\theta_n \leq 0}^L, \tilde{X}_{z_q \leq 0}^L, \tilde{X}_{z_q^* < 0}^{L*} \rangle, & (4.9a') \end{cases}$$

$$\mathcal{G}_\Theta = \langle \tilde{X}_{\theta_0}^L, \tilde{X}_{z_0}^L, \tilde{X}_{z_0^*}^{L*} \rangle \rightarrow \mathcal{P}_b = \langle \tilde{X}_{\theta_n \leq 0}^L, \tilde{X}_{z_q \leq 0}^L, \tilde{X}_{z_q^* \leq 0}^{L*} \rangle, \quad (4.9b)$$

Mimicking the denomination used for Borel subalgebras [29] we call (4.9a) the *natural* polarization while (4.9a') will be referred to as the *standard* one. It must be noted that (4.9a') and (4.9b) have the same structure; (4.9b) is obtained from (4.9a') by adding to it the generator $\tilde{X}_{z_0^*}^{L*}$ which appears in the corresponding \mathcal{G}_Θ .

We shall be concerned in what follows with the more general case of (4.8a) and, in turn, restricted to the (4.9a) choice which is going to provide us with a set of non-explicitly worked out (Verma module) representations. Exactly the same procedure can be applied in the other possibilities. It will be done elsewhere.

Wave Functions. The wave functions are the $U(1)$ -functions on $\widehat{\Omega SU}(2)$ satisfying the Full Polarization conditions, i.e., functions on $\widehat{\Omega SU}(2)$ verifying $\Xi \cdot \Psi = i\Psi$ and

$$\tilde{X}_{\theta_n}^L \cdot \Psi = 0, \quad \tilde{X}_{z_q}^L \cdot \Psi = 0, \quad n, q \in \mathbb{Z}, n \leq 0. \quad (4.10)$$

Equations (4.10) can be solved order by order leading to the following set of “Fock” states:

$$\begin{aligned}
 |\alpha, N\rangle &\sim \zeta \cdot W(\theta, z, z^*; \alpha, N), \quad (\text{the vacuum}) \\
 |\alpha, N; \theta_n\rangle &\sim \zeta \cdot W \cdot \left\{ \theta_n + i \sum_p z_{n-p} z_p^* + O(4) \right\}, \quad n > 0, \\
 |\alpha, N; \theta_n \theta_m\rangle &\sim \zeta \cdot W \cdot \left\{ \theta_n + i \sum_p z_{n-p} z_p^* + \dots \right\} \\
 &\quad \times \left\{ \theta_m + i \sum_p z_{m-p} z_p^* + \dots \right\}, \quad n, m > 0, \\
 \text{---} \\
 |\alpha, N; z_q^*\rangle &\sim \zeta \cdot W \cdot e^{-i\theta_0} \times \left\{ z_q^* - i \sum_{q-m < 0} \theta_{q-m} z_m^* - \frac{1}{2} \sum_{\substack{q-p-m < 0 \\ p < 0}} \theta_{q-p-m} \theta_p z_m^* \right. \\
 &\quad \left. + \sum_{q-p \leq 0} z_{q-p-m} z_p^* z_m^* + \dots \right\}, \quad q \in \mathbb{Z}, \\
 |\alpha, N; z_q^* z_k^*\rangle &\sim \zeta \cdot W \cdot e^{-2i\theta_0} \\
 &\quad \times \left\{ z_q^* - i \sum_{q-m < 0} \theta_{q-m} z_m^* - \frac{1}{2} \sum_{\substack{q-p-m < 0 \\ p < 0}} \dots \right\} \\
 &\quad \times \left\{ z_k^* - i \sum_{k-m < 0} \theta_{k-m} z_m^* - \frac{1}{2} \sum_{\substack{k-p-m < 0 \\ p < 0}} \dots \right\}, \quad q, k \in \mathbb{Z}, \\
 \text{---} \\
 |\alpha, N; \theta_n z_q^*\rangle &\sim \zeta \cdot W \cdot e^{-i\theta_0} \left\{ \theta_n + i \sum_p z_{n-p} z_p^* + \dots \right\} \\
 &\quad \times \left\{ z_q^* - i \sum_{q-m < 0} \theta_{q-m} z_m^* + \dots \right\}, \quad n > 0, q \in \mathbb{Z}, \\
 \text{---}
 \end{aligned}
 \tag{4.11}$$

where the weight function takes the form

$$\begin{aligned}
 W(\theta, z, z^*; \alpha, N) &= 1 - \frac{N}{2} \sum_k z_k z_{-k}^* + i \frac{\alpha}{2} \sum_{m < 0} m \theta_m \theta_{-m} \\
 &\quad + i \alpha \sum_m m z_m z_m^* - \alpha \sum_{m < 0} m \theta_m z_p z_{-m-p}^* + \dots
 \end{aligned}
 \tag{4.12}$$

Once the wave functions are known, the group law [(3.2), (4.4)] can be used to obtain group representations parametrized by α and N . This is accomplished by means of the left translations acting on Ψ ,

$$L_{\tilde{g}} \Psi(\tilde{g}) = \Psi(\tilde{g}' * \tilde{g}), \quad \tilde{g}', \tilde{g} \in \widehat{\Omega SU}(2).
 \tag{4.13}$$

As mentioned earlier the right-invariant vector fields \tilde{X}^R (which generate the finite left translations) act on the wave functions as ordinary derivations and define the basic (Lie algebra) operators.

Right-Invariant Vector Fields: Basic Operators and Lie Algebra Representations:

$$\begin{aligned}
 \tilde{X}_{z_q}^R &= \frac{\partial}{\partial z_q} - i\theta_{n-q} \frac{\partial}{\partial z_n} + iz_n^* \frac{\partial}{\partial \theta_n} \\
 &\quad - z_m z_n^* \frac{\partial}{\partial z_n} - \frac{1}{2} \theta_m \theta_{n-m-q} \frac{\partial}{\partial z_n} \\
 &\quad - z_m^* z_n^* \frac{\partial}{\partial z_n^*} + z_m^* \theta_{n-m-q} \frac{\partial}{\partial \theta_n} + \dots + \left[\left(\alpha q + i \frac{N}{2} \right) z_{-q}^* \right. \\
 &\quad \left. + \left(i \frac{\alpha}{2} (n-q) + \frac{N}{2} \right) z_n^* \theta_{-n-q} + \dots \right] \Xi, \\
 \tilde{X}_{z_n^*}^R &= \frac{\partial}{\partial z_q^*} + i\theta_{n-q} \frac{\partial}{\partial z_n^*} - iz_{n-q} \frac{\partial}{\partial \theta_n} - z_m^* z_{n-m-q} \frac{\partial}{\partial z_n^*} - \frac{1}{2} \theta_m \theta_{n-m-q} \frac{\partial}{\partial z_n^*} \\
 &\quad - z_m z_{n-m-q} \frac{\partial}{\partial z_n} + z_m \theta_{n-m-q} \frac{\partial}{\partial \theta_n} + \dots + \left[\left(\alpha q - i \frac{N}{2} \right) z_{-q} \right. \\
 &\quad \left. + \left(-i \frac{\alpha}{2} (n-q) + \frac{N}{2} \right) z_n \theta_{-n-q} + \dots \right] \Xi, \\
 \tilde{X}_{\theta_q}^R &= \frac{\partial}{\partial \theta_q} + \left[\frac{\alpha}{2} q \theta_{-q} + \dots \right] \Xi. \tag{4.14}
 \end{aligned}$$

The derivative of the wave functions with respect to (4.14) provide an approximate expression for the basic operators (indeed a 3rd order one) that, of course, could be improved by going further in the group law series. Nevertheless this 3rd order calculation allows us to conjecture a full expansion which is a posteriori tested by making use of the Lie algebra commutators and the fact that part of the operators are exactly known. As an example we give the action of \tilde{X}^R on the basic states, i.e., $|\alpha, N\rangle$, $|\alpha, N; \theta_n\rangle$, $n > 0$, and $|\alpha, N; z_q^*\rangle$, $\forall q \in \mathbb{Z}$:

$$\begin{aligned}
 \tilde{X}_{\theta_q}^R |\alpha, N\rangle &= +i\alpha q |\alpha, N; \theta_{-q}\rangle, \quad q \leq 0, \\
 \tilde{X}_{\theta_q}^R |\alpha, N\rangle &= 0, \quad q > 0, \\
 \tilde{X}_{\theta_q}^R |\alpha, N; \theta_n\rangle &= +i\alpha q |\alpha, N; \theta_n \theta_{-q}\rangle, \quad q \leq 0, \\
 \tilde{X}_{\theta_q}^R |\alpha, N; \theta_n\rangle &= \delta_{q,n} |\alpha, N\rangle, \quad q > 0, \\
 \tilde{X}_{\theta_q}^R |\alpha, N; z_n^*\rangle &= -i |\alpha, N; z_{n-q}^*\rangle + i\alpha q |\alpha, N; z_n^* \theta_{-q}\rangle, \quad q \leq 0, \\
 \tilde{X}_{\theta_q}^R |\alpha, N; z_n^*\rangle &= 0, \quad q > 0, \tag{4.15} \\
 \tilde{X}_{z_q}^R |\alpha, N\rangle &= -2 \sum_s \frac{(-i)^s}{s!} \sum_{m_1 \dots m_s > 0} \\
 &\quad \times \left[\frac{N}{2} - i\alpha(q + m_1 + \dots + m_s) \right] |\alpha, N; \theta_{m_1} \dots \theta_{m_s} z_{-q-m_1 \dots -m_s}^*\rangle,
 \end{aligned}$$

$$\begin{aligned} \tilde{X}_{z_q}^R |\alpha, N; z_n^*\rangle &= -2 \sum_s \frac{(-i)^s}{s!} \sum_{m_1 \dots m_s > 0} \\ &\times \left[\frac{N}{2} - i\alpha(q + m_1 + \dots + m_s) \right] |\alpha, N; \theta_{m_1} \dots \theta_{m_s} z_n^* z_{-q-m_1 \dots m_s}^*\rangle \\ &+ \sum_p \operatorname{sgn}(n-p) \sum_s \frac{(-i)^s}{s!} \sum_{m_1 \dots m_s > 0} \\ &\times |\alpha, N; \theta_{m_1} \dots \theta_{m_s} z_p^* z_{n-p-q-m_1 \dots m_s}^*\rangle, \end{aligned}$$

$$\begin{aligned} \tilde{X}_{z_q}^R |\alpha, N; \theta_n\rangle &= -2 \sum_s \frac{(-i)^s}{s!} \sum_{m_1 \dots m_s > 0} \\ &\times \left[\frac{N}{2} - i\alpha(q + m_1 + \dots + m_s) \right] |\alpha, N; \theta_n \theta_{m_1} \dots \theta_{m_s} z_{-q-m_1 \dots m_s}^*\rangle \\ &+ 2i \sum_s \frac{(-i)^s}{s!} \sum_{m_1 \dots m_s > 0} \\ &\times |\alpha, N; \theta_{m_1} \dots \theta_{m_s} z_{n-q-m_1 \dots m_s}^*\rangle, \end{aligned}$$

$$\tilde{X}_{z_q^*}^R |\alpha, N\rangle = 0,$$

$$\begin{aligned} \tilde{X}_{z_q^*}^R |\alpha, N; z_n^*\rangle &= \delta_{n,q} |\alpha, N\rangle \\ &+ \sum_s \frac{(i)^{s+1}}{(s+1)!} \sum_{m_1 \dots m_s > 0} |\alpha, N; \theta_{m_1} \dots \theta_{m_s} \theta_{n-q-m_1 \dots m_s}\rangle, \end{aligned}$$

$$\tilde{X}_{z_q^*}^R |\alpha, N; \theta_n\rangle = 0.$$

The expressions for general states $[\alpha, N; \theta \dots \theta z^* \dots z^* \rangle$ are easily obtained because of their factor structure.

Equations (4.15) generalize the much simpler case of the $SU(2)$ representations on Fock states in terms of which coherent ones can be given. Putting $\alpha=0$ in the previous equations, the set of operators $\left(\tilde{X}_{\theta_0}^R - \frac{N}{2} \tilde{X}_{\zeta}^R, \tilde{X}_{z_q}^R, \tilde{X}_{z_{\pm q}}^R \right)$ for fixed $q \geq 0$ represents the $SU(2)$ Lie algebra on the subspace $H = \{ |N\rangle, [N; z_{\pm q}^* \rangle, [N; z_{\pm q}^* z_{\pm q}^* \rangle, \dots, [N; (z_{\pm q}^*)^m \rangle \dots \}$:

$$\begin{aligned} \tilde{X}_{\theta_0}^R |N; (z_{\pm q}^*)^m \rangle &= -i \left(m - \frac{N}{2} \right) |N; (z_{\pm q}^*)^m \rangle, \\ \tilde{X}_{z_q}^R |N; (z_{\pm q}^*)^m \rangle &= (m - N) |N; (z_{\pm q}^*)^{m+1} \rangle, \\ \tilde{X}_{z_{\pm q}^*}^R |N; (z_{\pm q}^*)^m \rangle &= m |N; (z_{\pm q}^*)^{m-1} \rangle, \quad m = 0, \dots, N, \end{aligned} \tag{4.16}$$

where $[N; z_{\pm q}^* \rangle$ can be now summed up (by ignoring the θ_n , $n \neq 0$, and $z_p, z_{\pm p}^*, p \neq q$):

$$\begin{aligned} |N; (z_{\pm q}^*)^m \rangle &\sim (1 + \varkappa)^{N/2} [e^{-i\theta_0} (1 + \varkappa)^{-1} z_{\pm q}^*]^m, \\ \varkappa &\equiv \sqrt{1 - 4z_{\pm q}^* z_q}. \end{aligned} \tag{4.17}$$

Equations (4.16) clearly reveal the “spin” character of the half-integer $N/2$.

The above operators (4.15) leave invariant the volume ($L_{\tilde{\chi}^R} \mu = 0$)

$$\mu = \prod_{n,k} \theta_{(\theta_n)}^L \wedge \theta_{(\theta_{-n})}^L \wedge \theta_{(z_n)}^L \wedge \theta_{(z_n^* k)}^L, \quad n, k \in \mathbb{Z}, n > 0, \tag{4.18}$$

where $\theta_{(\cdot)}^L$ are (horizontal) components of the canonical 1-form, in terms of which an invariant hermitian form could be defined.

We shall discuss in detail these representations as well as those associated with the case (4.8b) in a future publication where the $\widetilde{\Omega SU}(2)$ group will be enlarged to incorporate the 1 + 1 “space-time translations.” Here we shall only point out that physical models can be built immediately in this group quantization framework which allows us to construct Noether invariants ($i_{\tilde{\chi}^R} \Theta$), equations of motions, etc. [20, 30]. For instance, for the “Hamiltonian” $-i_{\tilde{\chi}^R} \Theta$ we obtain

$$\sum_n -i\alpha n z_n^* z_n + \dots + \frac{N}{2} \sum_n z_n^* z_n + \dots, \tag{4.19}$$

where a term depending on the winding number N does appear.

The meaning of the characteristic algebra \mathcal{G}_Θ is also worth mentioning. Apart from providing the equations of motions, the characteristic group (the generators of which generate \mathcal{G}_Θ) constitutes the symmetry of the vacuum ($[\alpha, N]$ in our case), as can be easily derived from the general formalism [12, 20, 30]. In this way the addition to the group law of $\widetilde{\Omega SU}(2)$ of that we could call a “Wess-Zumino” cocycle (pseudo-cocycle indeed) break the original symmetry (4.8b) to (4.8a).

To conclude, we mention that the above procedure can be applied to any Affine Kac-Moody group $\widetilde{\Omega G}$ by following the same steps given here for the $SU(2)$ case. The only peculiarity (apart from the difficulty of writing down the group law for an arbitrary simple group) is that in general the dimension of the pseudo-cohomology group is greater than one. Thus, several indices of the N type must appear. More concretely, for a general finite dimensional simple Lie group G with Cartan matrix A_{ij} ($\det A \neq 0$), we shall have the following basic commutation relations:

$$\begin{aligned} [t^n \otimes E_\beta, t^{-n} \otimes E_{-\beta}] &= H_\beta + \frac{2n\alpha}{(\beta, \beta)} \Xi + N_\beta \delta^{n,0} \Xi, \\ [t^n \otimes H_i, t^{-n} \otimes H_j] &= \frac{2n\alpha A_{ij}}{(\beta_i, \beta_i)} \Xi. \end{aligned} \tag{4.20}$$

A simple look at (4.20) permits us to write the first terms in the expansion of the presymplectic form $d\Theta$:

$$\begin{aligned} d\Theta \sim \frac{2n\alpha A_{ij}}{(\beta_i, \beta_i)} dh_i^n \wedge dh_j^{-n} + \frac{2n\alpha}{(\beta, \beta)} de_\beta^n \wedge de_\beta^{-n} + \dots \\ + N_\beta \delta^{n,0} de_\beta^n \wedge de_\beta^{-n} + \dots, \end{aligned} \tag{4.21}$$

where h_i^n stand for the parameters associated with the generators $t^n \otimes H_i$ and e_β^n for those associated with $t^n \otimes E_\beta$; there are only $l(\text{rank } \mathcal{G})$ independent winding numbers N_β and they characterize a highest weight of the Lie algebra of G , \mathcal{G} . We

are able from (4.20) to determine the characteristic subalgebra for the case analogous to that of (4.8a),

$$\mathcal{G}_\theta = \langle H \rangle, \quad (4.22)$$

and the essentially two unequivalent Full Polarizations,

$$\mathcal{P} = \langle H^n, n \leq 0; L_\beta^n, n \in \mathbb{Z}, \beta > 0 \rangle, \quad (4.23)$$

$$\mathcal{P}' = \langle H^n, n \leq 0; L_\beta^{n \leq 0}, L_{-\beta}^{n < 0}, \beta > 0 \rangle; \quad (4.24)$$

any other Full Polarization can be obtained by restricting some Lie algebra isomorphism to either (4.23) or (4.24). It must be stressed that in general only the class of \mathcal{P}' is suitable for unitary representations.

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