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Hyperscaling Inequalities for Percolation

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Abstract. A set of critical exponent inequalities for independent percolation which saturate under the hyperscaling hypothesis is proved. One of the consequences of the inequalities is the lower bound $d_C \geqq 6$ for the upper critical dimension. The proof is based on a rigorous version of the finite size scaling argument which extends easily to other systems such as Ising ferromagnets.

1. Introduction

In the present paper, we prove the following critical exponent inequalities for the independent percolation [1]

$$(d-2+\eta)\delta_r \geqq 2 \ , \tag{1.1}$$

$$(d-2+\eta)\nu' \geqq 2\beta \ , \tag{1.2}$$

$$d\nu' \geqq \gamma' + 2\beta \ , \quad d\nu_{\max} \geqq \gamma + 2\beta \ , \tag{1.3}$$

$$d\nu \geqq 2\Delta - \gamma \ , \tag{1.4}$$

$$d\nu' \geqq \Delta' + \beta \ , \quad d\nu_{\max} \geqq \Delta + \beta \ , \tag{1.5}$$

$$(d-2+\eta)\mu\delta \geqq 2 \ , \tag{1.6}$$

$$d\mu \geqq 1 + 1/\delta \ . \tag{1.7}$$

These inequalities are of particular interest because of their close relation to the so-called hyperscaling hypothesis. If the hyperscaling hypothesis is valid, all the inequalities (1.1)–(1.7) become exact equalities.

Usually it is believed that the hyperscaling relations hold only in sufficiently low dimensions. As for independent percolation in two dimensions, Kesten [2] has recently proved almost all of the expected hyperscaling relations. However the validity of the hyperscaling hypothesis in dimensions higher than two is still wide

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open, even on a heuristic level. It should be mentioned that the corresponding problem in the three dimensional Ising model has been an extremely difficult open problem for more than two decades [3].

On the other hand, in dimensions higher than the upper critical dimension, it is believed that the critical phenomena are governed by simple mean field theories. Therefore, in these dimensions, all the critical exponents assume dimension independent mean field values, and the hyperscaling relations are violated.

One of the most interesting features of our inequalities (1.1)–(1.7) is that they provide us with information about the upper critical dimension. More precisely, they are all *inconsistent* with the complete mean field type critical phenomena when the lattice dimension is smaller than six. This implies that the critical phenomena of percolation in dimensions two, three, four, and five are *inevitably not mean field-like*. In terms of the upper critical dimension d_C, our inequalities provide us with a rigorous lower bound $d_C \geqq 6$. Note that the lower bound $d_C \geqq 6$, as well as Aizenman and Newman's sufficient condition (triangle condition) for the mean field behavior [4], is consistent with the general belief $d_C = 6$.

Though there have already been some critical exponent inequlaties which imply $d_C \geqq 4$ [5], and $d_C \geqq 6$ [6], our inequalities are the first ones that are believed to be sharp in sufficiently low dimensions. See [7, 8] and the references therein for other interesting critical exponent inequalities for percolation.

It should be mentioned that the first rigorous critical exponent inequality which saturates under the hyperscaling hypothesis was proved by Fisher [9] for Ising ferromagnets. Moreover the inequalities corresponding to (1.1)–(1.3), (1.6), (1.7) and special cases of (1.4), (1.5) have already been proved for certain ferromagnetic spin systems [9]. Since those proofs are based on the correlation inequalities and some specific features of the spin systems, none of them extend easily to percolation.[1] Therefore instead of looking for possible extensions of the existing proofs, we here develop a new argument which is based on the finite size scaling idea [10]. It then turns out that the argument naturally leads us to the desired hyperscaling inequalities. Moreover our technique can be easily extended to other lattice systems such as Ising ferromagnets.

The organization of the present paper is as follows. In Sect. 2, we give precise definitions of the percolation system, some physical quantities, and various critical exponents. In Sect. 3, we discuss a consequence of our inequalities to the problem of the upper critical dimension. Then, in Sects. 4–7, we prove our critical exponent inequalities. In the Appendix, we extend the present method to Ising ferromagnets.

2. Definitions

For simplicity, we restrict ourselves to the neares neighbour bond percolation on the d-dimensional hypercubic lattice. However all of our results extend automatically to any translation invariant short range bond or site percolations.

[1] It is interesting that most of the existing proofs of the critical exponent inequalities which saturate under the *scaling hypothesis* (e.g., Fisher's $(2-\eta)\nu \geqq \gamma$ [9]) automatically extend to percolation without any modifications

Let Z^d be the d-dimensional hypercubic lattice whose elements $x, y, \ldots$ are called *sites*. We denote by 0 the origin of the lattice. A *bond* is an unordered pair $\{x, y\}$ of two sites satisfying $\|x-y\|_2=1$. In independent percolation, each bond in the lattice is *occupied* (*respectively*, *unoccupied*) independently with probability p (*respectively* $1-p$). The occupation probability p is our only model parameter. Let us denote by $\mathrm{Prob}_p(\ldots)$ and $\langle \ldots \rangle_p$ the probability and expectation value associated to the above process.

For a given configuration (i.e., occupation status of all the bonds in Z^d), we say that two sites x, y are *connected* if there exists path of occupied bonds which connects x and y. More precisely, x and y are connected if there exists a sequence of sites $\{x_1, \ldots, x_n\}$, where $x_1=x$, $x_n=y$, and each $\{x_i, x_{i+1}\}$ is an occupied bond. We also say that a site x is connected to a set (of sites) Y if there exists a site y in Y which is connected to x. Finally a *cluster* $C(x)$ is defined as a set of all the sites which are connected to x. $|C(x)|$ denotes the number of the sites in $C(x)$.

Now let us define some physical quantities of interest. The *connectivity function* $\tau_p(\ldots)$ and *truncated connectivity function* $\tau'_p(\ldots)$ are

$$\tau_p(x_1, x_2, \ldots, x_n)=\mathrm{Prob}_p\,(C(x_1) \ni x_2, \ldots, x_n)\ , \tag{2.1}$$

$$\tau'_p(x_1, x_2, \ldots, x_n)=\mathrm{Prob}_p\,(C(x_1) \ni x_2, \ldots, x_n, |C(x_1)|<\infty)\ . \tag{2.2}$$

The *mean (finite) cluster size* $\chi(p)$ is

$$\chi(\mathrm{p})=\langle |C(0)|\, X(|C(0)|<\infty)\rangle_p=\sum_x \tau'_p(0, x)\ , \tag{2.3}$$

where $X(A)=1$ (or 0) when A is true (or false). The *correlation length* $\xi(p)$ is

$$\xi(p)=\inf\{\xi \,|\, \tau'_p(0, x) \leqq e^{-|x|/\xi} \quad \text{for any} \quad x \in Z^d\}\ . \tag{2.4}$$

Finally the *order parameter* $M(p)$ and its finite volume counterpart $M(p; L)$ are

$$M(p)=\mathrm{Prob}_p\,(|C(0)|=\infty)\ , \tag{2.5}$$

$$M(p; L)=\mathrm{Prob}_p\,(0 \text{ is connected to } \partial S_L)\ , \tag{2.6}$$

where

$$\partial S_L=\{x|\ |x|=[L/2]\}\ . \tag{2.7}$$

Throughout the present paper, we use the metric defined by

$$|x|=\max\{|x_1|, |x_2|, \ldots, |x_d|\}=\|x\|_\infty\ .$$

In the system with $d \geqq 2$, it is known [8] that there exists a critical probability p_c $(0<p_c<1)$ which is characterized by

$$M(p)=0 \quad \text{if} \quad p<p_c\ ,$$

$$M(p)>0 \quad \text{if} \quad p>p_c\ .$$

It is also known that the mean cluster size $\chi(p)$ and the correlation length $\xi(p)$ diverge when p approaches p_c from below [1, 4]. Note that if $p<p_c$, the truncated connectivity function $\tau'_p(\ldots)$ is nothing but the connectivity function $\tau_p(\ldots)$, since the condition $|C(x_1)|<\infty$ is automatically satisfied.

For the values of p close to or equal to p_c, the percolation system is believed to exhibit various critical phenomena. More precisely, in this region many physical quantities are expected to show power law singularities characterized by the *critical exponents*. In the present paper, we do not assume the existence of the power law singularities, and introduce the critical exponents through the following formal definitions.

Let the relation $f(x) \lesssim x^{\lambda}$ as $x \searrow 0$ be an abbreviation for $f(x) \leqq s(x) x^{\lambda}$ for $x \geqq 0$ with a slowly varying function $s(x)$ (i.e., $\lim_{t \searrow 0} s(tx)/s(t) = 1$ for any $x > 0$, e.g., $s(x) = \text{const}$, $s(x) = |\ln x|$). Then the critical exponents $\gamma, \gamma', \bar{\gamma}, \nu, \nu', \beta, \Delta_n, \Delta_n', \eta$, and δ_r are defined as the *optimal* constants satisfying the following relations.[2]

As $p \nearrow p_c$,

$$\chi(p) \gtrsim (p_c - p)^{-\gamma}, \tag{2.8}$$

$$\chi(p) \lesssim (p_c - p)^{-\bar{\gamma}}, \tag{2.9}$$

$$\xi(p) \lesssim (p_c - p)^{-\nu}, \tag{2.10}$$

$$\langle |C(0)|^{n-1} \rangle_p / \langle |C(0)|^{n-2} \rangle_p \gtrsim (p_c - p)^{-\Delta_n}, \tag{2.11}$$

as $p \searrow p_c$,

$$\chi(p) \gtrsim (p - p_c)^{-\gamma'}, \tag{2.12}$$

$$\xi(p) \lesssim (p - p_c)^{-\nu'}, \tag{2.13}$$

$$M(p) \lesssim (p - p_c)^{\beta},$$

$$\langle |C(0)|^{n-1} X(|C(0)| < \infty) \rangle_p / \langle |C(0)|^{n-2} X(|C(0)| < \infty \rangle_p \gtrsim (p - p_c)^{-\Delta_n'}, \tag{2.14}$$

at $p = p_c$,

$$\tau_{p_c}(x, y) \gtrsim |x - y|^{-(d-2+\eta)}, \tag{2.15}$$

$$M(p_c; L) \lesssim L^{-1/\delta_r}. \tag{2.16}$$

Here the specific choice of upper or lower bound is merely from technical considerations. Usually one believes that the above relations with $\lesssim$ or $\gtrsim$ replaced by $\sim$ ($f \sim x^{\lambda}$ means $f \lesssim x^{\lambda}$ and $f \gtrsim x^{\lambda}$) are valid. (Thus, in particular, we have $\gamma = \bar{\gamma}$.) See Sect. 7 for the definitions of the other critical exponents.

3. Critical Dimension

In a suitable mean field theory (e.g., Cayley tree model, $d \to \infty$ limit) for percolation, one can easily calculate the critical behavior of many quantities explicitly. Then we find that many quantities exhibit strict power law behaviors [the relations like (2.8)–(2.16) with $\lesssim$ or $\gtrsim$ replaced by $\sim$] with the critical exponents $\gamma = \gamma' = 1$, $\nu = \nu' = 1/2$, $\beta = 1$, $\Delta_n = \Delta_n' = 2$, $\eta = 0$, $\delta_r = 1/2$, $\mu = 1/4$, and $\delta = 2$. (See Sect. 7 for the definitions of the exponents μ and δ.)

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[2] Note that our definition of the gap exponent Δ_n differs from that in some articles (such as [2]) in percolation. But ours is a natural extension of the standard definition in the spin systems

Let us substitute these mean field values of the critical exponents into our inequalities (1.1)–(1.7). Then we find that each of the inequalities leads us to a single inequality $d \geqq 6$. This implies that the complete mean field type critical phenomea are *inconsistent* with our critical exponent inequalities (and with their counterparts for the original physical quantities such as ξ, χ, and M) if the lattice dimension is smaller than six. In other words, the upper critical dimension d_c of the translation invariant short range independent percolation cannot be less than six!

Note that the lower bound $d_c \geqq 6$ was already mentioned in [6], where the bound was concluded from the other critical exponent inequalities (which were however not optimal as the present ones).

4. Basic Inequalities

In the present section, we prove our most basic inequalities (1.1) and (1.2). Although the derivation of these inequalities is rather elementary and straightforward, it contains some of the essential ideas of the present rigorous finite size scaling approach.

Proposition 4.1. *For arbitrary positive integer L and $x \in Z^d$ with $|x| = L$, we have*

$$\tau_p(0, x) \leqq M(p; L)^2 \ . \tag{4.1}$$

From the above inequality with $p = p_c$ and the definition of the critical exponents η and δ_r, we immediately get

Corollary 4.1. *The critical exponents η and δ_r satisfy*

$$(d - 2 + \eta)\delta_r \geqq 2 \ . \tag{4.2}$$

Proof of Proposition. Observe that when 0 and x are connected, each of them must be connected to some site at a distance $[L/2]$ of each (Fig. 1). Since the latter two events take place in the two separated regions in the lattice, we get the desired bound (4.1). □

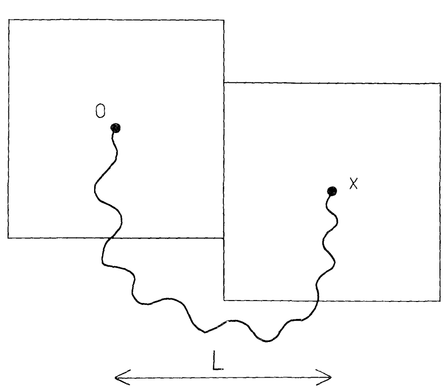

Fig. 1. The event that 0 and x are connected

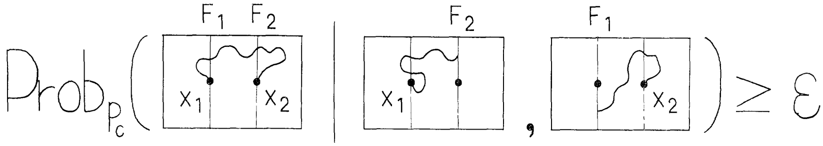

Fig. 2. A sufficient condition for the hyperscaling relation $(d-2+\eta)\delta_r=2$

Remark. Let $L=4n$ with a positive integer n. We consider the percolation system in a finite lattice $\{y|(-3/4)L\leqq y_1\leqq(3/4)L,\ |y_i|\leqq L/2\ (i=2,\ldots,d)\}$. We define (see Fig. 2)

$$F_1=\{y|y_1=-L/4,\ |y_i|\leqq L/2\ (i=2,\ldots,d)\}\ ,$$

$$F_2=\{y|y_1=L/4,\ |y_i|\leqq L/2\ (i=2,\ldots,d)\}\ ,$$

$x_1=(-L/4,0,0,\ldots)$, and $x_2=(L/4,0,0,\ldots)$. Consider the following condition (see Fig. 2):

$$\mathrm{Prob}_{p_c}(x_1 \text{ and } x_2 \text{ are connected} \mid x_1 \text{ is connected to } F_2,\ x_2 \text{ is connected to } F_1)\geqq\varepsilon$$

holds with $\varepsilon>0$ uniformly in L. Here $\mathrm{Prob}(A|B)=\mathrm{Prob}(A,B)/\mathrm{Prob}(B)$ denotes the conditional probability. Note that the above probability is nothing but the *intersection probability* of two large clusters at p_c. Then it is not difficult to show that the above condition [along with (4.1)] implies the relation $\tau_{p_c}(0,(L,0,0,\ldots))\sim M(p_c;L)^2$ which reduces to a *hyperscaling relation* $(d-2+\eta)\delta_r=2$. However we do not know any methods of proving (or disproving) the above condition in the dimensions higher than two.

In order to eliminate the rather unfamiliar exponent δ_r from (4.1), and to get our next inequality (1.2), we will make use of the idea of the heuristic finite size scaling theory [10].

In the finite size scaling theory, it is argued that the finite size order parameter $M(p_c;L)$ behaves almost similarly to the full order parameter $M(p)$ evaluated at the value of $p(p>p_c)$ which satisfies $\xi(p)=L$. Although it is very hard to justify this conjecture in general (but see [2] for the results in two dimensions), the following weaker version can be proved very easily.

Lemma 4.1. *For arbitrary $p>p_c$ with $\xi(p)\geqq C_1$, we have*

$$M(p)\leqq M(p;L)\leqq 2M(p) \tag{4.3}$$

when $L=2d\xi(p)|\ln\xi(p)|$. Here C_1 is a constant which depends only on the dimension.

Here the choice of constants $2d$ and 2 are rather arbitrary. In two dimensions, Nguyen [11] has proved the above bound without (unwanted) $\ln\xi$ factor.

Proof. The first inequality is trivial since whenever the origin is connected to infinity, it must be connected to ∂S_L. Let B_L be the event that the origin is connected to ∂S_L.

To prove the second inequality, note that

$$\begin{aligned} M(p;L) &= \mathrm{Prob}_p(B_L) \\ &= \{1-\mathrm{Prob}_p(B_L, |C(0)|<\infty)/\mathrm{Prob}_p(B_L)\}^{-1}\,\mathrm{Prob}_p(B_L, |C(0)|=\infty) \\ &\leqq \{1-\mathrm{Prob}_p(B_L, |C(0)|<\infty)/\mathrm{Prob}_p(B_L)\}^{-1} M(p) \ . \end{aligned} \tag{4.4}$$

Let us construct an upper bound for the prefactor. Recall that Simon's argument [12] combined with the percolation version of Simon's inequality [4] implies that

$$\sum_{x\in\partial S_L} \tau_p(0,x) \geqq 1 \quad \text{if} \quad p \geqq p_c \ . \tag{4.5}$$

Therefore one can find x in ∂S_L with the property $\tau_p(0,x) \geqq (2dL^{d-1})^{-1}$. Thus we get

$$\mathrm{Prob}_p(B_L) \geqq \tau_p(0,x) \geqq (2dL^{d-1})^{-1} \ . \tag{4.6}$$

On the other hand observe that

$$\mathrm{Prob}_p(B_L, |C(0)|<\infty) \leqq \sum_{x\in\partial S_L} \tau'_p(0,x) \leqq 2dL^{d-1} e^{-L/\xi(p)} \ , \tag{4.7}$$

where we used the defintion (2.4) of $\xi(p)$. Substituting (4.6), (4.7) into (4.4), we get

$$M(p;L) \leqq \{1-(2dL^{d-1})^2 e^{-L/\xi(p)}\}^{-1} M(p) \ .$$

For $L = 2d\xi(p)|\ln \xi(p)|$ and sufficiently large $\xi(p)$, the right-hand side of the above bound is bounded by $2M(p)$. □

Let us define the quantity $p(L)$ by the following formula:

$$p(L) = \inf\{p \,|\, p > p_c,\ 2d\xi(p)|\ln \xi(p)| \leqq L\} \ . \tag{4.8}$$

Note that if $\xi(p)$ is a monotone continuous function (as is expected), $p(L)$ is nothing but the inverse function of $2d\xi(p)|\ln \xi(p)|$.

Proposition 4.2. *For arbitrary $L \geqq C_1$ and x with $|x| = L$, we have*

$$\tau_{p_c}(0,x) \leqq 4M(p(L))^2 \ , \tag{4.9}$$

provided that $\xi(p) \nearrow \infty$ as $p \searrow p_c$.

Proof. Since $M(p;L) \geqq M(p_c;L)$ for $p \geqq p_c$, (4.9) follows immediately from Proposition 4.1 and Lemma 4.1. □

If we note that the relation $\xi(p) \lesssim (p-p_c)^{-\nu'}$ implies $p(L) - p_c \lesssim (p-p_c)^{-1/\nu'}$, we get

Corollary 4.2. *Whenever $\xi(p) \nearrow \infty$ as $p \searrow p_c$, the critical exponents η, ν', and β satisfy*

$$(d-2+\eta)\nu' \geqq 2\beta \ . \tag{4.10}$$

Remark. Combining the consequence of Simon's argument (4.5) and the present idea, we get the following strict lower bound for the finite size order parameter $M(p;L)$,

$$M(p;L) \geqq (2^d d)^{-1/2} L^{-(d-1)/2} \quad \text{if} \quad p \geqq p_c \ .$$

This bound, which was first proved by Aizenman [13], is a generalization of van den Berg and Kesten's result in two dimensions [14].

5. Inequalities for ν, γ, and β

In the present section, we prove the inequalities (1.3). We think that these are the best critical exponent inequalities among those proved in the present paper, since they only include simple and standard critical exponents which are also "approach exponents". Here the "approach exponents" means the critical exponents defined through the singular behavior which takes place when the system approaches its critical point.

Proposition 5.1. *For arbitrary $p>p_c$ with $\xi(p)\geqq C_2$, we have*

$$\chi(p)\leqq C_3 \sum_{L=\{(d/2)\xi(p)|\ln\xi(p)|\}^{1/d}}^{2d\xi(p)|\ln\xi(p)|} L^{d-1}M(p(L))^2 \ , \tag{5.1}$$

where C_2 and C_3 are constants which depend only on the dimension. ($p(L)$ is defined by (4.9).)

Let us again assume that $\xi(p)\nearrow\infty$ as $p\searrow p_c$. Then as p approaches p_c, any $p(L)$ in the summation in (5.1) also approaches p_c. Then we can substitute the critical behavior of the quantities into (5.1) to get

$$(p-p_c)^{-\gamma'}\lesssim \sum_{L=1}^{2d\xi(p)|\ln\xi(p)|} L^{d-1}L^{-2\beta/\nu'}\lesssim(p-p_c)^{-d\nu'+2\beta} \ .$$

This immediately implies

Corollary 5.1. *Whenever $\xi(p)\nearrow\infty$ as $p\searrow p_c$, the critical exponents ν', γ', and β satisfy*

$$d\nu'\geqq\gamma'+2\beta \ . \tag{5.2}$$

Proof of Proposition. Let us bound $\chi(p)$ by the following three terms:

$$\chi(p)=\sum_x \tau'_p(0,x)\leqq \sum_{x;|x|\leqq L_1} 1+\sum_{x;L_1<|x|\leqq L_2}\tau_p(0,x)+\sum_{x:L_2<|x|}\tau'_p(0,x) \ ,$$

where $L_1=\{(d/2)\xi(p)|\ln\xi(p)|\}^{1/d}$, $L_2=2d\xi(p)|\ln\xi(p)|$. Here we have used the trivial inequalities $\tau'_p(0,x)\leqq\tau_p(0,x)\leqq 1$. Using the bound (4.5) (which is a consequence of Simon's argument), the first and second terms in (5.2) can be related as

$$\sum_{x:|x|\leqq L_1} 1=d\xi(p)|\ln\xi(p)|\leqq \sum_{x;L_1<|x|\leqq L_2}\tau_p(0,x) \ .$$

From the definition of $\xi(p)$, the third term is bounded as

$$\sum_{x;L_2<|x|}\tau'_p(0,x)\leqq\sum_{L>L_2} 2dL^d e^{-L/\xi(p)}\leqq\xi(p)^{-a}$$

with $a>0$ and $\xi(p)\geqq C'$, where C' is a sufficiently large constant. Combining these bounds together we get an upper bound

$$\chi(p) \leqq \text{const} \sum_{x;\, L_1 < |x| \leqq L_2} \tau_p(0, x) \ ,$$

provided that $\xi(p) \geqq C_2$, where C_2 is a constant which depends only on the dimension. Now noting that $p(L) \geqq p$ for $L \leqq L_2$, we can repeat the arguments in the previous section to get

$$\tau_p(0, x) \leqq 4M(p(L))^2 \quad \text{for } x \text{ with } |x| = L \ ,$$

which leads us to the desired bound (5.1) when summed over L. □

By a slight modification of the above proof, we can also show the following result for the behaviour of the mean cluster size in the low density region.

Proposition 5.2. *For arbitrary $p < p_c$ with $\xi(p) \geqq C_2$, we have*

$$\chi(p) \leqq C_3 \sum_{L = \{(d/2)\xi(p)|\ln \xi(p)|\}^{1/d}}^{2d\xi(p)|\ln \xi(p)|} L^{d-1} M(p(L))^2 \ , \tag{5.3}$$

provided that $\xi(p) \nearrow \infty$ as $p \searrow p_c$.

Note that $\xi(p) \nearrow \infty$ as $p \nearrow p_c$ is known rigorously [1, 4].

As before we get the following critical exponent inequality from (5.3):

$$d\nu \geqq \gamma + 2\beta(\nu/\nu') \ . \tag{5.4}$$

If $\nu/\nu' \geqq 1$, (5.4) implies $d\nu \geqq \gamma + 2\beta$. On the other hand if $\nu'/\nu \geqq 1$, we multiply (5.4) by ν'/ν to find $d\nu' \geqq \gamma(\nu'/\nu) + 2\beta \geqq \gamma + 2\beta$. Therefore we get

Corollary 5.2. *Whenever $\xi(p) \nearrow \infty$ as $p \searrow p_c$, the critical exponents $\nu_{\max} = \max(\nu, \nu')$, γ and β satisfy*

$$d\nu_{\max} \geqq \gamma + 2\beta \ . \tag{5.5}$$

6. Inequalities for Gap Exponents

In the present section, we prove the inequalities (1.4), (1.5) for the gap exponents Δ_n and Δ'_n.

First let us state a simple inequality which will be used in the following proofs. (The inequality was also noted by Nguyen [4].)

Lemma 6.1. *For arbitrary p and $n \geqq 3$, we have*

$$\begin{aligned} &\langle |C(0)|^{n-1} X(|C(0)| < \infty)\rangle_p / \langle |C(0)|^{n-2} X(|C(0)| < \infty)\rangle_p \\ &\leqq \langle |C(0)|^{n} X(|C(0)| < \infty)\rangle_p / \langle |C(0)|^{n-1} X(|C(0)| < \infty)\rangle_p \ . \end{aligned} \tag{6.1}$$

Proof. By the Schwarz inequality we get

$$\begin{aligned} \langle |C(0)|^{n-1} X(|C(0)| < \infty)\rangle_p &= \langle |C(0)|^{n/2} |C(0)|^{(n-2)/2} X(|C(0)| < \infty)\rangle_p \\ &\leqq (\langle |C(0)|^{n} X(|C(0)| < \infty)\rangle_p \langle |C(0)|^{n-2} X(|C(0)| < \infty)\rangle_p)^{1/2} \ , \end{aligned}$$

which is nothing but (6.1). □

Remark. Note that if we *assume* the power law behaviors

$$\langle|C(0)|^{n-1}\rangle_p/\langle|C(0)|^{n-2}\rangle_p \sim (p_c-p)^{-\Delta_n} \quad \text{as} \quad p \nearrow p_c ,$$

$$\langle|C(0)|^{n-1}X(|C(0)|<\infty)\rangle_p/\langle|C(0)|^{n-2}X(|C(0)|<\infty)\rangle_p \sim (p-p_c)^{-\Delta'_n} \quad \text{as} \quad p \searrow p_c ,$$

the inequality (6.1) implies the critical exponent inequalities

$$\Delta_n \leqq \Delta_{n+1} , \quad \Delta'_n \leqq \Delta'_{n+1} . \tag{6.2}$$

These inequalities become equalities under the scaling hypothesis. Therefore they are believed to be sharp in any dimensions.

The following inequality (6.3) is essentially a simple consequence of the van den Berg, Kesten inequality [14]. But it leads us to our first hyperscaling inequality for the gap exponents.

Proposition 6.1. *For arbitrary $n \geqq 3$ and $p < p_c$ with $\xi(p) \geqq C_4$, we have*

$$(\langle|C(0)|^{n-1}\rangle_p/\langle|C(0)|^{n-2}\rangle_p)^2 \leqq C_5(\xi(p)|\ln \xi(p)|)^d \chi(p) , \tag{6.3}$$

where C_4 and C_5 are constants which depend only on the dimension.

Since we know rigorously that $\xi(p)$ diverges when p approaches p_c from below, this leads us to

Corollary 6.1. *The critical exponents ν, Δ_n $(n \geqq 3)$ and $\bar{\gamma}$ satisfy*

$$d\nu \geqq 2\Delta_n - \bar{\gamma} . \tag{6.4}$$

In order to prove the proposition, we have to state a simple geometric lemma. Let $\{x_1, \dots, x_n\}$ be an arbitrary set of sites (which need not to be distinct). For a configuration (i.e., occupation status of all the bonds) which satisfies $C(x_1) \ni x_2, \dots, x_n$, a pair of sites $\{y, y'\} \subset \{x_1, \dots, x_n\}$ is called a *separable pair* if there exists a path of occupied bonds ω which i) connects y and y', and ii) all the sites in $\{x_1, \dots, x_n\} \setminus \{y, y'\}$ remain connected with each other when we remove all the bonds on ω (see Fig. 3).

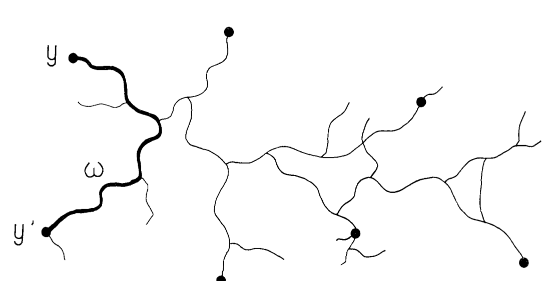

Fig. 3. A separabel pair $\{y, y'\}$ and a path ω

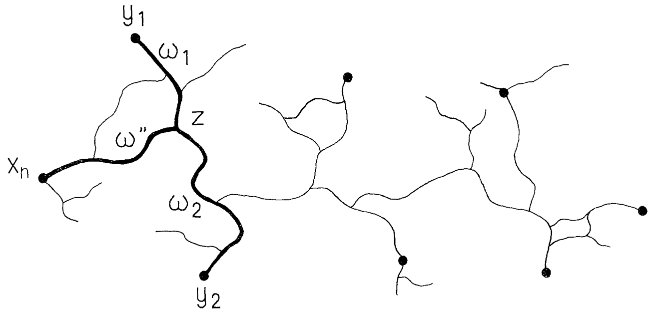

Fig. 4 $\{x_n, y_1\}$ becomes a new separable pair

Lemma 6.2. *For an arbitrary set of sites $\{x_1, \ldots, x_n\}$ and an arbitrary configuration where $C(x_1) \ni x_2, \ldots, x_n$, there exists at least one separable pair $\{y, y'\} \subset \{x_1, \ldots, x_n\}$.*

Proof. Since the statement is trivial when $n=2$, we proceed by induction. Let $n>2$. First, by omitting the site x_n, we may apply the lemma for $n-1$ to find a separable pair $\{y_1, y_2\} \subset \{x_1, \ldots, x_{n-1}\}$ and a path ω' which connects y_1 and y_2. When we remove all the bonds on ω' from the configuration, the site x_n may be i) still connected to $\{x_1, \ldots, x_{n-1}\} \backslash \{y_1, y_2\}$ or ii) disconnected from $\{x_1, \ldots, x_{n-1}\} \backslash \{y_1, y_2\}$. If i) is the case, we are done by taking $\{y, y'\} = \{y_1, y_2\}$. Let us assume ii). Then we can find a path of occupied bonds ω'' (with $\omega' \cap \omega'' = \phi$) which connects x_n to a site z in ω'. Decompose ω' as $\omega_1 \cup \omega_2$ by cutting it at z. (We assume $\omega_i \ni y_i$.) Observe that the set $\{x_1, \ldots, x_{n-1}\} \backslash \{y_1, y_2\}$ must be connected to at least one of ω_1 or ω_2 (which we call ω_i) without using the bonds in ω''. (If this is not the case, it contradicts with our assumption ii) (see Fig. 4). Then $\{y, y'\} = \{x_n, y_j\}$ $(i \neq j)$ is a separable pair $\{x_1, \ldots, x_n\}$ with the corresponding path $\omega = \omega'' \cup \omega_j$. $\square$

Proof of Proposition. From the expression $|C(0)| = \sum_x X(C(0) \ni x)$ and the translation invariance, we have

$$\langle |C(0)|^n \rangle_p = \sum_{x_2, \ldots, x_{n+1}} \tau_p(x_1, x_2, \ldots, x_{n+1}) \ .$$

Let $D_{\max}$ be an abbreviation for $\max\{|x_2 - x_1|, |x_3 - x_1|, |x_4 - x_1|, \ldots, |x_{n+1} - x_1|\}$. Then, since $\tau_p(x_1, \ldots, x_{n+1}) \leqq \tau_p(x_1, x_i)$ for any i, we get the following finite volume estimate for $\langle |C(0)|^n \rangle_p$.

$$\begin{aligned} &\langle |C(0)|^n \rangle_p - \sum_{\substack{x_2, \ldots, x_{n+1} \\ D_{\max} \leqq C_6 \xi(p) |\ln \xi(p)|}} \tau_p(x_1, x_2, \ldots, x_{n+1}) \\ &\leqq \sum_{L = C_6 \xi(p) |\ln \xi(p)|}^{\infty} \sum_{\substack{x_2, \ldots, x_{n+1} \\ D_{\max} = L}} \tau_p(x_1, \ldots, x_{n+1}) \\ &\leqq \sum_{L = C_6 \xi(p) |\ln \xi(p)|}^{\infty} n(2L)^{nd-1} e^{-L/\xi(p)} \leqq \xi(p)^{-a'} \ . \end{aligned}$$

Here $a'>0$, and the final inequality is valid when $\xi(p)\geqq C''$ with sufficiently large C''. Now by applying Lemma 6.2, we immediately get

$$\sum_{x_2,\ldots,x_{n+1},D_{\max}\leqq C_6\xi(p)|\ln\xi(p)|} \tau_p(x_1,\ldots,x_{n+1})$$
$$\leqq \sum_{\substack{\text{pairs}\\ \{i,j\}\subset\{1,\ldots,n+1\}}} \sum_{x_2,\ldots,x_{n+1},D_{\max}\leqq C_6\xi(p)|\ln\xi(p)|}$$
$$\mathrm{Prob}_p\{(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{n+1} \text{ are connected})\circ(x_i,x_j \text{ are connected})\} ,$$

where $\hat{x}_i$ denote omitted sites. Here $A\circ B$ stands for the event that two events A and B occur *disjointly* [14]. For positive events A, B, van den Berg and Kesten [14] have proved that $\mathrm{Prob}_p(A\circ B)\leqq \mathrm{Prob}_p(A)\,\mathrm{Prob}_p(B)$. Therefore the right-hand side of the above inequality is bounded as

$$\leqq \sum_{\text{pairs}\{i,j\}\subset\{1,\ldots,n+1\}} \sum_{\substack{x_2,\ldots,x_{n+1}\\ D_{\max}\leqq C_6\xi(p)|\ln\xi(p)|}} \tau_p(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{n+1})\tau_p(x_i,x_j)$$
$$\leqq\{n(n+1)/2\}\langle|C(0)|^{n-2}\rangle_p\chi(p)(2C_6\xi(p)|\ln\xi(p)|)^d .$$

Combining this result with (6.1), we get the desired inequality (6.3). □

Finally we state inequalities which follow from a combination of the methods in the present and previous sections.

Proposition 6.2. *For arbitrary $n\geqq 3$ and $p>p_c$ with $\xi(p)\geqq C_7$, we have*

$$(\langle|C(0)|^{n-1}X(|C(0)|<\infty)\rangle_p/\langle|C(0)|^{n-2}X(|C(0)|<\infty)\rangle_p)^2$$
$$\leqq C_8 \sum_{L=\{(C_6/2)\xi(p)|\ln\xi(p)|\}^{1/d}}^{C_6\xi(p)|\ln\xi(p)|} L^{d-1}M(p'(L))^2 , \tag{6.5}$$

where C_6, C_7 and C_8 are constants which depend only on the dimension. Also assume that $\xi(p)\nearrow\infty$ as $p\searrow p_c$. Then for arbitrary $n\geqq 3$ and $p<p_c$ with $\xi(p)\geqq C_7$, we have

$$(\langle|C(0)|^{n-1}\rangle_p/\langle|C(0)|^{n-2}\rangle_p)^2\leqq C_8 \sum_{L=\{(C_6/2)\xi(p)|\ln\xi(p)|\}^{1/d}}^{C_6\xi(p)|\ln\xi(p)|} L^{d-1}M(p'(L))^2 . \tag{6.6}$$

Here $p'(L)$ is defined by $p'(L)=\inf\{p|C_6\xi(p)|\ln\xi(p)|\leqq L\}$.

As in the previous section, these inequalities imply the following critical exponent inequalities.

Corollary 6.2. *Whenever $\xi(p)\nearrow\infty$ as $p\searrow p_c$, the critical exponents ν', $\nu_{\max}=\max\{\nu,\nu'\}$, Δ_n, Δ_n' $(n\geqq 3)$ and β satisfy*

$$d\nu'\geqq\Delta_n'+\beta , \tag{6.7}$$

$$d\nu_{\max}\geqq\Delta_n+\beta . \tag{6.8}$$

Proof of Proposition. The proof is almost a repetition of those of Propositions 6.1, 5.1, and 5.2. The only essential difference comes in when we bound the $n+1$ point connectivity function by the product of $n-1$ point and two point connectivity functions. (See the proof of Proposition 6.1.) The estimate used here is

$$\tau'_p(x_1,\ldots,x_{n+1}) \leqq \sum_{\substack{\text{pairs}\\ \{i,j\}\subset\{1,\ldots,n+1\}}} \mathrm{Prob}_p\{(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{n+1} \text{ are connected by a finite cluster}) \circ (x_i, x_j \text{ are connected})\}$$

$$\leqq \sum \tau'_p(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{n+1})\tau_p(x_i,x_j)\ ,$$

where the final inequality follows from the van den Berg, Fiebig inequality [15]. □

Remark. Note that, in the above proof, we cannot replace the upper bound for $\tau'_p(x_1,\ldots,x_{n+1})$ by

$$\mathrm{Prob}_p\{(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{n+1} \text{ are connected by a finite cluster}) \circ (x_i, x_j \text{ are connected by a finite cluster})\}\ .$$

This is the reason that we are not able to prove the inequality corresponding to (6.4) for the exponents Δ'_n.

7. Inequalities for Critical Isotherm Exponents

In the present section, we briefly describe our final inequalities (1.6) and (1.7) for the critical isotherm exponents.

First we define the percolation system under *positive external field* $h \geqq 0$. Let us add a "ghost site" g to our lattice Z^d. We assume that each ghost bond $\{g, x\}, x\in Z^d$ is occupied independently with probability $1-e^{-h}$. Then various physical quantities can be defined by regarding the "ghost site" g as "infinity", and replacing the condition $|C(x)|=\infty$ in (2.1)–(2.6) by the new condition $C(x)\ni g$.

When we fix p at its critical value p_c and let h approach zero, various critical phenomena are expected to take place. Let us define some critical exponents as the optimal constants satisfying the following relations when $h\searrow 0$.

$$\xi(p_c,h) \lesssim h^{-\mu}\ ,$$

$$M(p_c,h) \lesssim h^{1/\delta_1}\ ,$$

$$\partial^{n-1}M(p_c,h)/\partial h^{n-1} = \langle |C(0)|^{n-1}X(C(0)\ni g)\rangle_{p,h} \gtrsim h^{-1/\delta_n+1-n}\ , \qquad (n=2,3,\ldots)\ .$$

Then, by a straightforward modification of the methods in the previous sections, we can easily prove the following.

Proposition 7.1. *Whenever* $\xi(p_c,h)\nearrow\infty$ *as* $h\searrow 0$, *the critical exponents* η, μ, *and* δ_i $(i=1,2,4,6,8,\ldots)$ *satisfy*

$$(d-2+\eta)\mu\delta_1 \geqq 2\ , \tag{7.1}$$

$$d\mu \geqq 1 + (n/\delta_1 - 1/\delta_n)/(n-1) \quad \textit{for} \quad n=2,4,6,8,10,\ldots\ . \tag{7.2}$$

If $M(p_c, h)$ exhibits the following simple power law behavior (as is expected):

$$M(p_c, h) \sim h^{1/\delta} ,$$

we have $\delta_i = \delta$ for any i. Then (7.2) reduces to a single inequality

$$d\mu \geqq 1 + (1/\delta) . \tag{7.3}$$

Appendix. Extension to Ising Ferromagnets

In this appendix, we briefly describe the extension of our methods to the Ising ferromagnets.

First let us define Ising model. For an arbitrary positive integer L, let S_L and ∂S_L be

$$S_L = \{x \in Z^d | \, |x| \leqq [L/2]\}$$

and

$$\partial S_L = \{x \in S_L | \, |x| = [L/2]\} .$$

To each site x in S_L, we associate a spin variable $\sigma_x = \pm 1$. Then the *thermal expectation* with plus boundary condition is defined by

$$\begin{aligned} \langle \ldots \rangle_L &= Z_L^{-1} \sum_{\sigma_x = \pm 1 \, (x \in S_L \setminus \partial S_L)} (\ldots) \exp(-H) , \\ H &= -(\beta/2) \sum_{x, y \in S_L, \, \|x-y\|_2 = 1} \sigma_x \sigma_y - h \sum_{x \in S_L} \sigma_x , \\ \langle 1 \rangle_L &= 1 , \qquad \sigma_x = 1 \quad \text{if} \quad x \in \partial S_L . \end{aligned} \tag{A.1}$$

We consider the expectation in the *infinite volume limit* defined by

$$\langle \ldots \rangle = \lim_{L \nearrow \infty} \langle \ldots \rangle_L . \tag{A.2}$$

We define various physical quantities and critical exponents by simply replacing p by β, $\tau_p(x, y)$ by $\langle \sigma_x \sigma_y \rangle$, $\tau'_p(x, y)$ by $\langle \sigma_x ; \sigma_y \rangle = \langle \sigma_x \sigma_y \rangle - \langle \sigma_x \rangle \langle \sigma_y \rangle$, $M(p)$ by $\langle \sigma_0 \rangle$, and $M(p; L)$ by $\langle \sigma_0 \rangle_L$ in the definitions in Sects. 2 and 7. Thanks to the reflection positivity (which is not known for percolation), the correlation length defined as

$$\xi(p)^{-1} = - \lim_{|x| \to \infty} \ln \langle \sigma_0 ; \sigma_x \rangle / |x|$$

(limit exists) coincides with that defined by (2.4).

Then by a straightforward extension of the methods described in the text, we can prove

Proposition A.1. *The critical exponents of the d-dimensional Ising model satisfy*

$$(d - 2 + \eta) \delta_r \geqq 2 , \tag{A.3}$$

$$(d - 2 + \eta) \nu' \geqq 2\beta , \tag{A.4}$$

$$dv' \geqq \gamma' + 2\beta \ , \qquad dv_{\max} \geqq \gamma + 2\beta \ , \tag{A.5}$$

$$(d-2+\eta)\mu\delta_1 \geqq 2 \ , \tag{A.6}$$

$$d\mu \geqq 1 + (2/\delta_1) - (1/\delta_2) \ , \tag{A.7}$$

whenever $\xi(\beta, 0) \nearrow \infty$ *as* $\beta \searrow \beta_c$ [(A.4), (A.5)] *or* $\xi(\beta_c, h) \nearrow \infty$ *as* $h \searrow 0$[(A.6), (A.7)].

As we have mentioned in Sect. 1, most of these inequalities have already been proved by other methods. Our rigorous finite size scaling argument provides a new unified derivation. In particular our proof of the inequality $dv' \geqq \gamma' + 2\beta$ seems to be simpler than Sokal's highly technical proof [9].

Let us describe how two key arguments in our proof can be stated in the Ising model. Then the rest of the proofs will be just repetitions of those in the main text.

First we discuss the extension of Proposition 4.1 which was the main ingredient of most of our results. Let $L > 0$ and $|x| = L$. Define $\langle \ldots \rangle_{\tilde{h}}$ by the formula (A.1) with Hamiltonian H replaced by $H - \tilde{h} \sum_{x \in B} \sigma_x$, where $B = \{y \mid |y| = [L/2]$ or $|x-y| = [L/2]\}$. Note that when $\tilde{h} = \infty$, the whole system decouples into two finite systems in $L \times \ldots \times L$ cubes and one infinite system. Therefore by the Griffiths II inequality [16], we have

$$\langle \sigma_0 \sigma_x \rangle = \langle \sigma_0 \sigma_x \rangle_{\tilde{h}=0} \leqq \langle \sigma_0 \sigma_x \rangle_{\tilde{h}=\infty} = (\langle \sigma_0 \rangle_L)^2 \ ,$$

which is nothing other than the desired Ising model version of the inequlaity (4.1).

Next we describe the Ising model counterpart of Lemma 4.1. Let $\Theta_L(\beta, h)$ be

$$\Theta_L(\beta, h) = \sum_{y \in \partial S_L} \langle \sigma_0 \, ; \sigma_x \rangle \ .$$

Then we have the following.

Lemma A.1. *For arbitrary* β *and* h, *we have*

$$\langle \sigma_0 \rangle \leqq \langle \sigma_0 \rangle_L \leqq \langle \sigma_0 \rangle + \Theta_L(\beta, h) \{ |\ln \Theta_L(\beta, h)|/2 + 3dL^{d-1} e^{4d\beta} \} \ . \tag{A.8}$$

Now a rigorous finite size scaling argument corresponding to Lemma 4.1 can be proved very easily from the above inequalities. If we set $L = \text{const}\, \xi(\beta, h) |\ln \xi(\beta, h)|$, the upper bound in (A.8) reduces to

$$\langle \sigma_0 \rangle_L \leqq \langle \sigma_0 \rangle + \text{const}\, \xi(\beta, h)^{-a} \tag{A.9}$$

for sufficiently large $\xi(\beta, h)$. Since the constant a can be made arbitrarily large by choosing suitable constants, (A.9) is sufficient for carrying out our proofs of the critical exponent inequalities.

Remark. Inequality (A.8) also has a consequence on the problem of continuity of the magnetization [17]. By using the fact that $\langle \sigma_0 \rangle_L$ is a continuous function, we can show that $\langle \sigma_0 \rangle$ is continuous in β at $\beta = \beta_0$, $h = 0$ if $\lim_{L \nearrow \infty} \lim_{\beta \nearrow \beta_0} L^{d-1} \Theta_L(\beta, h) = 0$.

Proof of Lemma. The first inequality is a simple consequence of Griffiths II inequality. To prove the second inequality, note that

$$\langle\sigma_0\rangle_L-\langle\sigma_0\rangle=\int_0^\infty d\tilde{h}\,\partial\langle\sigma_0\rangle'_{\tilde{h}}/\partial\tilde{h}=\int_0^\infty d\tilde{h}\sum_{x\in\partial S_L}\langle\sigma_0\,;\sigma_x\rangle'_{\tilde{h}}$$
$$=\int_0^{h_0} d\tilde{h}\sum_{x\in\partial S_L}\langle\sigma_0\,;\sigma_x\rangle'_{\tilde{h}}+\int_{h_0}^\infty d\tilde{h}\sum_{x\in\partial S_L}\langle\sigma_0\,;\sigma_x\rangle'_{\tilde{h}}\,,$$

where $\langle\ldots\rangle'_{\tilde{h}}$ is defined by the formula (A.1) with Hamiltonian H replaced by $H-\tilde{h}\sum_{x\in\partial S}\sigma_x$. h_0 is a constant which will be determined later. By the GHS inequality [18], the small field part in the above bound can be bounded as

$$\int_0^{h_0} d\tilde{h}\sum_{x\in\partial S_L}\langle\sigma_0\,;\sigma_x\rangle'_{\tilde{h}}\leqq\int_0^{h_0} d\tilde{h}\sum_{x\in\partial S_L}\langle\sigma_0\,;\sigma_x\rangle=h_0\Theta(\beta,h)\ .$$

In order to bound the large field part, we simply decouple a site x in ∂S_L from the rest of the lattice by carefully bounding the local Boltzmann factor. Then tedious but elementary estimates show

$$\langle\sigma_0\,;\sigma_x\rangle'_{\tilde{h}}\leqq 3e^{-2\tilde{h}+4d\beta}\ .$$

Combining these results, we get

$$\int_0^\infty d\tilde{h}\sum_{x\in\partial S_L}\langle\sigma_0\,;\sigma_x\rangle'_{\tilde{h}}\leqq h_0\Theta(\beta,h)+2dL^{d-1}(3/2)e^{-2h_0+4d\beta}\ .$$

Setting $h_0=|\ln\Theta(\beta,h)|/2$, we get the desired inequality (A.8). □

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