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Asymptotic Inverse Spectral Problem for Anharmonic Oscillators

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Dedicated to V. P. Gurarii on his 50^{th} birthday

Abstract. We study perturbations $L=A+B$ of the harmonic oscillator $A=\frac{1}{2}(-\partial^2+x^2-1)$ on $\mathbb{R}$, when potential $B(x)$ has a prescribed asymptotics at ∞, $B(x)\sim|x|^{-\alpha}V(x)$ with a trigonometric even function $V(x)=\sum a_m \cos\omega_m x$. The eigenvalues of L are shown to be $\lambda_k=k+\mu_k$ with small $\mu_k=O(k^{-\gamma})$, $\gamma=1/2+1/4$.

The main result of the paper is an asymptotic formula for spectral fluctuations $\{\mu_k\}$,

$$\mu_k \sim k^{-\gamma}\tilde{V}(\sqrt{2k})+c/\sqrt{2k} \quad \text{as} \quad k\to\infty\,,$$

whose leading term $\tilde{V}$ represents the so-called "Radon transform" of V,

$$\tilde{V}(x)=\text{const}\sum\frac{a_m}{\sqrt{\omega_m}}\cos(\omega_m x-\pi/4)\,.$$

as a consequence we are able to solve explicitly the inverse spectral problem, i.e., recover asymptotic part $|x|^{-\alpha}V(x)$ of B from asymptotics of $\{\mu_k\}_1^\infty$.

The standard spectral problem for a perturbation $L=A+B$ of a differential operator A with the given spectrum $\{\lambda_k(A)\}_1^\infty$ asks to (approximately) calculate the eigenvalues of L in terms of $\{\lambda_k(A)\}$ and the perturbation. For a "relatively small" perturbation B, the k^{th} eigenvalue of L is

$$\lambda_k(L)=\lambda_k(A)+\mu_k\,,$$

so one is asked to calculate spectral fluctuations $\{\mu_k\}_1^\infty$. The corresponding inverse problem is then to recover $B(x)$ from the given (admissible) sequence of eigenvalues $\{\lambda_k(\mathrm{L})\}_1^\infty$ or fluctuations $\{\mu_k\}_1^\infty$.

Spectral problems were extensively studied in various contexts for both ordinary and partial differential operators. The best known example is the regular Sturm-Liouville problem: $L=\dfrac{d^2}{dx^2}+V(x)$ on $[0,1]$. The old result of Borg [Bo] gives the following asymptotics of λ_k,

$$\lambda_k \sim (\pi k)^2 + \int_0^1 V dx + V_{2k} + O\left(\frac{1}{k}\right), \tag{1}$$

where V_{2k} is the $2k^{\text{th}}$ Fourier coefficient of V.

Of course (1) by itself does not provide sufficient data for the inverse problem. The latter was resolved in [Bo] in the "two-spectra" setting: $\{\lambda_k\}_1^\infty$; $\{\lambda'_k\}_1^\infty$, which corresponds to two different values of the boundary-value parameter h: $u + hu'|_{x=0} = 0$.

It turned out that typically the inverse Sturm-Liouville problem does not have a unique solution. Large isospectral families of potentials V exist both in the periodic (Floquet) case [La, MM], where they are given by nonlinear KdV-type evolutions; and also for two-point boundary-value problems [IMT], where the isospectral families are characterized in terms of certain norming constants.

Turning to multivariable problems, i.e., Schrödinger operators $-\Delta + V(x)$, we shall mention two known examples: the "n-torus" [ERT] and "n-sphere" theory [We, Gu, Ur, Wi].

In both cases there exist natural "isospectral deformations" of V arising from symmetries of the problem: rotations on S_n, translations and reflections of T^n. Those were conjectured by Guillemin to be the only isospectral families, so-called "Spectral rigidity problem." This conjecture was proven for "generic potentials" on T^n [ERT] and for some classes of spherical harmonics on S_n [Gu].

The method of [Gu] was based on two sets of spectral invariants: the classical "heat-invariants" of Munakshisundaram-Pleijel $\{b_m(V)\}_{m=0}^\infty$, obtained by expanding the "heat-kernel" $\operatorname{tr}(e^{-tL})$ in powers of t, and a new class of spectral invariants, so called *band-invariant*, introduced Kac-Spencer and Weinstein [We].

Let us briefly outline some basic concepts and results of the S_n-theory.

The spectrum of the Laplacian $-\Delta$ on S_n is well known: the k^{th} eigenvalue $\lambda_k = k(k+n-1)$ has multiplicity $d_k = O(k^{n-1})$; $k = 0, 1, \ldots$. Introducing perturbation $V(x)$ destroys the underlying rotational symmetry. So each multiple eigenvalue λ_k splits into the cluster $\{\lambda_{kj} = \lambda_k + \mu_{kj}\}_{j=1}^{d_k}$ of simple eigenvalues of $L = -\Delta + V(x)$, whose distribution is described by the probability measure

$$d\varrho_k = \frac{1}{d_k} \sum_j \delta(t - \mu_{kj}). \tag{2}$$

As the size of the k^{th} cluster increases, one is interested in the asymptotic behavior of $\{d\varrho_k\}$ as $k \to \infty$.

It turned out [We], that the sequence $\{d\varrho_k\}_1^\infty$ converges to a continuous measure $\beta_0 dt$ on $\mathbb{R}$ expressed in terms of the Radon transform $\tilde{V}$ of V,

$$\langle f; \beta_0 \rangle = \iint_{S^*(S_n)} f \circ \tilde{V},$$

integration over the cosphere bundle of S_n.

Moreover, the following asymptotic expansion analogous to Borg's (1) was derived by Weinstein,

$$d\varrho_k \sim \beta_0 + \frac{1}{k}\beta_1 + \frac{1}{k^2}\beta_2 + \ldots. \tag{3}$$

Distributions $\{\beta_0; \beta_1, \ldots\}$ form a new class of spectral invariants, called band-invariants in [We]. The 0-th invariant β_0 has an immediate implication to the inverse problem. Namely, the Radon transforms of two isospectral potentials V_1; V_2 are equally distributed on $S^*(S_n)$. If we know the Radon transform $\tilde{V}$ itself, (rather than its distribution) we could easily solve the inverse problem. But a multivariable function cannot be recovered from its distribution, so β_0 by itself is not enough (cf. [Gu]).

The situation becomes different, however, in the context of the present work, namely perturbations $L=A+B(x)$ of the quantum mechanical harmonic oscillator $A=\frac{1}{2}(-\delta^2+x^2-1)$ on $\mathbb{R}$. We loosely call such operators anharmonic oscillators.

The harmonic oscillator is one of few examples (along with Laplacians on T^n or S_n) whose spectrum can be explicitly calculated: the k^{th} eigenvalue $\lambda_k=k$ and the eigenfunction $\varphi_k=k^{\text{th}}$ Hermite function $e^{x^2/2}\partial^k(e^{-x^2})$.

Our purpose in the present work is to establish the analogue of Borg's formula (1) for operators L and relate its first term to the so-called "Radon transform" of the perturbation. This relation (Theorem 1) enables us to efficiently solve the inverse problems for $A+B$ in the asymptotic context, namely, to link directly "large x-asymptotics of $B(x)$" on one hand and "large k-asymptotics of spectral fluctuations μ_k" on the other.

According to our basic "asymptotic" philosophy we shall consider a class of perturbations described by their behavior at ∞,

$$B(x) \sim |x|^{-\alpha} V(x) \quad \text{as} \quad x \to \infty\,, \tag{4}$$

with a trigonometric even function

$$V(x) = \sum a_m \cos \omega_m x\,. \tag{5}$$

So the input data for the direct problem consists of an exponential α in the algebraic factor as well as frequencies $\{\omega_m\}$ and Fourier coefficients $\{a_m\}$ of the trigonometric part. The inverse problem is then to recover this data (or a portion of it) from asymptotics of spectral fluctuations $\{\mu_k\}_1^\infty$. One can show that perturbation B (4) is small (compact) relative to A ([RS]), so the k^{th} eigenvalue of $L=A+B$, is

$$\lambda_k = k + \mu_k \quad \text{with "small" } \mu_k\,.$$

Our main result is the following

Theorem 1. *Let $L=A+B$ be the anharmonic oscillator with an even potential $B(x)\sim|x|^{-\alpha}V(x)$ whose trigonometric part $V(x)=\sum a_m\cos\omega_m x$. The k^{th} spectral fluctuation of L is asymptotic to*

$$\mu_k \sim k^{-\gamma}\tilde{V}(\sqrt{2k}) + \frac{c_\alpha}{\sqrt{2k}} \quad as \quad k\to\infty\,, \tag{6}$$

where constants $\gamma=\alpha/2+1/4$,

$$c_\alpha = \cos\left[\tfrac{\pi}{2}(1-\alpha)\right]\Gamma(1-\alpha)\sum a_m \omega_m^{\alpha-1}\,,$$

and $\tilde{V}$ denotes the "Radon transform" of the trigonometric part:

$$V \to \tilde{V}(x) = \sqrt{2/\pi} \sum \frac{a_m}{\sqrt{\omega_m}} \cos(\omega_m x - \pi/4). \tag{7}$$

Notice that (7) corresponds to a formal fractional derivative operation applied to V

$$\tilde{V} = \frac{1}{\sqrt{\pi}} |\partial|^{-1/2}(I - \partial/|\partial|)[V].$$

We call it "Radon transform" for no other reason than it formally resembles the Radon transform on S_n [We] and plays a similar role in our discussion.

Formula (8) gives the leading coefficient $b_0(k) = \tilde{V}(\sqrt{2k})$ is the asymptotic expansion of μ_k $\left(\text{or } \mu_k - \frac{c}{\sqrt{k}}\right)$, namely

$$\mu_k \sim k^{-\gamma}\left(b_0 + \frac{1}{k} b_1 + \dots\right);. \tag{8}$$

Other coefficients $b_1, b_2; \dots$ could also be calculated, (cf. [Ur]) but we shall not pursue it here.

Expansion (8) is analogous to Borg's (1) or Weinstein (3). One notable difference however is the k-dependence (oscillatory behavior) of coefficients $b_0; \dots$ determined by function $V(x)$.

The latter provides a crucial link to the inverse problem. Precisely, given an admissible sequence of fluctuations $\mu_k \sim k^{-\gamma} F(\sqrt{k})$ with a trigonometric function $F(x)$, we proceed in three steps.

1) Exponential $\gamma = \frac{\alpha}{2} + \frac{1}{4}$ (consequently $\alpha = 2\gamma - 1/2$) can be found as an upper bound

$$\gamma = \sup\left\{p > 0\colon \lim_{k \to \infty} k^p \mu_k = 0\right\}. \tag{9}$$

2) Assuming periodicity (or quasiperiodicity) of $F(x)$ we can reconstruct the period T (or quasiperiods $T_1 \dots T_n$). Notice that the sequence $\{\sqrt{k}\}$ is dense (and uniformly distributed [KN]) modulo any $T > 0$ or a tuple $T_1; \dots T_n$.

So any continuous function $F(x)$ on the torus $T = [0, T]$ or $T^n = [0, T_1] \times \dots \times [0, T_n]$ is uniquely determined by its values at $\{\sqrt{k}\}$. Moreover, the oscillation of F on any subsequence

$$N(x, \varepsilon) = \{j\colon |x - \sqrt{j}| < \varepsilon \operatorname{mod}(T_1; \dots T_n)\}$$

has to diminish as $\varepsilon \to 0$, i.e.,

$$O(x, \varepsilon) = \sup\{|i^\gamma \mu_i - j^\gamma \mu_j|\colon \text{all pairs } i, j \in N(x, \varepsilon)\} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thus the effective reconstruction procedure will screen all values of T (or a tuple $T_1 \dots T_n$) at which the oscillation $O(x; \varepsilon)$, as a function of parameters $T(T_1 \dots T_n)$, drops down as illustrated in Fig. 1.

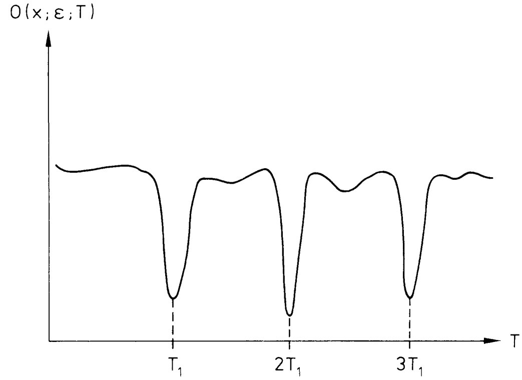

Fig. 1. Oscillation as a function of T in the periodic case

3) Once the periods $\{T_1; T_2; \ldots\}$ are found (or prescribed) we can reconstruct a trigonometric function $F(x)$, i.e., its frequencies and Fourier coefficients.

Namely, we write

$$F(x) = \sum b_m \cos(\omega_m x - \pi/4) = \sum \left(\frac{b_m}{\sqrt{2}} \cos \omega_m x + \frac{b_m}{\sqrt{2}} \sin \omega_m x \right),$$

m being a tuple of integers $(m_1, \ldots, m_n \ldots)$ and ω_m meaning the corresponding frequency

$$\omega_m = 2\pi \sum_i \frac{m_i}{T_i}. \tag{10}$$

From the uniform distribution property of $\{\sqrt{k}; \operatorname{mod}(T_1; \ldots T_n \ldots)\}$ we recover the m^{th} coefficient as

$$b_m = \sqrt{2} \lim_{k \to \infty} \frac{1}{k} \sum_1^k j^\gamma \mu_j \cos(\omega_m \sqrt{j}). \tag{11}$$

Let us remark that $\cos \omega_m x$ could be replaced with $\sin \omega_m x$ in (11), as sin and cos Fourier coefficients of F must be equal by the definition of the Radon transform (Theorem 1) for any even perturbation B. This provides an additional compatibility condition on the admissible spectral data $\{\mu_k\}_1^\infty$.

Finally, the trigonometric part V of perturbation B is obtained by inverting the Radon transform

$$F \to V(x) = \sqrt{\tfrac{\pi}{2}} \sum b_m \sqrt{\omega_m} \cos \omega_m x,$$

where frequencies $\{\omega_m\}$ and coefficients $\{b_m\}$ are given by (10), (11).

We shall summarize the inversion procedure in the following.

Corollary. (i) *The admissible spectral data for the inverse problem consists of sequences*

$$\mu_k \sim k^{-\gamma} F(\sqrt{k})$$

with a trigonometric function $F(x)$ whose sin *and* cos *Fourier coefficients* (11) *are equal.*

(ii) *The inverse problem has a unique solution* $B(x) \sim |x|^{-\alpha} \sum a_m \cos \omega_m x$, *whose parameters:* $\alpha = 2\gamma - 1/2$; *frequencies* $\{\omega_m\}$ *and coefficients* $\{a_m = \sqrt{\omega_m} b_m\}$ *are given by* (9)–(11).

Remark 1. The above uniqueness result can be compared to [IMS] and [MT]. The first paper showed that isospectral classes of the Sturm-Liouville problem have typically a unique even representative, whereas the second derived the same result for the isospectral class of the harmonic oscillator $\frac{d^2}{dx^2} + x^2$.

Our corollary extends these results to "asymptotic" isospectral classes of the harmonic oscillator perturbed by even potentials $B(x)$.

In the rest of the paper we shall outline the proof of Theorem 1. Our argument is based on the "averaging method" of Weinstein [We], whose origins go back to the classical work on celestial mechanics. Precisely, we observe that the spectrum of the harmonic oscillator $A = \frac{1}{2}(-\partial^2 + x^2 - 1)$ consists of integers $\{0; 1; \ldots\}$ and replace perturbation B by the average of its conjugates

$$\bar{B} = \frac{1}{2\pi} \int_0^{2\pi} e^{itA} B e^{-itA} dt. \tag{12}$$

So instead of $L = A + B$ we study operator $\bar{L} = A + \bar{B}$. The main advantage of averaging is that both terms A and $\bar{B}$ now commute. So spectral fluctuations $\{\bar{\mu}_k\}$ become nothing but eigenvalues of $\bar{B}$, which greatly facilitates calculations.

In order to pass from $\{\mu_k\}$ to $\{\bar{\mu}_k\}$ we need to show that spectra of L and $\bar{L}$ become approximately equal as $k \to \infty$. The reason for the asymptotic proximity of spectra is "almost unitary" equivalence of L and $\bar{L}$. Namely,

Lemma (cf. [We]). (i) *There exists a skew symmetric operator* Q *so that*

$$e^Q (A + B) e^{-Q} = A + \bar{B} + \text{“small remainder } R\text{.”} \tag{13}$$

(ii) *The remainder* R *satisfies the following operator inequality (in the sense of comparison of selfadjoint operators)*

$$|R| = (R^* R)^{1/2} \leqq c A^{-(1/4 + \alpha)} \quad \textit{with constant} \quad c > 0. \tag{14}$$

From the lemma we immediately get an estimate of proximity of eigenvalues $\lambda_k = k + \mu_k$ and $\bar{\lambda}_k = k + \bar{\mu}_k$,

$$|k^{\alpha/2 + 1/4} (\mu_k - \bar{\mu}_k)| \leqq c k^{-\alpha/2} \to 0 \quad \text{as} \quad k \to \infty. \tag{15}$$

So large k-asymptotics of μ_k and $\bar{\mu}_k$ are equal modulo small (higher order) error

$$\mu_k - \bar{\mu}_k = O(k^{-\gamma - \alpha/2}).$$

To prove the lemma and to calculate the leading asymptotics of $\{\bar{\mu}_k\}_1^\infty$ we shall use a form of pseudodofferential calculus to be introduced now.

Symbol classes $S^m (-\infty < m < \infty)$ consist of smooth functions $\sigma(x, \xi)$ on the phase-plane $\{(x, \xi)\} = \mathbb{R}^2$, which at large radius $r = \sqrt{x^2 + \xi^2}$ admit an asymptotic expansion

$$\sigma \sim \sum_{j=0}^{\infty} a_j r^{m-j} e^{i\omega_j r}.$$

Coefficients $\{a_j\}$ and phase-factors $\{\omega_j\}$ are assumed to depend smoothly on the polar angle $\theta = \arccos x/r$. One example is the potential

$$B(x) \sim |x|^{-\alpha} V(x) = r^{-\alpha} |\cos\theta|^{-\alpha} \sum a_j e^{ir\omega_j \cos\theta}$$

which belongs to $S^{-\alpha}$.

The harmonic oscillator A has an elliptic symbol $a(r,\theta) = r^2/2$, whose fractional powers can serve to "gauge" operators of classes S^m. With each symbol $\sigma(x,\xi)$ we associate a pseudodifferential (or Fourier integral) operator $K = \sigma(x,D)$; $D = -i\partial_x$, defined by the Weyl convention

$$(Ku)(x) = \frac{1}{2\pi} \iint e^{i\xi \cdot (x-y)} \sigma\left(\frac{x+y}{2};\, \xi\right) u(y) d\xi dy .$$

Two basic results of pseudodifferential calculus will be used here.

Proposition. (i) *Operators* $B = b(x,D)$ *of order zero,* $(b \in S^0)$ *are* L^2*-bounded. Consequently* $b \in S^{-m}$ *implies*

$$\|BA^{m/2}u\| \leqq \text{const}\,\|u\|, \quad \textit{equivalently} \quad |B| = (B^*B)^{1/2} \leqq CA^{-m/2} . \tag{16}$$

(ii) *Product (composition) of two operators* $B_1 \in S^{m_1}$ *and* $B_2 \in S^{m_2}$ *has "A-order"* $= m_1 + m_2$, *in the sense of* (16), *namely*

$$|B_1 B_2| \leqq \text{const}\, A^{(m_1+m_2)/2} .$$

The first statement follows from the Calderon-Vaillancourt Theorem and the fact that our class S^0 is included in the standard class $S_{0,0} = \{b(x,\xi) : |\partial_x^\alpha \partial_\xi^\beta b| \leqq C_{\alpha\beta}$ all $\alpha,\beta\}$ (see [Ta, Chap. 13]).

The proof of the second statement is somewhat longer and will be outlined in the Appendix.

Now we proceed to the lemma.

The intertwining operator Q is constructed following [We] as

$$Q = \frac{1}{2\pi i} \int_0^{2\pi} (2\pi - t) B(t) dt , \tag{17}$$

where $B(t)$ denotes the conjugate $e^{itA} B e^{-itA}$. Both operators $\bar{B}$ (12) and Q (17) are ψDO's of classes S^m, whose principal symbols can be calculated from symbols of conjugates $\{B(t)\}$. The latter according to the so-called Egorov Theorem is obtained by composing "symbol B" with the Hamiltonian flow $\{\exp tH_a\}$ of symbol A, $a(x,\xi) = \frac{1}{2}(x^2 + \xi^2)$, i.e.,

$$\sigma_{B(t)} = \sigma_B \circ \exp tH_a ,$$

where $H_a = x\partial_\xi - \xi\partial_x$. Then

$$\sigma_{\bar{B}} = \frac{1}{2\pi} \int_0^{2\pi} \sigma_B \circ \exp tH_a dt = \frac{1}{2\pi} \int_0^{2\pi} B(x\cos t + \xi \sin t) dt ,$$

which yields in polar coordinates (r,θ)

$$\sigma_{\bar{B}} = \sigma_{\bar{B}}(r) = \frac{1}{2\pi} \int_0^{2\pi} B(r\cos t) dt .$$

Remembering that $B \sim |x|^{-\alpha} V(x)$ we can write σ_B as

$$\sigma_B = r^{-\alpha} \frac{2}{\pi} \int_0^{\pi/2} \cos^{-\alpha} t\, V(r \cos t) dt. \tag{18}$$

Given a trigonometric function $V(x) = \sum a_m \cos \omega_m x$ asymptotics of integral (18) at large r is computed by the stationary phase method. Both end-points of integral (18) contribute to its asymptotics: critical point, $t=0$, and singular point, $t=\pi/2$.

The contribution of the first ($t=0$) is

$$r^{-\alpha-1/2} \sqrt{\frac{2}{\pi}} \sum \frac{a_m}{\sqrt{\omega_m}} \cos(\omega_m r - \pi/4) = r^{-\alpha-1/2} \tilde{V}(r),$$

the trigonometric series representing the Radon transform $\tilde{V}$ of V as defined in Theorem 1.

The second point ($t=\pi/2$) contributes

$$\frac{c'_\alpha}{r} \sum a_m \omega_m^{\alpha-1} = \frac{c_\alpha}{r}, \quad \text{with constant} \quad c_\alpha = \int_0^\infty \cos z \frac{dz}{z^\alpha} = \cos\left[\frac{\pi}{2}(1-\alpha)\right] \Gamma(1-\alpha).$$

Combining two contributions we get

$$\sigma_B = r^{-\alpha-1/2} \tilde{V}(r) + \frac{c_\alpha}{r}. \tag{19}$$

Formula (19) will be essential in calculating the eigenvalues of $\bar{B}$. It also shows that the symbol of $\bar{B}$ belongs to our class $S^{-\alpha-1/2}$.

Similarly one calculates the principal symbol of the ψDO Q, (17),

$$\begin{aligned} \sigma_Q(r,\theta) &= \frac{1}{2\pi i} \int_0^{2\pi} B(r\cos(t-\theta))(2\pi - t) dt \\ &= \frac{2\pi}{i} \sigma_B - \frac{1}{2\pi i} \int_0^{2\pi} B(r\cos(t-\theta)) t\, dt, \end{aligned}$$

whose order is also $-(\alpha + 1/2)$.

To establish the intertwining relation (13) for e^Q we start with an easily verified commutation formula

$$[A; Q] = \frac{1}{2\pi} \int_0^{2\pi} (2\pi - t) i B'(t) dt = i(\bar{B} - B).$$

Application of standard algebra yields

$$A e^Q - e^Q A = i(\bar{B} - B) + \frac{i}{2} \{(\bar{B} - B) Q + Q(\bar{B} - B)\} + R_1. \tag{20}$$

The remainder $R_1 = \sum_3^\infty \frac{1}{n!} \sum_{j=0}^{n-1} Q^{n-1-j} (\bar{B} - B) Q^j$ in (20) consists of products of ψDO's, Q, and $\bar{B} - B$, which belong to our classes S^{-m}. We can apply the Proposition to calculate the "A-order" in the sense of (16) of each product $Q^{n-1-j}(\bar{B} - B) Q^j$.

Notice that the second statement of the proposition extends from two to any number of factors $B_1 B_2 \dots B_n$, so

$$|Q^{n-1-j}(\bar{B}-B)Q^j| \leqq CA^{-s}; \quad \text{with} \quad s=(n-1)(\alpha+1/2)+\alpha.$$

We continue algebraic transformations and rewrite (20) as

$$Ae^Q - e^Q A = e^Q \bar{B} - Be^Q + R. \tag{31}$$

The new remainder $R=\frac{1}{2}[\bar{B}-B;\, Q]+\dots$ has "A-order" $=2\alpha+1/2$.

Now the first statement of the proposition (Calderon-Vaillancourt) applies to show that

$$|R| \leqq cA^{-(\alpha+1/4)},$$

which proves the lemma.

The main lemma reduces the problem of asymptotics of $\{\mu_k\}$ to asymptotics of eigenvalues $\{\bar{\mu}_k\}$ of the average operator $\bar{B}$. To complete the proof of Theorem 1 it remains to observe that the principle symbol (19) of operator $\bar{B}$ is

$$\sigma_B = r^{-\alpha-1/2}\tilde{V}(r) + \frac{c_\alpha}{r}.$$

In other words operator $\bar{B}$ represents a function of the operator $\sqrt{2A}$, which is approximately

$$\bar{B} \approx (2A)^{-\gamma}\tilde{V}(\sqrt{2A}) + c_\alpha (2A)^{-1/2} = \bar{B}_0.$$

The error term $\bar{B}-\bar{B}_0$ has a lower "A-order" according to the proposition. Therefore, the k^{th} eigenvalue $\bar{\mu}_k$ of $\bar{B}$, consequently the k^{th} fluctuation μ_k of L, is approximated by

$$(2k)^{-\gamma}\tilde{V}(\sqrt{2k}) + \frac{c_\alpha}{\sqrt{2k}},$$

as was claimed in Theorem 1.

Remark 2. Asymptotic formula (6) for μ_k also yields the limiting distribution of "average" spectral fluctuations. This result was obtained in the earlier version of our work [Gur]. Namely, by analogy with (2) we introduced measures

$$d\varrho_k(t) = \frac{1}{k}\sum_1^k \delta(t-j^\gamma \mu_j), \quad \text{or} \quad \frac{1}{k}\sum_1^k \delta\left(t-j^\gamma\left(\mu_j - \frac{c_\alpha}{\sqrt{j}}\right)\right) \quad \text{if} \quad \alpha>1/2. \tag{22}$$

The weight factors $\{j^\gamma\}$ in (22) take into account the algebraic rate of decay of $\mu_k = O(k^{-\gamma})$.

Then we have proved the following

Theorem 2. *Sequence $d\varrho_k$ converges to a continuous measure $\beta_0(t)dt$ on $\mathbb{R}$ (0^{th} band invariant) whose density $\beta_0(t)$ is equal to the distribution function of the "Radon transform" $\tilde{V}$, considered on the torus T or T^n.*

Here the trigonometric part $V(x)$ *is assumed to be periodic or quasiperiodic.*

This result follows immediately from Theorem 1 and the equidistribution property [KN] of sequence $\{\sqrt{k}\}$ modulo any period T or a tuple $(T_1; \dots T_n \dots)$.

Indeed, as $k \to \infty$,

$$\langle f; d\varrho_k \rangle \approx \frac{1}{k}\sum_1^k f \circ \tilde{V}(\sqrt{2j}) \to \frac{1}{T}\int_0^T f \circ \tilde{V} dx \quad \text{or} \quad \frac{1}{T_1 \dots T_n}\int_0^{T_1} \dots \int_0^{T_n} f \circ \tilde{V},$$

But Theorem 2 can also be established directly in the following steps (see [Gur]):

(i) Replacing fluctuations $\{\mu_j\}$ of $A+B$ in (22) by eigenvalues $\{\bar{\mu}_j\}$ of $\bar{B}$ via the lemma, i.e., approximating $d\varrho_k \approx d\bar{\varrho}_k = \frac{1}{k}\sum_1^k \delta(t-j^\gamma \bar{\mu}_j)$.

(ii) Interpreting $\langle f; d\bar{\varrho}_k\rangle$ as $\frac{1}{k}$ trace $[f(A^\gamma \bar{B})|E_k]$ [or $A^\gamma(\bar{B}-c_\alpha A^{-1/2})$ in case $\alpha \geqq 1/2$]. Here operator $f(A^\gamma B)$ is restricted on the linear span E_k of the first k eigenfunctions of A (Hermite functions).

(iii) Applying the Szego limit theorem to approximate trace $[f(A^\gamma \bar{B})|E_k]$ by the phase-space integral $\iint_{\frac{1}{2}(x^2+\xi^2)\leqq k}$ symbol $f(A^\gamma \bar{B})$, and finally

(iv) Explicitly calculating the principal symbol of $\bar{B}$ as in (19).

Appendix. Proof of the Proposition

We want to show that the product of two ψDO's, B, and B', in classes S^m and $S^{m'}$ has "A-order" $=m+m'$, i.e.,

$$|BB'| \leqq cA^{1/2(m+m')}. \tag{A1}$$

The standard way to establish (A1) would be a composition (product) formula for Weyl symbols

$$\text{symbol } (BB') = b \# b' \sim \sum_{k=0}^{\infty} \frac{1}{k!} \left\langle b \left(\frac{\overleftarrow{\partial}_x \overrightarrow{\partial}_\xi - \overrightarrow{\partial}_\xi \overleftarrow{\partial}_x}{2i} \right)^k b' \right\rangle . \tag{A2}$$

Here we adopt the convention of equipping the derivative operations ∂_x, ∂_ξ with arrows that indicate which of two functions, b or b', is subjected to it. Unfortunately expansion (A2) does not apply to oscillatory symbols $b=r^m e^{ir\omega}$ of classes S^m, since differentiations ∂_x, ∂_ξ do not reduce their r-order.

The proper "composition rule" of such oscillatory symbols should involve the whole machinery of "Fourier integral operators." However, we are able to circumvent the ensuing technical difficulties by introducing a form of "abstract symbolic calculus" based on the operator A. Let us note that (A2) still makes sense (consists of decreasing-order terms) if one of two symbols is of "classical type,"

$$|\partial_x^\alpha \partial_\xi^\beta b| \leqq c_{\alpha\beta} r^{m-\alpha-\beta}; \qquad r=\sqrt{x^2+\xi^2},$$

in particular, $b=r^s=$ symbol $A^{s/2}$.

Moreover, the commutator of two operators $B \in S^m$ (of classical type) and $B' \in S^{m'}$ belongs to $S^{m+m'-1}$. Indeed, the principal symbol of $[B; B']$ is equal by (A2) to the Poisson bracket

$$\{b; b'\} = \left\langle b \left(\frac{\overleftarrow{\partial}_x \overrightarrow{\partial}_\xi - \overrightarrow{\partial}_x \overleftarrow{\partial}_\xi}{i} \right) b' \right\rangle,$$

which in polar coordinates becomes

$$\frac{1}{r}(b_r b'_\theta - b_\theta b'_r) \in S^{m+m'-1}.$$

It follows, in particular, that

$$\|A^{s-m/2+1/2}[A^{-s};B]\| \leqq C, \quad \text{for all} \quad B \quad \text{in} \quad S^m. \tag{A3}$$

Guided by (16) and (A3) we shall introduce classes $\mathscr{S}_m$ of (compact) operators on the Hilbert space $L^2(\mathbb{R})$ that satisfy

(a) $\|BA^{m/2}\| \leqq C$, consequently $|B|=(B^*B)^{1/2} \leqq CA^{-m/2}$,

(b) iterated commutators $B_k=[A^{-s_1}[A^{-s_2}\ldots[A^{-s_k}; B]\ldots]$ satisfy $\|B_kA^{-m/2-s-k/2}\| \leqq C_k$ for all $k=1,2,\ldots$, where $s=s_1+s_1+\ldots+s_k$.

Condition (b) essentially means that each commutation operation $B \to [A^{-s};B]$ sends class $\mathscr{S}_m$ into $\mathscr{S}_{m+s}$. We have already shown that class $\mathscr{S}_m$ contains all operators in S^{-m}, m playing the role of the "formal A-order" of such B.

The advantage of embedding S^{-m} into larger classes $\mathscr{S}_m$ is that the latter are easy to multiply, maintaining the right order. Namely,

$$\mathscr{S}_m \cdot \mathscr{S}_{m'} \subseteqq \mathscr{S}_{m+m'}. \tag{A4}$$

In particular,

$$|B \cdot B'| \leqq \text{const}\, A^{-1/2(m+m')},$$

as was claimed in part (ii) of the proposition.

To demonstrate (A4) we observe that property (a) in the definition of $\mathscr{S}_m$ is equivalent to

$$\|Bu\| \leqq C\|A^{-m/2}u\| \quad \text{for all} \quad u \in L^2(\mathbb{R}).$$

Choosing any pair $B \in \mathscr{S}_m$; $B' \in \mathscr{S}_{m'}$ we need to show

(a′) $\|BB'u\| \leqq \text{const}\,\|A^{-\frac{m+m'}{2}}u\|$,

(b′) $\|[A^{-s_1}\ldots[A^{-s_k}; BB']\ldots]u\| \leqq C_k\|A^{-\frac{m+m'}{2}-s-k/2}u\|$, with $s=s_1+\ldots s_k$.

To demonstrate (a′) we write

$$\|BB'u\| \leqq C\|A^{-m/2}B'u\| \leqq C(\|B'A^{-m/2}u\| + \|[A^{-m/2};B']u\|), \tag{A5}$$

and then apply (a) and (b) with $k=1$ to both norms in (A5).

Similarly (b′) with $k=1$ is established by writing $[A^{-s};BB']=[A^{-s};B]B' + B[A^{-s};B']$, then estimating each of two terms as in (A5). For instance

$$\|B[A^{-s};B']u\| \leqq C\|A^{-m/2}[A^{-s};B']u\| \leqq C(\|[A^{-s};B']A^{-m/2}u\| + \|[A^{-m/2};[A^{-s};B']]u\|.$$

In a similar fashion one estimaties higher commutators $[A^{-s_1}\ldots[A^{s_k};BB']\ldots]$. Thus BB' belong to $\mathscr{S}_{m+m'}$, which proves the proposition.

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