Commun. Math. Phys. 112, 51–74 (1987)

Communications in
Mathematical
Physics

Semi-Infinite Ising Model

II. The Wetting and Layering Transitions

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Dedicated to Walter Thirring on his 60th birthday

Abstract. We study the equilibrium statistical mechanics of the semi-infinite Ising model, interpreted as a model of a binary system near a wall. In particular, the wetting transition is analyzed. In dimensions $d \geqq 3$ and at low temperature, we prove the existence of a layering transition which is of first-order.

1. Introduction

In this paper we continue our analysis of the effect of a wall in the Ising model. In our first paper [1], we analyze the particular case where the wall *does not* adsorb preferentially one of the phases. In the magnetic interpretation of the model this means that there is no boundary field. If the boundary field, h, is positive the wall adsorbs preferentially the $+$ phase. If in the bulk the $-$ phase prevails a film of the $+$ phase between the wall and the bulk phase forms. The thickness of the film can be *microscopic* or *macroscopic* in the coexistence region.

In the first part of this paper we study this wetting phenomenon (for $h>0$, $\lambda=0$)[1]. Our results which are rigorous are summarized in Sect. 2.1. If $h>0$, but $\lambda<0$, it is impossible to have a film of the $+$ phase of *macroscopic* thickness. However, if we vary h, at fixed λ and temperature, the local magnetization on the first layer of the system makes a jump at a particular value $h_l(\lambda)$ of h. This phenomenon occurs in dimensions $d \geqq 3$ and at low temperature. This first-order prewetting-, or *layering* transition is analyzed in the second part of the paper. Our results on the layering transition are described in Sect. 2.2.

The main tools for our analysis are correlation inequalities. In Sect. 3 of our previous paper [1], here called paper I, the relevant correlation inequalities have been summarized. That section of I is basic for understanding the proofs. Lemm 3.1 (I) refers to Lemma 3.1 of paper I, and [(4.8), I] refers to formula (4.8) of paper I.

Theoretical work on the wetting transition is reviewed in Binder [2], Fisher [3], de Gennes [4], and Abraham [5].

[1] h denotes the boundary magnetic field, λ the bulk field

In the last reference, rigorous results are emphasized, in particular those obtained by exact computation. Our approach follows the thermodynamic analysis of the problem, in terms of surface tensions; see e.g. [6].

2. Results

2.1. The Wetting Transition, Phase Diagram at $\lambda=0$ and $h>0$

Let us first recall the definition of the model. We consider a lattice $\mathbb{L}$, $\mathbb{L}\subset Z^d$, $\mathbb{L}=Z^{d-1}\times Z^+$. The points of $\mathbb{L}$ are denoted by $i=(i^1,\ldots,i^d)$ or by $i=(x,z)$, with $x\in Z^{d-1}$ and $z=i^d\in Z^+$.

The Hamiltonian is

$$H(J,K,h,\lambda)+=\sum_{\langle ij\rangle\subset\mathbb{L}}K(i,j)\sigma(i)\sigma(j)-\sum_{i\in\mathbb{L}}\lambda\sigma(i)-\sum_{i\in\Sigma}h\sigma(i),\tag{2.1}$$

where $\Sigma=\{i\in\mathbb{L};\, i^d=1\}$ is the boundary layer of $\mathbb{L}$, and $\langle ij\rangle$ indicates that we sum over all two-point subsets $\{i,j\}$, with $|i-j|=1$. The coupling constants $K(i,j)$ are chosen as follows: If i *and* $j\in\Sigma$, $K(i,j)=J$, otherwise $K(i,j)=K$. We may define the Hamiltonian in a slightly different way: We add another layer Σ_0 to the lattice $\mathbb{L}$, $\Sigma_0=\{i\in Z^d: i^d=0\}$. $\bar{\mathbb{L}}=\mathbb{L}\cup\Sigma_0$. On this new lattice we consider the Hamiltonian

$$\bar{H}(J,K,h,\lambda)=-\sum_{\langle ij\rangle\subset\bar{\mathbb{L}}}\bar{K}(i,j)\sigma(i)\sigma(j)-\sum_{i\in\mathbb{L}}\lambda\sigma(i),\tag{2.2}$$

where $\bar{K}(i,j)=J$, when i and j belong to Σ, $\bar{K}(i,j)=h$, when $i\in\Sigma$ and $j\in\Sigma_0$, otherwise $\bar{K}(i,j)=K$. We recover the Hamiltonian $H(J,K,h,\lambda)$ by setting $\sigma(i)=1$, for all $i\in\Sigma_0$.

The Hamiltonian (2.1) describes the effect of a wall in the Ising model on Z^d with Hamiltonian

$$H=-\sum_{\langle ij\rangle\subset Z^d}K\sigma(i)\sigma(j)-\sum_{i\in Z^d}\lambda\sigma(i).\tag{2.3}$$

The properties of the wall are given by J and h. We always choose $K>0$ and $J>0$ (ferromagnetic case). We assume that $h>0$, which means that the wall adsorbs preferentially the $+$ phase. The analysis of the model with $h<0$ can be obtained from the one given below by symmetry.

In paper I we have analyzed the *symmetric* model, with $h=\lambda=0$, and the model for which the signs of λ and of h are the *same*. By symmetry it is then sufficient to suppose that $h\geqq 0$ and $\lambda\geqq 0$. Therefore it is natural to consider the Gibbs state $\langle\cdot\rangle^+$, which is defined by $+$ boundary conditions ($+$b.c.) (see Sect. 2.2 of I). When $\mathrm{Re}h>0$, $\mathrm{Re}\lambda\geqq 0$, or when $\mathrm{Re}\lambda>0$ and $\mathrm{Re}h\geqq 0$ this state is *analytic* in h, in λ, respectively, in the sense that all correlation functions are analytic in these variables. (For further properties of $\langle\cdot\rangle^+$ see also Sect. 2 of I.)

When $\lambda=h=0$, symmetry breaking occurs at low enough temperature: There exists a temperature T_s, such that, above T_s, the local magnetization, $\langle\sigma(i)\rangle^+(J,K,0,0)$, in the first layer Σ, $i\in\Sigma$, is zero; below T_s it is positive. When $d\geqq 3$, and for J/K large enough, T_s is larger than $T_c(d)$, the critical temperature of the model on Z^d with Hamiltonian (2.3).

But if $d=2$, or if $d\geqq 3$ and $J\leqq K$, we always have $T_s=T_c(d)$.

For $\lambda=0$ and $K>K_c(d)$, $K_c(d)$ being the critical coupling constant of the model defined by (2.3) on Z^d, the bulk system is in the coexistence region. For $h>0$, the results described above only cover the situation where the bulk phase is the + phase. A more interesting situation arises when the bulk phase is in the − phase. We already noticed in paper I that the surface free energy of the model depends on the choice of boundary conditions (b.c.) when $\lambda=0$, i.e. it depends on the type of the bulk phase. It is likely that there is a well-defined surface free energy $F(J,K,h,\lambda)$ which is *independent* of b.c., for $\lambda \neq 0$, although we are able to prove this statement only for some range of the parameters of the model. We *assume* that this is true in this paper. But this assumption is only used in the discussion of the layering transition; see Sect. 4. Under this assumption, the surface free energy $F(J,K,h,\lambda)$ is discontinuous in λ at $\lambda=0$ and $h\neq 0$. We define

$$F^+(J,K,h)=\lim_{\lambda\downarrow 0} F(J,K,h,\lambda) \tag{2.4}$$

and

$$F^-(J,K,h)=\lim_{\lambda\uparrow 0} F(J,K,h,\lambda), \tag{2.5}$$

see [(2.5), I] and [(2.6), I]. F^+ and F^- are the surface tensions of the wall against the + phase and the − phase, respectively.

The functions F^+ and F^- can be defined directly at $\lambda=0$ using + b.c. and − b.c. Let us consider, in more detail, the difference between + b.c. and − b.c., for $h>0$. We choose a box $\Lambda(L,M)\subseteqq \mathbb{L}$,

$$\Lambda(L,M)=\{i\in\mathbb{L};|i^k|\leqq L,\ k=1,\ \ldots,\ d-1,\ 1\leqq i^d\leqq M\}. \tag{2.6}$$

The + b.c. for the box $\Lambda(\mathbb{L},M)$ means that we set $\sigma(j)=1$ for all $i\in\mathbb{L}\backslash\Lambda(L,M)$.

Using the Hamiltonian $\bar{H}$, see (2.2), this implies that $\sigma(i)=1$, for all $i\in\bar{\mathbb{L}}\backslash\Lambda(L,M)$. If we describe the configurations inside $\Lambda(L,M)$ by Peierls contours, all contours are closed. [The contours are dual to all bonds $\langle ij\rangle$, with

$$\sigma(i)\sigma(j)=-1\,.]$$

In contrast, if we choose − b.c. for $\Lambda(L,M)$, we have $\sigma(i)=1$ for all $i\in\Sigma_0$, and $\sigma(i)=-1$ for all $i\in\mathbb{L}\backslash\Lambda(L,M)$. In each configuration inside $\Lambda(L,M)$ all contours are closed *except one*, which we denote by Γ. This contour Γ is imposed by the − b.c. and describes a defect, or Bloch wall, between the adsorbed + phase and the bulk-phase. Such a defect makes a nonzero contribution to the surface free energy F^-, provided $K>K_c(d)$. Therefore the surface free energy, $\tau_w^{\pm}$, of the interface between the adsorbed + phase and the − phase in the bulk is

$$\tau_w^{\pm}(J,K,h)=F^-(J,K,h)-F^+(J,K,h) \tag{2.7}$$

in the thermodynamic limit, $L\to\infty$ and $M\to\infty$.

Let $\tau^{\pm}(K)$ be the usual surface tension between the two pure phases, + and −, of the model on Z^d. The quantity $\tau^{\pm}(K)$ is a difference of two free energies, too. It is defined as follows: Let $\Omega(L,M)$ be the box

$$\Omega(L,M)=\{i\in Z^d:|i^k|\leqq L,\ k=1,\ldots,d-1,\ -M<j^d\leqq M\}. \tag{2.8}$$

In $\Omega(L,M)$ we consider the model with Hamiltonian (2.3), $\lambda=0$ and + b.c., i.e. $\sigma(i)=1$ for all $i\in Z^d\backslash\Omega(L,M)$. The corresponding partition function is denoted by $Q^+_{L,M}$. If we choose the ± b.c., i.e. $\sigma(i)=-1$ for $i\in Z^d\backslash\Omega(L,M)$ and $i^d>0$, $\sigma(i)=1$ for

$i \in Z^d \setminus \Omega(L, M)$ and $i^d \leqq 0$, we obtain a partition function denoted by $Q^{\pm}_{L,M}$. The difference between these two b.c. is that the $\pm$b.c. force a defect, or Bloch wall, through the system, which is, however, far away from the boundary of the box when M is large. The free energy of this defect is precisely $\tau^{\pm}$, i.e.

$$\tau^{\pm}(K) = \lim_{\substack{L \to \infty \\ M \to \infty}} -\frac{1}{|\Sigma(L)|} \ln\left(\frac{Q^{\pm}_{L,M}}{Q^{+}_{L,M}}\right), \tag{2.9}$$

where $|\Sigma(L)| = |\Sigma \cap \Omega(L, M)|$ is the number of points inside $\Sigma \cap \Omega(L, M)$. We now summarize our results concerning surface free energies which are proven in Sect. 3:

a) $0 \leqq \tau^{\pm}_w(J, K, h) \leqq \tau^{\pm}(K)$ *if* $\lambda = 0$ and $h \geqq 0$; $\tau^{\pm}_w(J, K, 0) = 0$.
b) $\tau^{\pm}_w(J, K, h)$ is a monotone increasing function of h, K, and J.
c) $\tau^{\pm}_w(J, K, h)$ is a concave function of h, for $h \geqq 0$.
d) If $J \geqq K$ and $h \geqq K$, $\tau^{\pm}_w(J, K, h) = \tau^{\pm}(K)$.
e) For arbitrary J and K, $\lim_{h \to \infty} \tau^{\pm}_w(J, K, h) = \tau^{\pm}(K)$.

Point d) follows from a) and the results of [7]. Notice also that $\tau^{\pm}_w \equiv 0$ if $K \leqq K_c(d)$, since $\tau^{\pm}(K) = 0$ [8]. The main result is of course a), which can be written as

$$F^{-} - F^{+} \leqq \tau^{\pm}. \tag{2.10}$$

One says that there is *partial* wetting of the wall, when the adsorbed film has a *microscopic* thickness. There is complete wetting of the wall, when the adsorbed film has a *macroscopic* thickness. The transition between these two regimes is the *wetting transition.* Partial wetting occurs when $\tau^{\pm}_w < \tau^{\pm}$; complete wetting occurs when $\tau^{\pm}_w = \tau^{\pm}$. We are able to establish a direct connection with the equilibrium states of our model. Indeed, from c) we get

$$\tau^{\pm}_w(J, K, h) = \int_0^h ds(\langle \sigma(i) \rangle^{+}(J, K, s) - \langle \sigma(i) \rangle^{-}(J, K, s)), \tag{2.11}$$

where, in (2.11), $i \in \Sigma$ and $\langle \cdot \rangle^{+(-)}(J, K, s)$ is the equilibrium state with $+$b.c., ($-$b.c.), boundary field $h = s$ and bulk field $\lambda = 0$. The integrand in (2.11) is a nonnegative, monotone decreasing function of s. If we define $h_w(J, K)$ by

$$h_w(J, K) = \inf\{h : \tau^{\pm}_w(J, K, h) = \tau^{\pm}(K)\}, \tag{2.12}$$

we immediately get that

$$\langle \sigma(i) \rangle^{+}(J, K, h') = \langle \sigma(i) \rangle^{-}(J, K, h'), \qquad i \in \Sigma, \tag{2.13}$$

for all $h' > h_w(J, K)$, since

$$\tau^{\pm}(K) = \int_0^{h_w} ds(\langle \sigma(i) \rangle^{+}(J, K, s) - \langle \sigma(i) \rangle^{-}(J, K, s)). \tag{2.14}$$

We can prove that (2.13) actually implies

$$\langle \cdot \rangle^{+} = \langle \cdot \rangle^{-},$$

see Lemma 3.3 of I. This is a precise statement that the adsorbed film has a macroscopic thickness. In contrast, if $h < h_w(J, K)$, a) and e) imply that

$$\langle \sigma(i) \rangle^{+}(J, K, h) \neq \langle \sigma(i) \rangle^{-}(J, K, h). \tag{2.15}$$

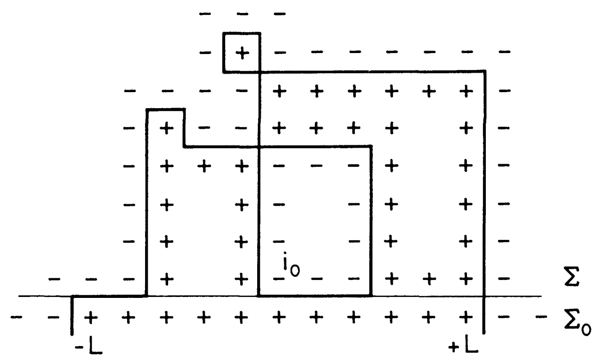

Fig. 1

In this case, we have several Gibbs states. The adsorbed film has a microscopic thickness. At $h=h_w$ two possibilities arise: If there is a unique Gibbs state, the transition is continuous, but if there are (at least) two Gibbs states, it is of 1st order.

Let us consider the state $\langle\cdot\rangle^-$. One may also construct $\langle\cdot\rangle^-$ as follows. We fix L and choose $L'>L$. Instead of taking the $-$b.c. for $\Lambda(L',M)$, we use the b.c. $\sigma(i)=1$, for $i\in\Sigma_0$ and $|i^k|\leqq L$, otherwise $\sigma(i)=-1$. This b.c. forces a defect through the system, which is described by an open contour Γ (see Fig. 1). The $-$b.c. corresponds to the special choice $L=L'$. We let $L'\to\infty$ and $M\to\infty$, and we obtain a state $\langle\cdot\rangle_L^-$ defined on the lattice $\mathbb{L}$. This state describes a situation where the wall Σ adsorbs the $+$ phase only on the finite subset $\Sigma(L)$.

If the contour Γ crosses the segment defined by $(0,\dots,0)$ and $(0,\dots,0,1)=i_0$, we say that $\Gamma<i_0$ (see Fig. 1). Let

$$P_L(\Gamma<i_0)=\mathrm{Prob}\{\Gamma<i_0\}, \tag{2.16}$$

the probability being computed in the state $\langle\cdot\rangle_L^-$. It is not difficult to show that

$$\lim_{L\to\infty} =\langle\cdot\rangle_L^- \tag{2.17}$$

(F.K.G. inequalities) and

$$P_L(\Gamma<i_0)\geqq\tfrac{1}{2}(\langle\sigma(i_0)\rangle^+-\langle\sigma(i_0)\rangle^-). \tag{2.18}$$

As soon as there is partial wetting, in the sense that $\tau_w^\pm<\tau^\pm$, $P_L(\Gamma<i_0)$ is *strictly positive, uniformly in L*. But we can also prove, at least at low temprature, that

$$\lim_{L\to\infty} P_L(\Gamma<i_0)=0 \tag{2.19}$$

if the Gibbs state is unique.

It is easy to derive a lower bound for $h_w(J,K)$. From (2.14) we have

$$h_w(J,K)\geqq\tfrac{1}{2}\tau^\pm(K). \tag{2.20}$$

If $J\geqq K$, we also have an upper bound, derived from d), namely

$$h_w(J,K)\leqq K, \qquad J\geqq K. \tag{2.21}$$

If we introduce explicitly the inverse temperature β the parameters of the model are multiplied by β, and we multiply all free energies by β^{-1}. As a function of β, $h_w(\beta)$ is defined by

$$h_w(\beta)=\inf\{h:\tau_w^{\pm}(\beta J,\beta K,\beta h)=\tau^{\pm}(\beta K)\}, \tag{2.22}$$

and the bounds (2.20) and (2.21) then become

$$(2\beta)^{-1}\tau^{\pm}(\beta K)\leqq h_w(\beta)\leqq K, \quad \text{for} \quad J\geqq K. \tag{2.23}$$

It is interesting to notice that

$$\lim_{\beta\to\infty}\frac{1}{\beta}\tau^{\pm}(\beta K)=2K. \tag{2.24}$$

For $K=J$ and $d=2$, we know the exact value of h_w, since the model has been solved by Abraham [9]: h_w is a solution of the equation

$$e^{2K}(\mathrm{ch}\,2K-\mathrm{ch}\,2h_w)=\mathrm{sh}\,2K. \tag{2.25}$$

Abraham found this equation by computing the magnetization profile. In [10], Abraham and de Coninck rederived this equation using the definition (2.12) of h_w. In Fig. 2, we have compared the exact value of $h_w(\beta)$ for $d=2$ with the lower bound $(2\beta)^{-1}\tau^{\pm}(\beta K)$.

McCoy and Wu computed the surface free energy $F^+(h)$ for $h\geqq 0$, $J=K$ and $d=2$. (Their definition of the surface free energy is different, but we have proved in I that it yields F^+ in the thermodynamic limit.) They found that there is an analytic continuation of $F^+(h)$ to $h\leqq 0$, and therefore also of $\langle\sigma(i)\rangle^+(h)$, $i\in\Sigma$; [11].

We can interpret their results in terms of the wetting phenomenon. The analytic continuation of $\langle\sigma(i)\rangle^+(h)$ to $h<0$ is very likely $\langle\sigma(i)\rangle^+(h)$, which is different from $\langle\sigma(i)\rangle^-(h)$, when there is partial wetting.

By symmetry, we get

$$\begin{aligned} -\langle\sigma(i)\rangle^-(-h)&=\langle\sigma(i)\rangle^+(h), \\ -\langle\sigma(i)\rangle^+(-h)&=\langle\sigma(i)\rangle^-(h), \end{aligned} \tag{2.26}$$

with $h>0$.

In this interpretation, h_w is given by the intersection point of the two curves $\langle\sigma(i)\rangle^+(h)$ and $\langle\sigma(i)\rangle^-(h)$. This can be checked using the results of [11].

Let us now discuss the special case $J=K=h$. For $d=2$, this model has been analyzed in [12, 13]. It has been conjectured that this model has a unique Gibbs state for any nonzero temperature, and the conjecture has been verified in the SOS limit [14]. In Sect. 3.5 we prove that there is a unique Gibbs state for *any* finite temperature and *any* dimension d. For $J>K$ and $h=K$, there is of course a unique Gibbs state, since $h_w(J,K)$ is decreasing in J, and we define

$$h_w(J,K)=\inf\{h:\langle\sigma(i)\rangle^+(J,K,h)=\langle\sigma(i)\rangle^-(J,K,h)\}. \tag{2.27}$$

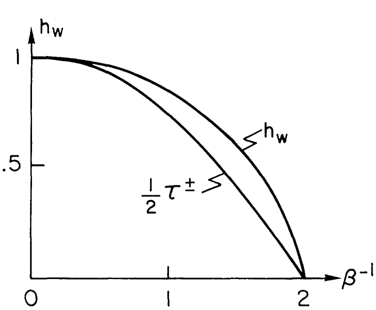

Fig. 2

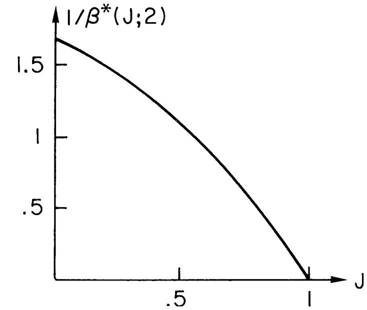

Fig. 3

A natural question is whether the same situation is met for $J<K$ and $h=K$. The answer is negative: For $d \geqq 2$, there exists $\beta^*(J,d)$, with $\beta^*(J,d)<\infty$ for $J<K$, such that there are two Gibbs states, for $\beta>\beta^*$. A lower bound for $\beta^*(J,2)$ is indicated in Fig. 3. Here $K=h=1$; the critical temperature of the two-dimensional model is $\cong 2.27$ in our units, and the exact value of $\beta^*(J,2)$ for $J=O$ is $\cong 2.0$ (see Fig. 3).

The last regime to be discussed is $J<K$ and h arbitrary. We consider the extreme situation, where $J=0$ and $d=2$. It is possible to study the model, for this particular choice of parameters, by integrating over the spin variables of the first layer Σ, and computing an effective boundary field h_{ef}. The effective field is a solution of the equation

$$\tanh(h_{\mathrm{ef}}) = \tanh(K)\tanh(h). \tag{2.28}$$

The value of h_{ef} is the same for $d \geqq 2$. For $d=2$ and $J=K$ we know the exact value of $h_w(\beta)$, which is given by [see (2.25)]

$$e^{2\beta K}(\mathrm{ch}\,2\beta K - \mathrm{ch}\,2\beta h_w(\beta)) = \mathrm{sh}(2\beta K). \tag{2.29}$$

This permits us to study $h_w(\beta; J=0)$, since this model is equivalent to a model with $J=K$ and $h=h_{\mathrm{ef}}$. In particular

$$\tau_w^{\pm}(0, K, h) = \tau_w^{\pm}(K, K, h_{\mathrm{ef}}). \tag{2.30}$$

The results are as follows: For $d=2$ and $\beta \gg 1$,

$$h_w(\beta) < 2K, \qquad J=0 \tag{2.31}$$

and

$$\lim_{\beta\to\infty} h_w(\beta) = 2K, \qquad J=0. \tag{2.32}$$

For $\beta \gg 1$, we can also analyze the model in three or more dimensions. For $J=0$, $\beta \gg 1$, and $d=3$, we prove that there is only partial wetting provided $h<(2d-1)K$.

Figure 4 displays the phase diagram, in the (h, T)-plane, for a model with $d \geqq 3$, $\lambda=0$ and J/K large enough, so that there is a surface transition at $T_s > T_c(3)$. The line $T=T_c(3)$ is a critical line, since, for $T>T_c(3)$, the transverse correlation length, $\xi_\perp(h, T)$, defined by

$$\xi_\perp^{-1} = \lim_{z\to\infty} -\frac{1}{z} \ln \langle \sigma(0,1); \sigma(0,z)\rangle^+(h, T) \tag{2.33}$$

is equal to $\xi_{\mathrm{Is}}(T)$, the correlation length of the Ising model. Therefore $\xi_\perp(h, T)$ diverges, with exponent ν, when $T \downarrow T_c(3)$ (see I). However, on this line we find only

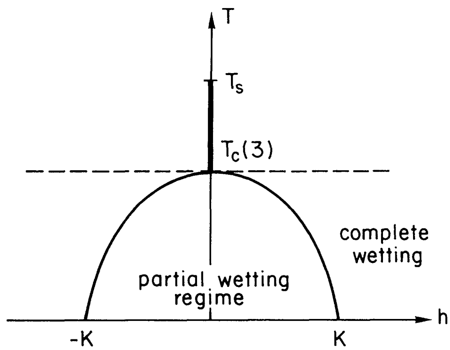

Fig. 4

one Gibbs state, except at $h=0$. Moreover, the surface free energy F and the unique state, $\langle\cdot\rangle$, are analytic in h, for $\mathrm{Re}\,h \neq 0$ and the boundary susceptibility χ_Σ is finite. There are several Gibbs states in the partial wetting region and on the segment $(T_c(3), T_s)$ of Fig. 4.

The results of this section are proved in Sect. 3. The existence of further Σ-invariant Gibbs states is briefly discussed in Sect. 3.5.

2.2. *The Layering Transition, Phase diagram at $\lambda<0$ and $h>0$*

We consider the model with $h \geqq 0$ and $\lambda \neq 0$. We assume that the same surface free energy is obtained from $+$b.c. and from $-$b.c. (see Sect. 2 of I). This hypothesis, i.e. $F^+ = F^-$, implies that there is a unique Gibbs state for $\lambda>0$ and $h \geqq 0$ (Lemma 3.2). It is useful to consider the structure of the (*Σ-invariant*) ground states (g.s.) indicated in Fig. 5. There are three regions with a unique g.s.: On the vertical line $\lambda=0$, there is coexistence of the g.s. $\{\sigma(i)\equiv 1\}$ and $\{\sigma(i)\equiv -1\}$, for $0 \leqq h < K$. At $h=K$, there are infinitely many g.s.: $\{\sigma(i)\equiv 1\{$, $\{\sigma(i)\equiv -1\}$ and $\{\sigma(i)=1\}$, for the first m layers, $1 \leqq i^d \leqq m$, $\sigma(i)=-1$, otherwise. Here m is arbitrary. For $h>K$, there are still infinitely many g.s., as above, but the g.s. $\{\sigma(i)\equiv -1\}$ has disappeared. On the line $h=K-\lambda$, $\lambda<0$, there is coexistence of the g.s. $\{\sigma(i)\equiv -1\}$ and $\{\sigma(i)=1\}$, $i\in\Sigma$, $\sigma(i)=-1$, otherwise.

We show in Sect. 4 that, at low temperature and for $d \geqq 3$, there exists a line of first order transitions in the region $\{(\lambda,h);\ \lambda \leqq 0,\ h \geqq 0\}$ which is asymptotically given by the line $h=K-\lambda$.

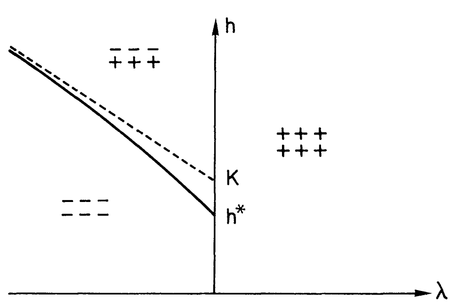

Fig. 5

More precisely, let $2J > K$ and let J, K and λ be fixed, with $\lambda < 0$. If the inverse temperature β is large enough, there exist a constant $C(\beta) > 0$ (independent of λ and of h), and a value of h, $h_l(\beta)$, such that

$$\langle \sigma(i) \rangle^{-}(h,\beta) \leqq -C(\beta), \qquad h \leqq h_l(\beta), \qquad i \in \Sigma, \tag{2.34}$$

and

$$\langle \sigma(i) \rangle^{-}(h,\beta) \geqq C(\beta), \qquad h > h_l(\beta), \qquad i \in \Sigma. \tag{2.35}$$

The function $h_l(\beta)$ depends on the parameters J, K and λ. We can show that

$$0 < h_l(\beta; J, K, \lambda) \leqq K - \lambda. \tag{2.36}$$

This transition line ends at $h_*(\beta)$ when $\lambda = 0$. From (2.36) we have $0 < h_* \leqq K$. Since $C(\beta)$ is independent of λ, h_* is still a point of first-order transition. This line of first-order transitions is a line of layering transitions. Unfortunately, we cannot find the exact value of $h_*(\beta)$, and therefore we cannot decide whether $h_* = h_w$ or not. However, if $K/2 < J < K$, and if β is large enough, we know that $h_w(\beta) > K \geqq h_*(\beta)$.

We have no results on the existence of further layering transitions when $\lambda < 0$ tends to zero. Such transitions are predicted in the mean-field theory of the model (see e.g. [15]). The layering transition line in the (h, λ) plane, at fixed low temperature, is displayed in Fig. 5. Our proofs appear in Sect. 4.

3. The Wetting Transition

In Sect. 3.1, below, we prove the basic relation (2.10) and the formula giving $\tau_w^{\pm}$ in terms of the local magnetization of the first layer of the model. Section 3.2 contains the proofs of several results concerning the unicity of the Gibbs state. In Sect. 3.3, we derive three upper bounds on $\tau_w^{\pm}$: (3.18), (3.22) – which is an upper bound in terms of the free energy of a $(d-1)$-dimensional system – and (3.28). The bound (3.22) is particularly useful for $d=2$, since it can be computed explicitly. Bounds (3.22) and (3.28) are of the same type. Using these upper bounds, we establish lower bounds on $h_w(J, K)$. This is done in Sect. 3.4. Lemma 3.4 shows that h_w is a strictly decreasing function of J. Formula (3.34) is a general bound, independent of J. Using the bounds (3.22), respectively (3.28), we can show that, for $h = K$ and $J = K$, $\tau_w^{\pm}(\beta J, \beta K, \beta K) < \tau^{\pm}(\beta K)$ at high β. This proves the existence of several Gibbs states, since we have partial wetting. Finally, we consider the extreme case $J = 0$. To distinguish between partial wetting and complete wetting, we must verify one of the inequalities $h_{\mathrm{ef}} < h_w(K, K)$, or $h_{\mathrm{ef}} > h_w(K, K)$. For $d=2$ this can be done easily since we know $h_w(K, K)$ explicitly. At the end of the section we prove the equivalence of the definitions (2.12) and (2.27) for $h_w(J, K)$. In Sect. 3.5, we prove that the model with $K = J = h$ has always a unique Gibbs state at any finite temperature.

Finally, in Sect. 3.6, we discuss the behavior of the defect described by the contour Γ which is imposed by the b.c.

3.1. The Basic Inequality $F^- - F^+ \leqq \tau^{\pm}$

We recall the definition of F^+ (see Sect. 2 of I). Let $\Lambda(L, M)$ be the box

$$\Lambda(L, M) = \{ i \in \mathbb{L} : |i^k| \leqq L,\ 1 \leqq k \leqq d-1,\ 1 \leqq i^d \leqq M \}. \tag{3.1}$$

We consider the model in $\Lambda(L, M)$ defined by the Hamiltonian (2.1) and $+$b.c. The corresponding partition function is denoted by $Z^{+}_{L,M}$. We also consider the box $\Omega(L, M)$

$$\Omega(L,M)=\{i\in Z^{d}: |i^{k}|\leqq L,\ 1\leqq k\leqq d-1,\ -M<i^{d}\leqq M\}. \tag{3.2}$$

In this box, we consider the model defined by the Hamiltonian (2.3) and $+$b.c. The corresponding partition function is $Q^{+}_{L,M}$. By definition

$$F^{+}=\lim_{\substack{L\to\infty\\ M\to\infty}} -\frac{1}{2|\Sigma(L)|}\ln\left[\frac{(Z^{+}_{L,M})^{2}}{Q^{+}_{L,M}}\right]. \tag{3.3}$$

The surface free energy F^{-} is defined in a similar way, with $-$b.c. If $\lambda=0$, $Q^{+}_{L,M}=Q^{-}_{L,M}$ by symmetry. Therefore

$$F^{-}(J,K,h,0)-F^{+}(J,K,h,0)\equiv\tau^{\pm}_{w}(J,K,h)=\lim_{\substack{L\to\infty\\ M\to\infty}}\frac{-1}{|\Sigma(L)|}\ln\left(\frac{Z^{-}_{L,M}}{Z^{+}_{L,M}}\right). \tag{3.4}$$

We modify the $-$b.c. for the box $\Lambda(L,M)$. We set $\sigma(i)=1$ if $i\in\mathbb{L}\backslash\Lambda(L,M)$ and $i^{d}\leqq M/2$, and $\sigma(i)=-1$ if $i\in\mathbb{L}\backslash\Lambda(L,M)$ and $i^{d}>M/2$ (see Fig. 6). We denote the partition function by $Z^{\pm}_{L,M}$. The difference between $\ln Z^{-}_{L,M}$ and $\ln Z^{\pm}_{L,M}$ is of order $O(M\cdot L^{d-2})$. We have seen, in Sect. 4.1 of I, that F^{+} or F^{-} can be defined as the limit of (3.3) when $L\to\infty$ with $M=L^{\alpha}$, $0<\alpha<1$. If in (3.4) we replace $Z^{-}_{L,M}$ by $Z^{\pm}_{L,M}$ and set $M=L^{\alpha}$, $0<\alpha<1$, then we have

$$\tau^{\pm}_{w}(J,K,h)=\lim_{L\to\infty}\frac{-1}{|\Sigma(L)|}\ln\left(\frac{Z^{\pm}_{L,M}}{Z^{+}_{L,M}}\right), \tag{3.5}$$

since $|\Sigma(L)|=O(L^{d-1})$. We take the derivative of the right-hand side of (3.5) with respect to h. For $L<\infty$, the result is

$$|\Sigma(L)|^{-1}\sum_{i\in\Sigma(L)}(\langle\sigma(i)\rangle^{+}_{L,M}-\langle\sigma(i)\rangle^{\pm}_{L,M})\geqq 0. \tag{3.6}$$

The positivity of (3.6) follows from inequality [(3.23), I]. For fixed L and M, we let $h\uparrow\infty$. In this way we obtain an upper bound,

$$\tau^{\pm}_{w}(J,K,h)\leqq\lim_{L\to\infty}\frac{-1}{|\Sigma(L)|}\ln\left(\frac{Z^{\pm}_{L,M-1}}{Z^{+}_{L,M-1}}\right). \tag{3.7}$$

If we compare (3.7) and (2.9) we see that the right-hand side of (3.7) is nothing but $\tau^{\pm}$. The fact that one may take $0<\alpha<1$, is implicitly proved in [7]. The proof is

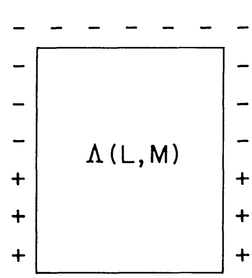

Fig. 6

similar to the proof of the existence of F^+ in Sect. 2.1 of I. However, in this case we introduce a self-duplication of the model with respect to the hyperplane $i^d = \frac{1}{2}$ [we adopt the definition (2.9) of $\tau^\pm$]. Using Lemma 3.4 (I), part b), we obtain the desired result. Next, we prove (2.11). From (3.4)

$$-\frac{1}{|\Sigma(L)|} \ln\left(\frac{Z^-_{L,M}}{Z^+_{L,M}}\right) = \frac{1}{|\Sigma(L)|} \sum_{i\in\Sigma(L)} \int_0^h dh' (\langle\sigma(i)\rangle^+_{L,M}(J,K,h') - \langle\sigma(i)\rangle^-_{L,M}(J,K,h')) . \tag{3.8}$$

In a duplicate system we can write

$$\langle\sigma(i)\rangle^+_{L,M} - \langle\sigma(i)\rangle^-_{L,M} = \langle t(i)\rangle^t_{L,M} . \tag{3.9}$$

By Lemma 3.4 (I), part b), the limit $L\to\infty$ gives

$$\tau^\pm_w = \int_0^h dh' (\langle\sigma(i)\rangle^+(h') - \langle\sigma(i)\rangle^-(h')) , \qquad i\in\Sigma . \tag{3.10}$$

The derivative of (3.8) with respect to h is given by

$$\frac{1}{|\Sigma(L)|} \sum_{i\in\Sigma(L)} \langle t(i)\rangle^t_{L,M} . \tag{3.11}$$

The derivative of (3.11) with respect to h is

$$\frac{1}{|\Sigma(L)|} \sum_{i\in\Sigma(L)} \sum_{j\in\Sigma(L)} (\langle t(i)s(j)\rangle^t_{L,M} - \langle t(i)\rangle^t_{L,M}\langle s(j)\rangle^t_{L,M}) . \tag{3.12}$$

This expression is negative, by [(3.23), I]. Therefore (3.8) is a concave function of h, and this also true for $\tau^\pm_w$.

3.2. Criteria for the Uniqueness of Gibbs States

We adapt to our case a well-known criterion [16] for uniqueness:

Lemma 3.1. *Let $\lambda\in R$, $h\in R$, $\lambda\cdot h\geqq 0$. If at h_0*

$$\frac{dF^+}{dh}(J,K,h_0,\lambda) = \frac{dF^-}{dh}(J,K,h_0,\lambda) , \tag{3.13}$$

then there is a unique Gibbs state.

Proof. We can suppose that $h\geqq 0$ and $\lambda\geqq 0$. F^+ and F^- are concave functions of h. If they are differentiable at h_0 then

$$\frac{dF^+}{dh}(h_0) = \lim_{L,M\to\infty} \frac{dF^+_{L,M}}{dh}(h_0) , \tag{3.14}$$

and similarly for F^-. From (3.13) and (3.14) we get

$$\langle\sigma(i)\rangle^+(J,K,h_0,\lambda) = \langle\sigma(i)\rangle^-(J,K,h_0,\lambda) , \qquad i\in\Sigma . \tag{3.15}$$

The proof now follows from Lemma 3.3, (I). Notice that (3.15) is valid without the assumption that $h\cdot\lambda\geqq 0$. This is used in Sect. 4.

Lemma 3.2. *If $\lambda\geqq 0$ and $h>0$, and if $F^+(J,K,h,\lambda) - F^-(J,K,h,\lambda)$ is independent of h, then there is a unique Gibbs state.*

Proof. F^+ is concave and, for almost all $h>0$, $-\dfrac{dF^+}{dh}=\langle\sigma(i)\rangle^+$, $i\in\Sigma$. But if $\lambda\geqq 0$ and $h>0$ $\langle\sigma(i)\rangle^+$ is a bounded concave function of h (G.H.S. inequality). Therefore F^+ is a C^1-function, for $h>0$. The proof is completed by using Lemma 3.1.

Lemma 3.3. *Let $\lambda_0\geqq 0$ and $h_0\geqq 0$. If there is a unique Gibbs state, for (J,K,h_0,λ_0), then there is a unique Gibbs state for all (J,K,h,λ), with $h\geqq h_0$ and $\lambda\geqq\lambda_0$.*

Proof. By hypothesis $\langle\sigma(i)\rangle^+(h_0,\lambda_0)=\langle\sigma(i)\rangle^-(h_0,\lambda_0)$. By duplicate variables inequalities (see I), we have, for $h\geqq h_0$ and $\lambda\geqq\lambda_0$,

$$0\leqq\langle\sigma(i)\rangle^+(h,\lambda)-\langle\sigma(i)\rangle^-(h,\lambda)\leqq\langle\sigma(i)\rangle^+(h_0,\lambda_0)-\langle\sigma(i)\rangle^-(h_0,\lambda_0). \tag{3.16}$$

This clearly proves the lemma.

3.3. Upper Bounds on $\tau_w^{\pm}(J,K,h)$

We suppose that $\lambda=0$ and $h\geqq 0$. We derive three upper bounds for $\tau_w^{\pm}(J,K,h)$.

A) From the expression

$$\tau_w^{\pm}(J,K,h)=\int_0^h dh'(\langle\sigma(i)\rangle^+(h')-\langle\sigma(i)\rangle^-(h')), \tag{3.17}$$

we get the upper bound

$$0\leqq\tau_w^{\pm}(J,K,h)\leqq 2h\langle\sigma(i)\rangle^+(J,K,0), \qquad i\in\Sigma. \tag{3.18}$$

Indeed, (3.17) can be written in a duplicate system as

$$\tau_w^{\pm}(J,K,h)=\int_0^h dh'\langle t(i)\rangle^t(h'), \qquad i\in\Sigma, \tag{3.19}$$

where, in (3.19), h' is a "s-field." Thus $\langle t(i)\rangle^t(h')$ is decreasing in h'.

B) In the duplicate system we add a t-field on the second layer

$$-\sum_x \mu t(x,2). \tag{3.20}$$

We recall that $t(i)=\sigma(i)-\sigma'(i)$ and $s(i)=\sigma(i)+\sigma'(i)$. The expectation value $\langle t(i)\rangle^t$, $i\in\Sigma$, increases when μ increases. Letting $\mu\to\infty$, we get

$$\langle t(i)\rangle^t(J,K,h)\leqq\langle\sigma(i)\rangle_{d-1}(J,h+K)-\langle\sigma(i)\rangle_{d-1}(J,h-K), \tag{3.21}$$

where $\langle\sigma(i)\rangle_{d-1}(J,\lambda')$ is the expectation value in the $(d-1)$-dimensional Ising model, defined in Σ, with coupling constant J and external field λ'. Therefore we obtain

$$\begin{aligned}\tau_w^{\pm}(J,K,h)&\leqq\int_0^h[\langle\sigma(i)\rangle_{d-1}(J,h'+K)-\langle\sigma(i)\rangle_{d-1}(J,h'-K)]\,dh'\\&=-F_{d-1}(J,h+K)+F_{d-1}(J,K)+F_{d-1}(J,h-K)-F_{d-1}(J,-K)\\&=-F_{d-1}(J,h+K)+F_{d-1}(J,h-K),\end{aligned} \tag{3.22}$$

where $F_{d-1}(J,\lambda')$ is the free energy of the $(d-1)$-dimensional Ising model with couplings J and external field λ'. Let us consider the two-dimensional model. The exact expression for $F_1(J,\lambda')$ is known explicitly. For $h=K$, we find

$$\tau_w^{\pm}(J,K,K)\leqq -F_1(J,2K)+F_1(J,0), \tag{3.23}$$

or, using the explicit expression for F_1,

$$\begin{aligned}\tau_w^{\pm}(J,K,K) &\leqq \ln(\operatorname{ch}2K+(\operatorname{sh}^2 2K+e^{-4J})^{1/2}) \\ &\quad -\ln(1+e^{-2J}) \leqq \ln(2\operatorname{ch}2K)-\ln(1+e^{-2J}).\end{aligned} \tag{3.24}$$

If $h>K$ we get

$$\tau_w^{\pm}(J,K,h) \leqq -F_1(J,h+K)+F_1(J,h-K). \tag{3.25}$$

C) These two upper bounds are not explicit for $d \geqq 3$. Let us compute the conditional expectation value of $t(i)$, $i \in \Sigma$, given the values of the variables $t(j)$ and $s(j)$, $|i-j|=1$:

$$\begin{aligned}&\langle t(i) \,|\, t(j), s(j);\, |i-j|=1\rangle \\ &= \frac{2\operatorname{sh}\left(\sum_j K(i,j)t(j)\right)}{\operatorname{ch}\left(\sum_j K(i,j)t(j)\right)+\operatorname{ch}\left(\sum_j K(i,j)s(j)+2h(j)\right)},\end{aligned} \tag{3.26}$$

where $K(i,j)=J$ if i and $j \in \Sigma$, $K(i,j)=K$ if $i \in \Sigma$ and $j \notin \Sigma$; $h(j)=h$ for $j \in \Sigma$ and zero otherwise. We get an upper bound if we take $t(j)=2$, $|i-j|=1$. Let

$$2\Omega = 4(d-1)J+2K. \tag{3.27}$$

Using (3.19), (3.26) and the fact that $t(j) \neq 0$ implies $s(j)=0$, we have that

$$\begin{aligned}\tau_w^{\pm}(J,K,h) &\leqq 2\cdot\operatorname{sh}2\Omega \int_0^h dx \frac{1}{\operatorname{ch}2\Omega+\operatorname{ch}2x} \\ &= \ln\frac{\operatorname{ch}(\Omega+h)}{\operatorname{ch}(\Omega-h)} = (\Omega+h)-|\Omega-h|+\ln\frac{1+e^{-2(\Omega+h)}}{1+e^{-2|\Omega-h|}}.\end{aligned} \tag{3.28}$$

Remark. If $J=0$ then this bound coincides with the bound (3.24).

3.4. *Properties of* $h_w(J,K)$

By definition, we have that

$$h_w(J,K)=\inf\{h : \tau_w^{\pm}(J,K,h)=\tau^{\pm}(K)\}. \tag{3.29}$$

Since $\tau_w^{\pm}(J,K,h)$ is increasing in J, $h_w(J,K)$ is decreasing in J.

Lemma 3.4. *Let* $\lambda=0$. *If*

$$0<h<h_w(J,K),$$

then

$$\langle\sigma(i)\sigma(j)\rangle^+(J,K,h) > \langle\sigma(i)\sigma(j)\rangle^-(J,K,h)$$

for all $i, j \in \Sigma$, $|i-j|=1$. *In particular,* $h_w(J,K)$ *is a strictly decreasing function of* J.

Proof. Were this not the case, then by the method of [17], $\langle\sigma(i)\sigma(j)\rangle^+ = \langle\sigma(i)\sigma(j)\rangle^-$, for all $i, j \in \Sigma$. These two states have clustering properties [(3.10), I].

This implies that

$$\langle\sigma(i)\sigma(j)\rangle^+ = \langle\sigma(i)\sigma(j)\rangle^- \to (\langle\sigma(i)\rangle^+)^2 = (\langle\sigma(i)\rangle^-)^2$$

when $|i-j|\to\infty$. But this is impossible, since $\langle\sigma(i)\rangle^{+}-\langle\sigma(i)\rangle^{-}>0$ and $\langle\sigma(i)\rangle^{+}+\langle\sigma(i)\rangle^{-}>0$, for $0<h<h_w(J,K)$. [To prove the last statement, we add an external field $-\mu t(j)$, for all j, $|i-j|=1$, and let $\mu\to\infty$; we get a lower bound which is strictly positive.] We have that

$$\tau_w^{\pm}(J_2,K,h)-\tau_w^{\pm}(J_1,K,h)$$
$$=\tfrac{1}{2}\sum_{j:\langle ij\rangle\subset\Sigma}\int_{J_1}^{J_2}dx(\langle\sigma(i)\sigma(j)\rangle^{+}(x,K,h)-\langle\sigma(i)\sigma(j)\rangle^{-}(x,K,h)). \tag{3.30}$$

Equation (3.30) is therefore strictly positive and, from (3.19), $h_w(J,K)$ is strictly decreasing.

Let us fix the coupling constants and let us introduce explicitly the inverse temperature β. We define

$$h_w(\beta)=\inf\{h:\tau_w^{\pm}(\beta J,\beta K,\beta h)=\tau^{\pm}(\beta K)\}. \tag{3.31}$$

We want to discuss the function $h_w(\beta)$ as a function of β.

For $J\geqq K$ and $h\geqq K$, we have (see (2.10) and [7]) that

$$\tau_w^{\pm}(J,K,h)=\tau^{\pm}(K). \tag{3.32}$$

This gives

$$h_w(\beta)\leqq K, \quad \text{for} \quad J\geqq K. \tag{3.33}$$

Using the upper bound (3.18), we get

$$\tfrac{1}{2}\tau^{\pm}(\beta K)\leqq h_w(\beta). \tag{3.34}$$

This bound is independent of J. Notice that $\beta\tau^{\pm}(\beta K)-2\beta K$ is an analytic function of the variable $e^{-2\beta K}$, for $\mathrm{Re}\,\beta$ large, [18]

$$\beta\tau^{\pm}(\beta K)-2\beta K=G_d(t), \qquad t=e^{-2\beta K}. \tag{3.35}$$

For $d=2$, Onsager computed $G_2(t)$,

$$G_2(t)=-\ln\frac{1+t}{1-t}=-2t+O(t^3). \tag{3.36}$$

For $d=3$, we have (see [18])

$$G_d(t)=-2t^{2(d-1)}+O(t^{2d}). \tag{3.37}$$

From these results and (3.34) we get

$$\lim_{\beta\to\infty} h_w(\beta)\geqq K. \tag{3.38}$$

Let $0\leqq J\leqq K$ and $h=K$. Let us first consider the two-dimensional model. We propose to prove that

$$\tau_w^{\pm}(\beta J,\beta K,\beta K)<\tau^{\pm}(\beta K) \tag{3.39}$$

for $J<K$ and β large enough. This follows from (3.24). Since we know the exact value of $\tau^{\pm}$, for $d=2$, it is possible to give a good upper bound for $\beta^{*}(J)$, such that

$\beta > \beta^*(J)$ implies (3.39). Let $K = h = 1$. From (3.24) we get the upper bound

$$\begin{aligned}\frac{1}{\beta}\tau_w^{\pm}(\beta J, \beta, \beta) &\leqq \frac{1}{\beta}\ln(2\,\mathrm{ch}\,2\beta) - \frac{1}{\beta}\ln(1+e^{-2\beta J}) \\ &= \frac{1}{\beta}\ln\frac{e^{2\beta}+e^{-2\beta}}{1+e^{-2J\beta}} = 2 + \frac{1}{\beta}\ln\frac{1+e^{-4\beta}}{1+e^{-2J\beta}}.\end{aligned} \tag{3.40}$$

Furthermore,

$$\frac{1}{\beta}\tau^{\pm}(\beta) = 2 + \frac{1}{\beta}\ln \mathrm{th}\,\beta = 2 + \frac{1}{\beta}\ln\frac{1-e^{-2\beta}}{1+e^{-2\beta}}. \tag{3.41}$$

Given J, with $0 \leqq J < 1$, we must find β such that

$$\ln\frac{1-e^{-2\beta}}{1+e^{-2\beta}} > \ln\frac{1+e^{-4\beta}}{1+e^{-2J\beta}}. \tag{3.42}$$

Inequality (3.42) is equivalent to

$$\frac{1-e^{-2\beta}}{1+e^{-2\beta}} > \frac{1+e^{-4\beta}}{1+e^{-2J\beta}}, \tag{3.43}$$

which is equivalent to

$$e^{-2J\beta}(1-e^{-2\beta}) > e^{-2\beta}(2+e^{-2\beta}+e^{-4\beta}). \tag{3.44}$$

Taking the logarithm in (3.44), we get

$$-2\beta J + \ln(1-e^{-2\beta}) > -2\beta + \ln(2+e^{-2\beta}+e^{-4\beta}), \tag{3.45}$$

which gives finally

$$J < 1 + \frac{1}{2\beta}\ln\frac{(1-e^{-2\beta})}{(2+e^{-2\beta}+e^{-4\beta})} \equiv 1 + G(\beta). \tag{3.46}$$

The function $G(\beta)$ is negative, monotone increasing in β, and $\lim_{\beta\to\infty} G(\beta) = 0$. Let $\tilde{\beta}$ be defined by $J = 1 + G(\tilde{\beta})$. From the above results we conclude that $\tilde{\beta}(J) \geqq \beta^*(J)$. The value of $\tilde{\beta}(J)$ is given in Fig. 3 of Sect. 2.1.

Remark. Let $d = 2$ and $J < K/2$. The wetting transition occurs for $h_w(\beta) \equiv h_w(\beta; J, K)$. If $0 < 2K - 2J - h$, then, for β large enough, there is only partial wetting, and consequently $h_w(\beta) \geqq h$. This can be seen from (3.24). In particular, for $J < K/2$, $\lim_{\beta\to\infty} h_w(\beta) \geqq 2K - 2J$.

Let $d \geqq 3$ and $h = K$. We use the upper bound (3.28),

$$\begin{aligned}\frac{1}{\beta}\tau^{\pm}(\beta J, \beta K, \beta K) \leqq 2K &+ \frac{1}{\beta}\ln(1+\exp[-4(d-1)\beta J - 4\beta K]) \\ &- \frac{1}{\beta}\ln(1+\exp(-4(d-1)\beta J)).\end{aligned} \tag{3.47}$$

From (3.35) and (3.37) we have

$$\frac{1}{\beta}\tau^{\pm}(\beta K) = 2K - \frac{1}{\beta}O(e^{-4(d-1)\beta K}). \tag{3.48}$$

Comparison of (3.47) and (3.48) shows the existence of $\beta^*(J) < \infty$, for all $J < K$, such that $\beta > \beta^*(J)$ implies inequality (3.39) and hence partial wetting.

The last case we propose to discuss is $J = 0$ and $d = 2$. We can analyze this system by integrating over the spins $\sigma(i)$, $i \in \Sigma$, and computing the resulting effective field. In this way, we reduce the analysis for $J = 0$ to the analysis of the model with $J = K$ and $h = h_{\mathrm{ef}}$. This first step is valid for $d \geqq 2$. Then we must compare h_{ef} with $h_w(K, K)$. For $d = 2$, this can be done, since $h_w(K, K)$ is known explicitly. Let us compute h_{ef}. We must find a constant c and h_{ef} such that

$$e^h e^{K\sigma(j)} + e^{-h} e^{-K\sigma(j)} = c e^{h_{\mathrm{ef}}\sigma(j)}, \qquad \sigma(j) = \pm 1. \tag{3.49}$$

The value of c is not important for us, since it does not affect the value of $\tau_w^{\pm}$. One shows that h_{ef} is a solution of the equation

$$\tanh(h_{\mathrm{ef}}) = \tanh(K) \cdot \tanh(h). \tag{3.50}$$

One then finds

$$e^{-2h_{\mathrm{ef}}} = \frac{1 - \tanh(K)\tanh(h)}{1 + \tanh(K)\tanh(h)} = \frac{e^{-2K} + e^{-2h}}{1 + e^{-2K} e^{-2h}}. \tag{3.51}$$

Now, let $K = 1$, $h = (1 + \alpha)$, and let $h_{\mathrm{ef}} = h^*(\beta) \cdot \beta$, with

$$e^{-2h^*(\beta) \cdot \beta} = \frac{e^{-2\beta}(1 + e^{-2\alpha\beta})}{1 + e^{-2\alpha\beta} e^{-4\beta}}. \tag{3.52}$$

From (3.52) we have

$$h^*(\beta) = 1 - \frac{1}{2\beta} \ln \frac{1 + e^{-2\beta\alpha}}{1 + e^{-2\beta\alpha} e^{-4\beta}} \geqq 1 - \frac{1}{2\beta} \ln(1 + e^{-2\beta\alpha}) \geqq 1 - \frac{1}{2\beta} e^{-2\beta\alpha}. \tag{3.53}$$

So far, our analysis is valid for any $d \geqq 2$. If $J = K = 1$, $d = 2$, $h_w(\beta)$ is a solution of

$$e^{2\beta}(\operatorname{ch} 2\beta - \operatorname{ch} 2h_w \cdot \beta) = \operatorname{sh} 2\beta. \tag{3.54}$$

An analysis of the solution of (3.54) shows that for large β,

$$h_w(\beta) \leqq 1 - \frac{1}{2\beta} e^{-2\beta}. \tag{3.55}$$

From (3.53) and (3.55) we see that, for large β and $\alpha \geqq 1$, $h^*(\beta)$ is always larger than $h_w(\beta)$. Therefore, for those β, there is complete wetting and thus, by (3.38),

$$\lim_{\beta \to \infty} h_w(\beta; J = 0) = 2K. \tag{3.56}$$

Finally we prove the equivalence of the two definitions (2.12) and (2.27) of $h_w(J, K)$. If $\tau_w^{\pm}(h) = \tau^{\pm}$, then, for all $h' > h$, we must have that $\langle \sigma(i) \rangle^+(h') - \langle \sigma(i) \rangle^-(h') = 0$, since $\langle \sigma(i) \rangle^+(h') - \langle \sigma(i) \rangle^-(h')$ is a non-negative, monotone decreasing function [see (2.11)]. We must prove that $\tau_w^{\pm}(h) < \tau^{\pm}$ implies that $\langle \sigma(i) \rangle^+(h) - \langle \sigma(i) \rangle^-(h) > 0$. The statement follows from (2.11) and the fact that $\tau_w^{\pm}(h)$ is a concave function of h, with $\lim_{h \uparrow \infty} \tau_w^{\pm}(h) = \tau^{\pm}$ (property e) of $\tau_w^{\pm}$). It remains to prove property e). Since $\tau_w^{\pm}(J, K, h) \geqq \tau_w^{\pm}(0, K, h)$, it is sufficient to prove property e) for $J=0$. But $\tau_w^{\pm}(0,K,h)=\tau_w^{\pm}(K,K,h_{\mathrm{ef}})$ and $\lim\limits_{h\uparrow\infty} h_{\mathrm{ef}}(h,K)=K$, see (3.50). Since by (3.32) $\tau^{\pm}=\tau_w^{\pm}(K,K,K)$, the proof is complete.

3.5. Uniqueness of the Gibbs State for the Model with $J=K=h$

We first prove the uniqueness of the Gibbs state for $J\geqq K$ and $h\geqq K$. We know that $\tau^{\pm}(K)=\tau_w^{\pm}(J,K,h)$ if $h\geqq K$ and $J\geqq K$, see (3.32). Therefore, if $h>K$ we can apply Lemma 3.1. Using duplicate variables, as in Sect. 3.3, we have

$$\begin{aligned}&\langle\sigma(0,2)\rangle^{+}(K,K,h)-\langle\sigma(0,2)\rangle^{-}(K,K,h)\geqq\langle\sigma(0,1)\rangle^{+}(K,K,K)\\&-\langle\sigma(0,1)\rangle^{-}(K,K,K).\end{aligned} \tag{3.57}$$

Indeed, we must simply take $h\uparrow\infty$ to get (3.57). (More precisely this can be done in a finite volume, and then we take the thermodynamic limit.) But, for $h>K$, we have a unique Gibbs state, and therefore

$$\langle\sigma(i)\rangle^{+}(K,K,K)=\langle\sigma(i)\rangle^{-}(K,K,K),\qquad i\in\Sigma, \tag{3.58}$$

and this implies $\langle\cdot\rangle^{+}=\langle\cdot\rangle^{-}$ by Lemma 3.3 (I).

The next lemma summarizes these results.

Lemma 3.5. *Let $\lambda=0$.*

a) *If $J\geqq K, h\geqq K$, there is a unique Gibbs state for any nonzero temperature and any $d\geqq 2$.*

b) *If $h=K, J<K$, there exists a finite $\beta^*(J,d)$ such that, for $\beta>\beta^*(J,d)$, there are two Σ-invariant Gibbs states.*

When there are two ergodic Gibbs states a natural question is whether there are further ergodic Gibbs states. (We consider only states which are Σ-invariant.) We have some partial results on this question. The state $\langle\cdot\rangle^{-}$ is obtained as

$$\langle\cdot\rangle^{-}=\lim_{L,M\to\infty}\langle\cdot\rangle_{L,M}^{-} \tag{3.59}$$

with the boundary condition $\sigma(i)=-1$, for all $i\in\mathbb{L}\backslash\Lambda L,M)$. It is natural to consider other boundary conditions, for example the following ones: $\sigma(i)=+1$ if $i\in\mathbb{L}\backslash\Lambda(L,M)$ and $1\leqq i^d\leqq p$, and $\sigma(i)=-1$, otherwise. Notice that for $h\geqq K$ this boundary condition gives a ground state such that the first p layers have $\sigma(i)=+1$ and otherwise $\sigma(i)=-1$ (the bulk field $\lambda=0$). Let $\langle\cdot\rangle'$ be the corresponding Gibbs state.

We notice that the free energy $F'(J,K,h)$ computed with this boundary condition is equal to $F^{-}(J,K,h)$, since for any $d\geqq 2$, the energy difference between these two boundary conditions is $O(L^{d-2})$ for a finite system in $\Lambda(L,M)$. As concave functions of h, F' and F^{-} are differentiable in h, for almost all h. (The exceptional values form a set which is at most countable.) Therefore we can apply Lemma 3.1, and we have, for almost all h,

$$\langle\sigma(i)\rangle^{-}=\langle\sigma(i)\rangle',\qquad i\in\Sigma. \tag{3.60}$$

[We have only used the concavity in this argument.] For $d=2$, we can prove that the two states coincide, since the two states $\langle\cdot\rangle'$ and $\langle\cdot\rangle^{-}$ are equivalent (see [19]). They must therefore coincide, because $\langle\cdot\rangle^{-}$ is an extremal Gibbs state.

3.6. The Contour Γ

We consider the state $\langle \cdot \rangle_L^-$ defined in Sect. 2.1; (with $\lambda = 0$). We recall that this state is obtained by imposing the b.c.: $\sigma(i) = 1$, for all $i^d = 0$ and $|i^k| \leqq L$, $k = 1, \ldots, d-1$, and $\sigma(i) = -1$, otherwise. Using F.K.G. inequalities one proves existence of $\langle \cdot \rangle_L^-$ and that

$$\lim_{L \to \infty} \langle \cdot \rangle_L^- = \langle \cdot \rangle^- .$$

Let $P_L(\Gamma < i_0) \equiv \text{Prob}\{\Gamma < i_0\}$, computed in the state $\langle \cdot \rangle_L^-$ [see (2.16)].

$$\langle \sigma(i_0) \rangle_L^- = -P_L(\Gamma < i_0) + \sum_{\Gamma > i_0} \langle \sigma(i_0) | \Gamma \rangle_L^- P_L(\Gamma) . \tag{3.61}$$

If $\Gamma > i_0$, by F.K.G. inequalities,

$$\langle \sigma(i_0) | \Gamma \rangle_L^- \geqq \langle \sigma(i_0) \rangle^+ , \qquad \Gamma > i_0 . \tag{3.62}$$

Therefore

$$P_L(\Gamma < i_0) \geqq \tfrac{1}{2}(\langle \sigma(i_0) \rangle^+ - \langle \sigma(i_0) \rangle_L^-) \geqq \tfrac{1}{2}(\langle \sigma(i_0) \rangle^+ - \langle \sigma(i_0) \rangle^-) . \tag{3.63}$$

We can express $\langle \sigma(i_0) \rangle_L^-$ in another way: If $\sigma(i_0) = -1$, either $\Gamma < i_0$ or there is a unique closed contour γ surrounding i_0. Clearly

$$\langle \sigma(i_0) \rangle_L^- = 1 - 2P_L(\sigma(i_0) = -1)$$

and

$$P_L(\sigma(i_0) = -1) = P_L(\Gamma < i_0) + \sum_{\gamma \ni i_0} P_L(\gamma) , \tag{3.64}$$

where, in (3.64), we sum over all contours, γ, that are closed and contain i_0. For any $h \geqq 0$, the Peierls estimate yields

$$P_L(\gamma) \leqq e^{-\beta k |\gamma|} \tag{3.65}$$

with $k = \min(J, K)$, and $|\gamma|$ is the number of faces of γ. At low temperature

$$\sum_{\gamma \ni i_0} e^{-\beta k |\gamma|} < \infty . \tag{3.66}$$

and by the dominated convergence theorem

$$\lim_{L \to \infty} \sum_{\gamma \ni i_0} P_L(\gamma) = \sum_{\gamma \ni i_0} P^-(\gamma) , \tag{3.67}$$

where $P^-(\gamma)$ is the probability of γ in the state $\langle \cdot \rangle^-$. [We have that

$$\lim_{L \to \infty} P_L(\gamma) = P^-(\gamma) ,$$

because only finite contours contribute to the sum (3.67) when (3.66) holds.] Since the left-hand side of (3.64) is decreasing in L, we have proved existence of a limit of $P_L(\Gamma < i_0)$, as L tends to infinity. At low temperature, the probability that $\sigma(i_0) = -1$, computed in the state $\langle \cdot \rangle^+$, is given by

$$\sum_{\gamma \ni i_0} P^+(\gamma) , \tag{3.68}$$

where, in (3.68), only finite, closed contours occur. If $\langle\sigma(i_0)\rangle^+ = \langle\sigma(i_0)\rangle^-$, then, at low temperature, we get from (3.64), (3.67), and (3.68)

$$\lim_{L\to\infty} P_L(\Gamma < i_0) = 0 \tag{3.69}$$

since $P^+(\gamma) = P^-(\gamma)$.

4. A Layering Transition

To establish the existence of a layering transition, we must derive three bounds: We require a lower bound, (4.1), on $\langle\sigma(i)\rangle^-(h)$, which implies that $\langle\sigma(i)\rangle^-(h) > 0$, for h large enough, (here $h > K - \lambda$); an upper bound, (4.12), on $\langle\sigma(i)\rangle^-(h)$, which implies that $\langle\sigma(i)\rangle^-(h) < 0$, for small h; finally we need a lower bound on $\langle\sigma(i)\sigma(j)\rangle(h)$ which is uniform in i, j and $h \geqq 0$. This last bound is derived by using reflection positivity in a standard Peierls argument (Lemma 4.1 and 4.2). On the basis of these bounds the proof of existence of the layering transition is given after the proof of Lemma 4.2.

Let $d \geqq 3$ and let $h \geqq 0$, $\lambda \leqq 0$. We derive two bounds on the local magnetization of the first layer Σ in the state $\langle\cdot\rangle^-$. Let $n(i) = \frac{1}{2}(1 + \sigma(i))$. The function $n(i)$ is positive and increasing. If we set $\sigma(j) = -1$, for all j of the second layer ($j^d = 2$), F.K.G. inequalities give

$$\langle n(i)\rangle^-(J, K, h, \lambda) \geqq \langle n(i)\rangle^-_{d-1}(J, h + \lambda - K), \qquad i \in \Sigma, \tag{4.1}$$

where $\langle\cdot\rangle^-_{d-1}(J, \mu)$ is the Gibbs state of the $(d-1)$-dimensional Ising model on Σ, with coupling constant J and external field μ. The upper index indicates that we impose $-$b.c. If, in (4.1), $h + \lambda - K > 0$, there is only one Gibbs state for the $(d-1)$-dimensional Ising model on Σ, and

$$\langle n(i)\rangle^-(J, K, h, \lambda) \geqq \langle n(i)\rangle^+_{d-1}(J, h + \lambda - K) > \tfrac{1}{2}. \tag{4.2}$$

Next, we derive an upper bound. Let us consider the model in the box $\Lambda(L, M)$. By F.K.G. inequalities,

$$\left\langle \prod_{i\in\Sigma(L)} n(i)\right\rangle^-_{L,M} \geqq \prod_{i\in\Sigma(L)} \langle n(i)\rangle^-_{L,M}. \tag{4.3}$$

Furthermore, we have

$$\left\langle \prod_{i\in\Sigma(L)} n(i)\right\rangle^-_{L,M} \leqq \frac{\left\langle \prod_{i\in\Sigma(L)} n(i)\right\rangle^-_{L,M}}{\left\langle \prod_{i\in\Sigma(L)} (1 - n(i))\right\rangle^-_{L,M}}. \tag{4.4}$$

Up to a constant, $\left\langle \prod_i n(i)\right\rangle^-_{L,M}$ is nothing but the partition function of the model on $\Lambda(L, M-1)$, with $J = K$, $h = K$ and bulk field λ.

Similarly, $\left\langle \prod_i (1 - n(i))\right\rangle^-_{L,M}$ is, up to a constant, the partition function of the model on $\Lambda(L, M-1)$, with $J = K$, $h = -K$ and bulk field λ. Thus

$$\left\langle \prod_{i\in\Sigma(L)} n(i)\right\rangle^-_{L,M} \leqq e^{2|\Sigma(L)|(h - |\lambda|)} \frac{Z^-_{L,M-1}(h = K, \lambda)}{Z^-_{L,M-1}(h = -K, \lambda)} e^{O(L^{d-2})}. \tag{4.5}$$

The last factor in (4.5) comes from boundary terms. The ratio of the partition functions can be written as

$$\frac{Z^-_{L,M-1}(K,\lambda)}{Z^-_{L,M-1}(-K,\lambda)} \equiv \exp(-|\Sigma(L)|\alpha^{\pm}_{L,M-1}(K,\lambda)). \tag{4.6}$$

More generally, we define

$$\alpha^{\pm}_{L,M}(h,\lambda) = F^-_{L,M}(K,K,h,\lambda) - F^+_{L,M}(K,K,h,-\lambda). \tag{4.7}$$

The thermodynamic limit, $L\to\infty$ and $M\to\infty$, exists, and we set

$$\alpha^{\pm}(h,\lambda) = \lim_{\substack{L\to\infty \\ M\to\infty}} \alpha^{\pm}_{L,M}(h,\lambda). \tag{4.8}$$

The continuity properties of F^- and F^+ imply that

$$\lim_{\lambda\uparrow 0} \alpha^{\pm}(h,\lambda) = \tau^{\pm}_w(K,K,h). \tag{4.9}$$

For $h \geqq 0$, and $\lambda \leqq 0$, $\alpha^{\pm}(h,\lambda)$ is monotone increasing in h, and it is monotone decreasing in λ. Consequently, for $h \geqq 0$ and $\lambda \leqq 0$,

$$\tau^{\pm}_w(K,K,h) \leqq \alpha^{\pm}(h,\lambda). \tag{4.10}$$

We define

$$\hat{h}_1 = h + \lambda - \tfrac{1}{2}\alpha^{\pm}(K,\lambda). \tag{4.11}$$

Using Lemma 3.1 (I), (4.3), (4.5), and (4.11), we obtain the upper bound

$$\langle n(i)\rangle^-(h,\lambda) = \lim_{L\to\infty} \Big(\prod_{j\in\Sigma(L)} \langle n(j)\rangle^-_{L,M}\Big)^{1/|\Sigma(L)|} \leqq e^{2\hat{h}_1}. \tag{4.12}$$

Remarks. 1) In the same way we prove

$$\langle 1 - n(i)\rangle^-(h,\lambda) \leqq e^{-2\hat{h}_1}. \tag{4.13}$$

2) If we explicitly introduce the temperature, β^{-1}, we define

$$\alpha^{\pm}(\beta) \equiv \frac{1}{\beta}\alpha^{\pm}(\beta K, \beta\lambda) \tag{4.14}$$

and

$$\hat{h}_1(\beta) \equiv h + \lambda - \tfrac{1}{2}\alpha^{\pm}(\beta). \tag{4.15}$$

The bound (4.12) then becomes

$$\langle n(i)\rangle^-(\beta h, \beta\lambda) \leqq e^{2\beta\hat{h}_1(\beta)}. \tag{4.16}$$

We now turn to the proof of a lower bound on $\langle\sigma(i)\sigma(j)\rangle$.

Let us consider the three-dimensional model. We introduce periodic boundary conditions on the boundary of the box $\Lambda(L,M)$: The spins $\sigma(i)$ and $\sigma(j)$ are identified if $i^k = j^k \bmod 2L$, $k=1,2$. We choose $\sigma(i) = -1$ if $i^3 > M$.

Let $\langle\cdot\rangle_{L,M}$ be the Gibbs state defined by these b.c. Let $\alpha = \langle ij\rangle \subset \Sigma$. We define

$$f(\alpha) = 1, \quad \text{if} \quad \sigma(i)\sigma(j) = -1; \qquad f(\alpha) = 0, \quad \text{if} \quad \sigma(i)\sigma(j) = 1. \tag{4.17}$$

Lemma 4.1. *Let $\alpha_1, \ldots, \alpha_n$ be given, in such a way that the segments $\langle i_k j_k \rangle$, $k=1,\ldots,n$, are all parallel. Let $2L=2^P$, for some P. Then*

a) $\langle f(\alpha_k)\rangle_{L,M} \leqq 2^{1/2L}\exp(-2J+K-|h+\lambda|)$,

b) $\left\langle \prod_{k=1}^{n} f(\alpha_k)\right\rangle_{L,M} \leqq (2^{1/2L}\exp(-2J+K-|h+\lambda|))^n$.

Proof. The proof follows from reflection positivity and chessboard estimates [20]. We can use reflection positivity with respect to vertical planes passing through the lines of the lattice Σ or passing through the lines of the lattice Σ^* obtained by a shift of (1/2, 1/2, 0).

In the first case, reflection positivity holds, because the model determines a Markov field. In the second case, reflection positivity holds, because the coupling constants K and J are positive.

From chessboard estimates we get

$$\langle f(\alpha_k)\rangle_{L,M} \leqq \left\langle \prod_{\substack{\alpha \subset \Sigma(L) \\ \alpha \| \alpha_k}} f(\alpha)\right\rangle^{1/|\Sigma(L)|}, \tag{4.18}$$

where $\alpha \| \alpha_k$ means that α is parallel to α_k. The only configurations which contribute to the right-hand side of (4.18) are configurations with the property that, for all lines of $\Sigma(L)$ parallel to α_k, we have alternating spins ± 1. Therefore there are at most $\exp((2L)^{d-2}\ln 2)$ configurations in $\Sigma(L)$ which yield a non-zero congribution. We bound the denominator on the right-hand side of (4.18) from below. If $h-\lambda>0$, we set $\sigma(i)\equiv 1$ for all $i\in\Sigma(L)$. If $h-\lambda<0$, we set $\sigma(i)\equiv -1$ for all $i\in\Sigma(L)$. This yields the lower bound

$$\exp[\{(d-1)J+|h+\lambda|\}\,|\Sigma(L)|]\hat{Z}_{L,M-1}, \tag{4.19}$$

where $\hat{Z}_{L,M-1}$ is the partition function of the model in $\Lambda(L,M-1)$ with $J=K$, $h=K$, if $h-\lambda>0$, and $h=-K$, if $h-\lambda<0$. The contribution of one configuration to the numerator is at most

$$\exp[-(J-(d-2)J)\,|\Sigma(L)|]\tilde{Z}_{L,M-1}, \tag{4.20}$$

where $\tilde{Z}_{L,M-1}$ is the partition function of a model in $\Lambda(L,M-1)$ with $J=K$ and alternating boundary field $\pm K$. We always have

$$\tilde{Z}_{L,M-1}/\hat{Z}_{L,M-1} \leqq \exp K|\Sigma(L)|. \tag{4.21}$$

From (4.18)–(4.21) we conclude Lemma 4.1 a). Part b) follows from chessboard estimates, as in part a).

Lemma 4.2. *Let $h\geqq 0$, $\lambda\leqq 0$. We define $\Theta\equiv 2J-K+|h+\lambda|$. If Θ is large enough there is a constant $C(\Theta)>0$, with $C(\Theta)\to 1$ if $\Theta\to\infty$, such that*

$$\langle\sigma(i)\sigma(j)\rangle(J,K,h,\lambda)\geqq C^2(\Theta), \qquad i,j\in\Sigma.$$

Moreover, either $\langle\sigma(i)\rangle^+(J,K,h,\lambda)\geqq C(\Theta)$ or $\langle\sigma(i)\rangle^-(J,K,h,\lambda)\leqq -C(\Theta)$, $i\in\Sigma$. $\left(\text{Here } \langle\cdot\rangle = \lim_{L,M\to\infty}\langle\cdot\rangle_{L,M}\right)$.

Proof. This follows from a standard Peierls argument. Clearly

$$\langle\sigma(i)\sigma(j)\rangle = 1-2\,\mathrm{Prob}\{\sigma(i)\neq\sigma(j)\}. \tag{4.22}$$

This probability can be estimated in the usual way by Lemma 4.1. If $\Theta \to \infty$, $\mathrm{Prob}\{\sigma(i) \neq \sigma(j)\} \to 0$. The second part of the lemma follows from the ergodic decomposition of $\langle \cdot \rangle$ into ergodic Σ-invariant states. We also use the inequalities

$$\langle \sigma(i) \rangle^{+}(J, K, h, \lambda) \geqq \langle \sigma(i) \rangle'(J, K, h, \lambda) \geqq \langle \sigma(i) \rangle^{-}(J, K, h, \lambda), \tag{4.23}$$

where in (4.23) $\langle \cdot \rangle'$ is an arbitrary Gibbs state. Indeed, for each ergodic state $\langle \cdot \rangle'$ which contributes to the decomposition of $\langle \cdot \rangle$, we have

$$\lim_{|i-j| \to \infty} \langle \sigma(i)\sigma(j) \rangle' = (\langle \sigma(i) \rangle')^2 . \tag{4.24}$$

Therefore there exists at least one ergodic state $\langle \cdot \rangle'$ in the decomposition of $\langle \cdot \rangle$, with the property that

$$(\langle \sigma(i) \rangle')^2 \geqq C(\Theta)^2 . \tag{4.25}$$

From (4.25) and (4.23) we conclude the second part of Lemma 4.2

We now have all bounds which we need in order to establish the existence of a layering transition line. Let J, K and $\lambda \leqq 0$ be given. We introduce the inverse temperature β. The parameter Θ becomes

$$\Theta(\beta) = 2\beta J - \beta K + \beta |h + \lambda| . \tag{4.26}$$

We choose β large enough, so that $\Theta(\beta)$ satisfies the hypothesis of Lemma 4.2, for any $h \geqq 0$, and any $\lambda \leqq 0$. This is possible as long as $2J > K$. We define, for fixed β, $i \in \Sigma$,

$$h^{+}(\beta) = \inf\{h : \langle \sigma(i) \rangle^{+}(\beta J, \beta K, \beta h, \beta \lambda) \geqq C(\Theta)\} \tag{4.27}$$

and

$$h^{-}(\beta) = \sup\{h : \langle \sigma(i) \rangle^{-}(\beta J, \beta K, \beta h, \beta \lambda) \leqq -C(\Theta)\} . \tag{4.28}$$

These two quantities are well-defined, because $\langle \sigma(i) \rangle^{+}$ and $\langle \sigma(i) \rangle^{-}$ are increasing in h. By Lemma 4.2, we must have $h^{+} \leqq h^{-}$. We now use the hypothesis

$$F^{+}(J, K, h, \lambda) = F^{-}(J, K, h, \lambda), \qquad \lambda \neq 0 . \tag{4.29}$$

By concavity in h we have, *for* $\lambda < 0$ (see proof of Lemma 3.1)

$$\langle \sigma(i) \rangle^{+}(J, K, h, \lambda) = \langle \sigma(i) \rangle^{-}(J, K, h, \lambda) \quad \text{a.s.} \tag{4.30}$$

Therefore we must have $h^{+} = h^{-}$ for $\lambda < 0$, since it is impossible that $h^{+} < h^{-}$ by (4.30).

We define, for $\lambda < 0$,

$$h_l(\beta) \equiv h^{+} = h^{-} . \tag{4.31}$$

By continuity of $\langle \cdot \rangle^{-}$,

$$\langle \sigma(i) \rangle^{-}(h = h_l) \leqq -C(\Theta) , \tag{4.32}$$

and

$$C(\Theta) \leqq \langle \sigma(i) \rangle^{+}(h = h_l) \leqq \langle \sigma(0) \rangle^{-}(h'), \qquad h' > h_l . \tag{4.33}$$

[Notice that (4.33) is valid only for $\lambda < 0$, as a consequence of (4.30). For $\lambda = 0$ (4.33) is wrong.] We have an upper bound on h_l, using (4.1).

$$h_l \leqq K + |\lambda| . \tag{4.34}$$

We also have a lower bound using (4.12): Let $\tilde{h}$ be such that

$$\tilde{h}\beta+\lambda\beta-\tfrac{1}{2}\alpha^{\pm}(\beta K,\beta\lambda)<\tfrac{1}{2}\ln\tfrac{1}{2}. \tag{4.35}$$

From (4.11), (4.12) and the definition of h_l, we get

$$h_l \geqq \tilde{h}. \tag{4.36}$$

We now consider the limiting case $\lambda=0$. We have chosen $\Theta(\beta)$ uniformly in $\lambda\leqq 0$. By continuity,

$$\lim_{\lambda\uparrow 0}\langle\cdot\rangle^{-}(J,K,h,\lambda)=\langle\cdot\rangle^{-}(J,K,h,0). \tag{4.37}$$

Therefore the first-order transition line $h_l(\lambda)$, $\lambda<0$ (β is fixed), ends at h_* for $\lambda=0$, and h_* is a point of first-order transition. Moreover,

$$\tilde{h}_* \leqq h_* \leqq K \tag{4.38}$$

with $\tilde{h}_*$ such that

$$\beta\tilde{h}_* - (1/2)\tau^{\pm}(\beta K) < (1/2)\ln(1/2). \tag{4.39}$$

[By F.K.G. inequalities $h_l=h_l(\lambda)$ is decreasing in λ for $\lambda<0$. Thus $h_*=\lim_{\lambda\uparrow 0} h_l(\lambda)$ is well-defined.]

As a consequence of the bounds (4.38), we see that, for J such that $2J>K>J$, and for β large enough, we must have

$$h_* \leqq K < h_w. \tag{4.40}$$

Acknowledgements. It is a pleasure to thank D. B. Abraham, E. Bonomi, and J. Bricmont for very helpful discussions.

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Communicated by E. Lieb

Received February 1, 1987; in revised form February 12, 1987