

Renormalization Group in the Local Potential Approximation

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Abstract. In the local potential approximation, renormalization group equations reduce to a semilinear parabolic partial differential equation. We derive this equation and show the relation with the hierarchical model. We construct a family of non-trivial fixed points u_{2n}^* , $n=2, 3, 4, \dots$, which have the form of n -well potentials and exist in the ranges of dimensions $2 < d < d_n = 2 + 2/(n-1)$. As $d \uparrow d_n$, u_{2n}^* tends to zero. For the Wilson fixed point u_4^* , we give bounds on critical exponents. In the case of dipole gas in this approximation we show that no non-trivial fixed points exist.

1. Introduction

Non-trivial fixed points of the renormalization group (RG) play a crucial role in the understanding of statistical mechanics systems in the vicinity of the critical point [1]. In the case of a symmetric scalar field (a classical statistical mechanics system with one-component order parameter) the non-trivial fixed points are expected to appear as bifurcating from the trivial massless fixed point as one varies continuously the dimension d of space [2]. These bifurcations occur at the thresholds $d_n = 2 + 2/(n-1)$, $n=2, 3, 4, \dots$, where the linearized RG acquires a zero mode (see Fig. 1. The dotted lines represent branches which are believed to be unphysical). The fixed point relevant for three dimensional physics is given by extrapolating to $d=3$ the branch bifurcating at $4=d_2$ dimensions.

This pattern is not well understood from a rigorous point of view, but some pieces of it were established in toy models like Dyson's hierarchical one [3]: Bleher and Sinai [4] proved for this model the existence of a non-trivial fixed point if $d=d_n-\varepsilon$, where $\varepsilon>0$ is small enough. Their result was refined by Collet and Eckmann who proved that the ε -expansion is asymptotic [5]. Gawedzki and Kupiainen considered the case of an N -component spin system in the hierarchical approximation and constructed a fixed point in three dimensions for N large enough [6]. Recently Witter and Koch [7] succeeded in constructing a fixed point

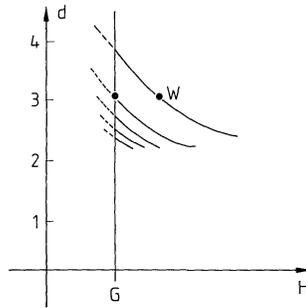


Fig. 1. Fixed points in the space of hamiltonians as functions of the dimension d . G is the Gaussian, W the Wilson fixed point

in three dimensions and $N = 1$ for Gallavotti’s [8] version of the hierarchical model with the help of a computer.

In this paper, we study a continuous scale version of the hierarchical model. It is given by the partial differential equation

$$u_t = \frac{1}{2}u_{xx} - \frac{d-2}{2}xu_x + du - \frac{1}{2}u_x^2, \tag{1.1}$$

describing the flow of the effective potential $u(t, x)$ on momentum scale e^{-t} as a function of the field $x \in \mathbb{R}$. A similar equation was studied numerically by Hasenfratz and Hasenfratz [9] who found a non-trivial fixed point in three dimensions. Brydges and Kennedy [10] also studied similar equations in connection with the Mayer expansion.

Equation (1.1) can be derived in two ways: either as a limit of infinitesimal step of the recursion relation of the hierarchical model, or as the local potential approximation (LPA) to Wilson’s “exact” renormalization group equations [11, 12]. These derivations are given in Sect. 2.

The spectrum of the linearization at $u = 0$ of (1.1) is

$$\{\lambda_n = d - n(d-2)/2, n = 0, 2, \dots\}$$

in

$$L^2_{\text{even}} \left(e^{-\frac{d-2}{2}x^2} dx \right),$$

and thus the bifurcation picture discussed at the beginning of this section should hold. Unfortunately conventional bifurcation analysis does not work here due to the nasty non-linearity, and we have to use other methods.

It turns out that RG fixed points must be identified with *global* stationary solutions of (1.1). Besides the trivial fixed points $u = 0$ and $u = x^2 - 1/d$ (high temperature fixed point), Eq. (1.1) has non-trivial global stationary solutions $u_{2n}^*(x)$ with the form of n -well potentials in the ranges $2 < d < d_n = 2 + 2/(n-1)$, $n = 2, 3, \dots$. These solutions converge pointwise to zero as $d \uparrow d_n$. Their large x behaviour is dictated by (1.1), $u_{2n}^*(x) \sim x^2$ as $x \rightarrow \pm \infty$, as one easily sees, making the ansatz $u(x) \sim Ax^p$ for $x \rightarrow \infty$. These results as well as some remarks on dynamics are contained in Sect. 3. Another equation which could be simpler to analyse as far as the

dynamics is concerned is the LPA of the dipole gas. In the sine-Gordon representation the field x of canonical dimension $\frac{d-2}{2}$ is replaced by a d -dimensional vector x^i (the gradient of the field) with canonical dimension $\frac{d}{2}$. The equation becomes

$$u_t = \frac{1}{2}\Delta u - \frac{d}{2}x \cdot \nabla u + du - \frac{1}{2}(\nabla u)^2. \quad (1.2)$$

(here d has to be taken integer). In this case the methods of Giga and Kohn [13] can be used to show that no non-trivial fixed points exist (Theorem 3.3).

The paper is organized as follows: in Sect. 2 we derive (1.1) from the LPA of Wilson's equation and from the hierarchical model as the scale factor $L \rightarrow 1$. In Sect. 3, we discuss the equation and state the results. The φ^4 fixed point u_4^* is constructed in Sect. 4, and the general construction of u_{2n}^* is in Sect. 5.

A short version of these results appeared in [14].

2. Derivation of the Partial Differential Equation

The P.D.E. we study in this paper can be understood in two ways: it is the local potential approximation to Wilson's continuous parameter renormalization group equation, and it is also the $L \rightarrow 1$ limit of the recursion relation of the hierarchical model, where L is the scale parameter (the size of the block).

(a) *The Local Potential Approximation.* Renormalization group equations describe the change of the effective hamiltonian as the scale is varied. The effective hamiltonian can be written as a sum of the Gaussian fixed point and a potential, which is, in general, non-local. The local potential approximation (LPA) consists in projecting at each renormalization step the potential onto its local part. We describe this approximation in detail for Wilson's "exact renormalization group" equations [11]

$$\frac{\partial H}{\partial t} = \int_q h(q^2) \left[\frac{\delta H}{\delta \varphi_q \delta \varphi_{-q}} - \frac{\delta H}{\delta \varphi_q} \frac{\delta H}{\delta \varphi_{-q}} + \varphi_q \frac{\delta H}{\varphi_q} \right] + \int_q \left(\frac{d}{2} \varphi_q + q \nabla_q \varphi_q \right) \frac{\delta H}{\delta \varphi_q}, \quad (2.1)$$

$$h(q^2) = 1 - \frac{1}{2}\eta + 2q^2,$$

but the same result can be obtained from other RG schemes with continuous scale parameter, such as Polchinski's [15] (with a suitable cut-off function). The notation in (2.1) is the following: $H = H(t, \{\varphi_q\})$ is the effective hamiltonian on scale e^{-t} expressed as function of the Fourier components φ_q of the field variable. The 2π -conventions are:

$$\int_q \equiv \int \frac{d^d q}{(2\pi)^d}, \quad \int_q \delta_{q-a_0} f(q) = f(q_0), \quad \frac{\delta \varphi_{q_1}}{\delta \varphi_{q_2}} = \delta_{q_1 - q_2}. \quad (2.2)$$

We neglect systematically φ independent quantities on the right-hand side of (2.1), which only affect the zero point energy and are thus irrelevant to the computation of expectations. We set furthermore $\eta = 0$ for consistency, since wave function renormalization is not present in this approximation.

The perturbative treatment of (2.1) was developed by Wilson and Kogut [11] and Wegner [12]. We recall here some of their calculations: the Gaussian critical fixed point of (2.1) is, for $\eta=0$,

$$H_G^* = \frac{1}{2} \int_q V_2^*(q) \varphi_q \varphi_{-q}, \quad V_2^*(q) = \frac{q^2}{q^2 + e^{-2q^2}}. \quad (2.3)$$

The linearization of the generator of (2.1) at H_G^* ,

$$L = \int_q h(q^2) \frac{\delta^2}{\delta \varphi_q \delta \varphi_{-q}} + \int_q \left(\frac{d}{2} h(q^2) (1 - 2V_2^*(q)) + q \nabla_q \varphi_q \right) \frac{\delta}{\delta \varphi_q}, \quad (2.4)$$

has the translation invariant eigenfunctions

$$\mathcal{O}_{nr} = \int_{q_1 \dots q_n} P_r(q_1, \dots, q_{n-1}) : \prod_{i=1}^n (f(q_i^2) \varphi_{q_i}) : \delta_{\sum_{i=1}^n q_i}, \quad (2.5)$$

to the eigenvalues

$$\lambda_{nr} = \frac{d-2}{2} n + |r| - d, \quad (2.6)$$

where r labels the homogeneous polynomials P_r , and $|r|$ is the degree of P_r . The function

$$f(q^2) = \frac{e^{-q^2}}{q^2 + e^{-2q^2}}$$

plays the role of the cut-off function in this approach. Wick-ordering is defined with respect to the covariance $V_2^*(q)^{-1}$:

$$: e^{\int_q S_q \varphi_{-q}} : = e^{\int_q S_q \varphi_{-q} + \frac{1}{2} \int_q S_q V_2^*(q)^{-1} S_{-q}}. \quad (2.7)$$

The LPA is given by a projection of Eq. (2.1) onto the space spanned by the local operators \mathcal{O}_{n0} . Acting on field monomials the projection operator is defined by taking the kernel at zero momentum:

$$P \int_{q_1 \dots q_n} V(q_1, \dots, q_{n-1}) \prod_{i=1}^n \varphi_{q_i} f(q_i^2) \delta_{\Sigma q_i} = V(0, \dots, 0) \int \prod_{i=1}^n \varphi_{q_i} f(q_i^2) \delta_{\Sigma q_i}. \quad (2.8)$$

The equation for the potential $V = H - H_G^*$ is

$$\begin{aligned} \frac{\partial V}{\partial t} = & \int_q h(q^2) \frac{\delta^2 V}{\delta \varphi_q \delta \varphi_{-q}} + \int_q \left[\left(\frac{d}{2} + h(1 - 2V_2^*) \right) \varphi_q + q \nabla_q \varphi_q \right] \frac{\delta V}{\delta \varphi_q} \\ & - \int h(q^2) \frac{\delta V}{\delta \varphi_q} \frac{\delta V}{\delta \varphi_{-q}}. \end{aligned} \quad (2.9)$$

To compute the LPA we assume that V is in the range of P and act with P on the right-hand side of (2.9). The practical way of doing this is to expand V into \mathcal{O}_n 's:

$$V = \sum \mu_n \mathcal{O}_n, \quad \mathcal{O}_n = \int \sum_{i=1}^n (\varphi_i f(q_i^2)) \delta_{\Sigma q_i} \quad (2.10)$$

and compute the various terms explicitly:

$$P \int_q h(q^2) \frac{\delta^2 \mathcal{O}_n}{\delta \varphi_q \delta \varphi_{-q}} = \left(\int_q h(q^2) f(q^2)^2 \right) n(n-1) \mathcal{O}_{n-2}, \quad (2.11a)$$

$$P \int_q g(q^2) \varphi_q \frac{\delta \mathcal{O}_n}{\delta \varphi_q} = g(0) n \mathcal{O}_n, \quad (2.11b)$$

$$\begin{aligned} P \int_q q V_q \varphi_q \frac{\delta \mathcal{O}_n}{\delta \varphi_q} &= P \left. \frac{d}{d\lambda} \right|_{\lambda=1} \mathcal{O}_n(\varphi_{\lambda q}) \\ &= P \left. \frac{d}{d\lambda} \right|_{\lambda=1} \lambda^{-d(n-1)} \mathcal{O}_n(\varphi_q) = -d(n-1) \mathcal{O}_n(\varphi_q), \end{aligned} \quad (2.11c)$$

$$P \int h(q^2) \frac{\delta \mathcal{O}_n}{\delta \varphi_q} \frac{\delta \mathcal{O}_m}{\delta \varphi_{-q}} = n \mathcal{O}_{n-1} m \mathcal{O}_{m-1}, \quad (2.11d)$$

where (2.11c) was computed by scaling variables in the integral over q in (2.10) and by using the fact that $f^{-1} V_q f = 0$ at $q=0$. We then identify $V(\{\varphi_q\}) = \sum_n \mu_n \mathcal{O}_n(\varphi)$ with the function in one variable $V(\varphi) = \sum_n \mu_n \varphi^n$ (the “density” of the local potential)

$$\frac{\partial V}{\partial t} = A \frac{\partial^2 V}{\partial \varphi^2} - \frac{d-2}{2} \varphi \frac{\partial V}{\partial \varphi} + dV - \left(\frac{\partial V}{\partial \varphi} \right)^2, \quad A = \int_q h(q^2) f(q^2)^2. \quad (2.12)$$

Equation (1.1) is obtained from (2.12) by the transformation

$$V(t, \varphi) = A^{-1} u(t, (A/2)^{1/2} \varphi).$$

(b) *The $L \rightarrow 1$ Limit of the Hierarchical Model.* The hierarchical model [3] in the version introduced by Gallavotti [8] reduces to the recursion relation which maps the effective potential on scale μ to the effective potential on scale $L^{-1}\mu$ given by

$$R_L : u(\mu, \cdot) \rightarrow u(L^{-1}\mu, \cdot), \quad (2.13)$$

$$u(L^{-1}\mu, x) = -\ln \int \exp[-L^d u(\mu, L^{1-d/2}x + z) - z^2/2(L-1)] \frac{dz}{\sqrt{2\pi(L-1)}}.$$

Here $x \in \mathbb{R}$ is the field (classical spin) variable. This recursion relation defines effective potentials on the discrete set of scales $L^{-k}\mu$, $k=0, 1, 2, \dots$, where μ is some cut-off. In the limit $L \rightarrow 1$ this recursion relation becomes a differential equation

$$\mu \frac{\partial u}{\partial \mu} = \lim_{L \rightarrow 1} \frac{u(L^{-1}\mu, x) - u(\mu, x)}{L^{-1} - 1} = - \left. \frac{d}{d\lambda} \right|_{\lambda=1} (R_L u)(\mu, x). \quad (2.14)$$

The factors $(L-1)$ in (2.13), (which are irrelevant as long as L is a fixed number larger than one) were chosen so that the limit $L \rightarrow 1$ of the right-hand side of (2.13) is $u(\mu, x)$. Taking the derivative of the right-hand side of (2.13) with respect to L at $L=1$ is straightforward. The result is (1.1) in the variable $\mu = e^{-t}$:

$$-\mu \frac{\partial u}{\partial \mu} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \frac{d-2}{2} x \frac{\partial u}{\partial x} + du - \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2. \quad (2.15)$$

The reader may be confused by the fact that we did not impose a normalization condition on the zero point energy. This could be done by adding a Lagrange multiplier

$$u_t = \frac{1}{2}u_{xx} - \frac{d-2}{2}xu_x + du - \frac{1}{2}u_x^2 + \theta(t) \tag{1.1'}$$

to enforce, e.g., the condition $u(t, 0) = 0$. But solutions of (1.1') are in a trivial one-to-one correspondence with solutions of (1.1) obeying $u(0, 0) = C$, for any fixed constant C . Indeed if u is a solution of (1.1) with this property,

$$\tilde{u}(t, x) = u(t, x) - u(t, 0), \quad \theta(t) = -u_t(t, 0) + du(t, 0) \tag{2.16}$$

is a solution of (1.1'). The inverse map is

$$u(t, x) = \tilde{u}(t, x) + e^{dt}C - \int_0^t e^{d(t-s)}\theta(s)ds. \tag{2.17}$$

3. Discussion of the Equation, Results

We begin this section by discussing some elementary properties of the partial differential equation derived in the previous section. The conclusion will be that non-trivial fixed points must be identified with the *global* stationary solutions of (1.1). We then state our main result (Theorem 3.2) and conclude with some remarks and further results.

At first sight, the fixed point equation

$$u''(x) - (d-2)xu'(x) + 2du(x) - u'(x)^2 = 0 \tag{3.1}$$

has a lot of solutions, parametrized by the initial condition $u(0)$ [$u'(0) = 0$ if u is even]. Note however that (3.1) admits solutions that blow up (like $-\ln|x - x_0|$) at finite value x_0 of the field variable x . A solution of (3.1) going to infinity at x_0 corresponds to a Gibbs factor $p = \exp(-u)$ which vanishes at x_0 . The following theorem states that all fixed points which are limits for $t \rightarrow \infty$ of effective Gibbs factors $q(x, t) = \exp(-u(x, t))$ with $q(x, 0) > 0$ are positive. Thus the relevant fixed points are global solutions of (3.1).

Theorem 3.1. (i) Let $T > 0$, $d > 2$, and let $q(t, x)$ be a bounded classical solution on $[0, T] \times \mathbb{R}$ of

$$q_t = \frac{1}{2}q_{xx} - \frac{d-2}{2}xq_x + dq \ln|q| \tag{3.2}$$

with positive initial data $q(0, x)$. Then $q(t, x)$ is positive.

(ii) Let q^* be a bounded non-negative stationary $C''(\mathbb{R})$ solution of (3.2). Then either q^* vanishes identically or $q^*(x) > 0$ for all $x \in \mathbb{R}$.

Proof. (i) Let $q = e^{(d-2)x^2/4}g$; then g obeys the equation

$$g_t = \frac{1}{2}g_{xx} - q(x)g, \quad q(x) = \frac{1}{2}\left(\frac{d-2}{2}\right)^2 x^2 - \frac{d-2}{4} - d \ln|q|. \tag{3.3}$$

Since ϱ is bounded, g is L^2 and the “potential” q is bounded below. The claim follows using the Feynman-Kac formula.

(ii) Suppose the claim is false. Then there exists an $x_0 \in \mathbb{R}$ and a $\delta > 0$ such that $\varrho^*(x_0) = 0$, and $\varrho^*(x) > 0$ for $x_0 - \delta < x < x_0$ [or for $x_0 < x < x_0 + \delta$ but the latter case can be reduced to the former by noting that if $\varrho^*(x)$ is a solution, so is $\varrho^*(-x)$]. Since ϱ^* is, by assumption, nonnegative, $\varrho^{*\prime}$ must vanish at x_0 . Let $r^2(x) = \varrho^*(x)^2 + \varrho^{*\prime}(x)^2$. By continuity $r(x) \rightarrow 0$ as $x \uparrow x_0$. We show that this is not possible: r obeys

$$rr' = (d-2)x(\varrho^{*\prime})^2 + \varrho^*\varrho^{*\prime} \left(2d \ln \frac{1}{\varrho^*} + 1 \right). \quad (3.4)$$

For $\varrho > 0$ small enough $\varrho \ln(1/\varrho)$ is positive and increasing in ϱ . Estimate the right-hand side of (3.4) by using $\varrho^* \leq r$, $|\varrho^{*\prime}| \leq r$, and, if δ' is small enough, $\varrho^* \ln(1/\varrho^*) \leq r \ln(1/r)$, for $x_0 - \delta' < x < x_0$,

$$r' \geq -(d-2)|x_0 - \delta'|r - r \left(2d \ln \frac{1}{r} + 1 \right) = -r 2d \ln \frac{D}{r}, \quad D > 1.$$

We get thus the estimate on $r(x_0)$ in terms of $r(x_0 - \delta')$,

$$r(x_0) \geq D \exp \left(e^{2d\delta'} \ln \frac{r(x_0 - \delta')}{D} \right).$$

Thus $r(x_0) > 0$ which is a contradiction. \square

We are now ready to state our results. The proofs are contained in the next two sections.

Theorem 3.2. *Let $2 < d < d_n = 2 + \frac{2}{n-1}$, $n = 2, 3, 4, \dots$. Then there exists an even fixed point u_{2n}^* , the “ ϕ^{2n} fixed point”, (i.e. an even global solution of (3.1)). The function $u_{2n}^*(x)$ has $2n-1$ critical points and grows to infinity as $x \rightarrow \pm \infty$.*

This picture of “ n -well potentials” is in agreement with (the lowest order) ε -expansion which predicts the existence of fixed points

$$u_{2n}^*(x) = c_n \varepsilon H_{2n}((d/2-1)^{1/2}x) + O(\varepsilon^2) \quad \text{for } d = d_n - \varepsilon,$$

where H_{2n} is the $2n^{\text{th}}$ Hermite polynomial.

We cannot establish that the ε -expansion is asymptotic, but we can show that, as $\varepsilon \rightarrow 0^+$, $u_{2n}^*(x) \rightarrow 0$ pointwise in x . We defer the proof of this fact (for $2n=4$) to the end of Sect. 4. Of course, no stronger convergence to zero is expected since the large x asymptotics of u_{2n}^* is x^2 as dictated by (3.1).

A numerical study around $d=4$ shows that the ε -expansion is accurate [in the sense that the initial condition $u_4^*(0)$ predicted by the ε -expansion does correspond to a global solution] for $\varepsilon > 0$. No fixed points corresponding to $\varepsilon < 0$ seem to exist.

Critical exponents are given in terms of eigenvalues of the linearized flow equation at the non-trivial fixed point. The information contained in Theorem 3.2 is sufficient to give some (rough) bound on these eigenvalues: consider the renormalization group equation linearized around the Wilson fixed point u_4^* in $2 < d < 4$ dimensions:

$$d\dot{u} = L A u, \quad L = \frac{1}{2} \frac{d^2}{dx^2} - \left(\frac{d-2}{2} x + u_4^{*\prime} \right) \frac{d}{dx} + d; \quad (3.5)$$

L is selfadjoint in $L^2\left(\mathbb{R}, e^{-\frac{d-2}{2}x^2 - 2u_4^*(x)} dx\right)$. The obvious eigenfunctions 1 to the eigenvalue $\lambda_0 = d$ (corresponding to a trivial rescaling of the zero point energy) and u_4^* to the eigenvalue $\lambda_3 = (d-2)/2$ have, respectively, no zero and three zeros. Therefore there exists an odd eigenfunction with one zero and eigenvalue λ_1 and an even eigenfunction with two zeros and eigenvalue λ_2 such that

$$d = \lambda_0 > \lambda_1 > \lambda_2 > \lambda_3 = \frac{d-2}{2}.$$

[The oscillation theorem holds since L can be represented as minus a Schrödinger operator on $L^2(\mathbb{R}, dx)$.] The critical exponents of the correlation length, susceptibility and magnetization

$$\xi \sim (T - T_c)^{-\nu}, \quad \chi \sim (T - T_c)^{-\gamma}, \quad M(T_c) \sim h^{1/\delta}$$

are given by ($\eta = 0$ in our approximation)

$$\nu = \frac{1}{\lambda_2}, \quad \gamma = 2\nu, \quad \delta = \frac{2\lambda_1}{d-2},$$

and obey the bounds

$$\frac{1}{d} < \nu < \frac{2}{d-2}, \quad 1 < \delta < \frac{2d}{d-2}.$$

Unfortunately, our control on the P.D.E. is not sufficient to say something about the flow in the vicinity of the fixed points beyond the linear approximation, or to construct the basin of attraction of the fixed point. A simple argument, though, gives a necessary condition for a potential to be in the basin of attraction of a fixed point u_{2n}^* . In fact, for “generic” solutions $u(t, x)$ of (1.1), the number of critical points

$$n(t) = \{x | u_x(t, x) = 0\}$$

is a *non-increasing* function of t . To see this consider the set

$$C = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R} | u_x(t, x) = 0\},$$

and let $v(t, x) = u_x(t, x)$. We make the “genericity” assumption that $(v_t, v_x) \neq (0, 0)$ on C . Then C is a collection of curves, locally represented by functions $x(t)$ or $t(x)$. If, at time t_0 , $n(t)$ jumps, there exists a point x_0 such that $t_0 = t(x_0)$ and $t'(x_0) = -v_x(t_0, x_0)/v_t(t_0, x_0) = 0$ and if $n(t)$ increases, $t''(x_0) \geq 0$. The derivative of (1.1)

$$v_t = \frac{1}{2}v_{xx} - \frac{d-2}{2}xv_x + \frac{d+2}{2}v - vv_x \tag{3.6}$$

gives that

$$t''(x_0) = -v_{xx}(t_0, x_0)/v_t(t_0, x_0) = -2 < 0.$$

Thus $n(t)$ cannot increase.

Actually, we cannot expect the condition $(v_t, v_x) \neq 0$ on C to hold for generic even solutions since at $x=0$ by symmetry $v = v_t = 0$ for all t . We can weaken this

condition by requiring $(v_t/x, (v/x)_x) \neq (0, 0)$ on $C^* = \{(t, x) | v(t, x)/x = 0\}$. For $x \neq 0$ this is equivalent to the previous condition. If a new pair of zeros appears at $x = 0$, C^* can be locally described by a function $t(x)$ with $t'(0)$ and $t''(0) \geq 0$. But

$$t''(0) = \lim_{x \rightarrow 0} - \frac{(v/x)_x}{v_t/x} = -\frac{1}{3} \frac{v_{xxx}(0)}{v_{tx}(0)},$$

and taking the partial derivative with respect to x of (3.6) yields $t''(0) = -2/3 < 0$, a contradiction. We conclude that, generically, a necessary condition for a function $u(0, x)$ to be in the basin of attraction of u_{2n}^* is that it has at least $2n - 1$ critical points.

We conclude by stating a result on the dipole gas case (1.2) about non-existence of fixed point. We use a method introduced by Giga and Kohn [13] to study self-similar blowing up solutions of a non linear heat equation.

Theorem 3.3. *Let u be a C'' global stationary solution of (1.2) which is bounded below. Then u vanishes identically.*

Proof. Let $\varrho = \exp[-u((2/d)^{1/2}x)]$. If u is a solution of (1.2) with u_t replaced by 0, ϱ is a positive solution of

$$\nabla(\alpha \nabla \varrho) + 2\alpha \varrho \ln \varrho = 0, \quad (3.7)$$

where

$$\alpha(x) = e^{-x^2/2}.$$

The key step is to use the identities

$$\begin{aligned} d \int (\nabla \varrho)^2 \alpha dx - 2d \int \varrho^2 \ln \varrho \alpha dx &= 0, \\ \int (d - x^2) \varrho^2 \alpha dx - \int x^2 (\nabla \varrho)^2 \alpha dx + 2 \int x^2 \varrho^2 \ln \varrho \alpha dx &= 0, \\ \int (d + x^2) (\nabla \varrho)^2 \alpha dx - \int (d - x^2) \varrho^2 \alpha dx + 2 \int (d - x^2) \varrho^2 \ln \varrho \alpha dx &= 0, \end{aligned} \quad (3.8)$$

which are obtained by multiplying (3.7) by ϱ , $x^2 \varrho$, and $x \cdot \nabla \varrho$, respectively, and integrating over \mathbb{R}^d . Summing up the three identities we get

$$\int (\nabla \varrho)^2 \alpha dx = 0.$$

Thus ϱ is constant and by (3.7) equal to one. \square

This ‘‘miraculous cancellation’’ is peculiar to the borderline case of Eq. (1.2): it depends crucially on the ratio between the coefficient of u and the coefficient of $x \cdot \nabla u$ being -2 . A similar property is shared by the Giga-Kohn equation.

4. The Non-Trivial φ^4 Fixed Point

This section is devoted to the construction of the non-trivial fixed point u_4^* (the φ^4 fixed point) in $2 < d < 4$ dimensions. As pointed out in Sect. 3, a fixed point is a global solution of the ordinary differential equation

$$u''(x) - (d - 2)xu'(x) + 2du(x) - u'(x)^2 = 0. \quad (4.1)$$

It is convenient to take the derivative of this equation and study the equation for $v(x)=u'(x)$:

$$v'' - (d-2)xv' + (d+2)v - 2vv' = 0. \tag{4.2}$$

After the rescaling $v \rightarrow \sqrt{d+2}v(\sqrt{d+2} \cdot)$, the equation is reduced to

$$v'' - \frac{x}{\sigma}v' + v - 2vv' = 0, \quad \sigma = \frac{d+2}{d-2}, \tag{4.3}$$

which we view as a dynamical system on the phase plane (x is “time”):

$$\begin{aligned} v' &= w, \\ w' &= \frac{x}{\sigma}w - v + 2vw. \end{aligned} \tag{4.4}$$

It is an easy exercise to solve this equation in closed form in the limit $\sigma \rightarrow \infty$ ($d \rightarrow 2$)¹, but if $d > 2$ the system is non-autonomous, and more difficult to handle. It is sufficient to construct a global solution $(v(x), w(x))$ for $x \geq 0$, since the continuation for negative x is given by

$$(v(-x), w(-x)) = (-v(x), w(x)),$$

provided the initial condition is such that $v(0) = 0$, (we restrict ourselves with even potentials). Divide the plane into six regions I, II, ..., VI (Fig. 2)

$$\begin{aligned} \text{I} &= \{(v, w) | w \geq \frac{1}{2}, v < 0\}, & \text{II} &= \{(v, w) | w \geq \frac{1}{2}, v \geq 0\}, \\ \text{III} &= \{(v, w) | 0 \leq w < \frac{1}{2}, v < 0\}, & \text{IV} &= \{(v, w) | 0 < w < \frac{1}{2}, v \geq 0\}, \\ \text{V} &= \{(v, w) | w < 0, v \leq 0\}, & \text{V} &= \{(v, w) | w \leq 0, v \geq 0\}. \end{aligned} \tag{4.5}$$

The arrows in Fig. 2 show the direction of the vector field (for $x > 0$), on the boundaries of these regions: one sees that these boundaries can be crossed in only one way by solutions of (4.4). We show more: the solutions leave all regions except IV within finite time.

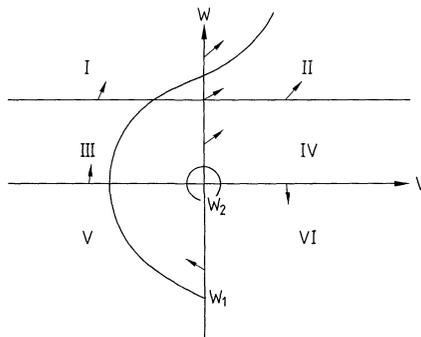


Fig. 2. The partition of the phase plane in six regions

¹ The $\sigma = \infty$ trajectories are given by the equation $(w - \frac{1}{2}) \exp(2(w - v^2)) = J, J \geq -\frac{1}{2}$. They are closed for $-\frac{1}{2} \leq J < 0$ and blow up at finite x for $J > 0$. The trivial fixed points correspond to $J = -\frac{1}{2}, 0$

Proposition 4.1. *Let $1 < \sigma < \infty$, $x_0 > 0$, and $(v_0, w_0) \in A = \text{I, II, III, V or VI}$. Then there exists a time $x_1 > x_0$ (depending continuously on σ, x_0, v_0, w_0) such that if $(v(x), w(x))$ is the solution of (4.4) with $v(x_0) = v_0$ and $w(x_0) = w_0$, $(v(x_1), w(x_1))$ is in a region $B \in \{\text{I, \dots, VI}\}$ such that there is an arrow from A to B in the following diagram:*



The arrow from II to ∞ indicates that all solutions going through II, and only these, blow up in finite time.

Proof. We prove all statements represented by (4.6) separately calling $v(x_0) = v_0$, $w(x_0) = w_0$.

(a) I \rightarrow II. In region I we have $v' \geq \frac{1}{2}$, $w' \leq \frac{x}{\sigma} w$. Thus, as long as $(v, w) \in \text{I}$, $v(x) \geq v_0 + \frac{1}{2}(x - x_0)$, $w \leq w_0 \exp[(x^2 - x_0^2)/\sigma]$, implying that the solution does not blow up in I and that after a time at most $-2v_0$, (v, w) enters II.

(b) II $\rightarrow \infty$. In II, $v' \geq \frac{1}{2}$, thus $v \geq \frac{1}{2}(x - x_0)$ and, if x is larger than $x_0 + \sqrt{2\sigma}$, $xv \geq \sigma$. Let us set, for later purposes, $x'_0 = \max(x_0 + \sqrt{2\sigma}, \sqrt{2\sigma})$. Now either (i) $w(x'_0) > v(x'_0)^2$ or (ii) $w(x'_0) \leq v(x'_0)^2$. In case (i), $w(x) > v(x)^2$ for all $x \geq x'_0$ since, if $w(x) \leq v(x)^2$,

$$\frac{d}{dx}(w - v^2) = \frac{x}{\sigma} w - v \geq \left(\frac{x}{\sigma} v - 1\right) v \geq 0. \tag{4.7}$$

Hence $v'(x) > v^2(x)$ for $x > x'_0$ with escape to infinity in finite time. In the second case, (ii), either at some x''_0 , $w(x''_0) > v^2(x''_0)$ and we are back to case (i), or $w(x) \leq v^2(x)$ for all $x > x'_0$. Then using (4.4) and the fact that $w \geq \frac{1}{2}$ in II, we get

$$w' \geq \frac{x}{\sigma} w + (2w - 1)\sqrt{w} \geq \left(\frac{x}{\sqrt{2\sigma}} - 1\right)\sqrt{w} + 2w^{3/2} \geq 2w^{3/2} \tag{4.8}$$

since, by assumption, $\frac{x}{\sqrt{2\sigma}} \geq 1$; the solution blows up again.

(c) III \rightarrow I, III \rightarrow II, III \rightarrow IV. In III, $v' \geq 0$ and solutions cannot blow up in this region. The bound $w' \geq \frac{x}{\sigma} w$ implies that, if $w_0 > 0$, $w(x) \geq w_0 \exp(x^2 - x_0^2)/2$, and the solution leaves III in finite time. If $w_0 = 0$, the bound $w' \geq -v > 0$ implies that for $x'_0 > x_0$, $w(x'_0) > 0$ and the previous argument can be used.

(d) V \rightarrow III. We only prove the statement for $x_0 = 0$, which is all we need to construct the φ^4 fixed point. The general case will be proven in Sect. 5. Let

$$S_x = \{(v, w) | I(x, v, w) \geq 0\} \cap V, \quad I(x, v, w) = \frac{xw}{1 - 2w} - v. \tag{4.9}$$

We claim:

- (i) For $x=0$, $S_x = V$.
- (ii) If $(v(x), w(x)) \in V$ for $x \in [x_1, x_2]$ and $(v(x_1), w(x_1)) \in S_{x_1}$ then $(v(x), w(x)) \in S_x$ for all $x \in [x_1, x_2]$.
- (iii) All solutions $(v(x), w(x)) \in S_x$ enter III after a finite time.

These properties immediately imply $V \rightarrow III$, if $x_0 = 0$. Proof of (i), (ii), (iii): (i) is obvious, (iii) follows from the fact that, in S_x ,

$$w' = \sigma^{-1}(1 - 2w)I(x, v, w) - (1 - \sigma^{-1})(1 - 2w)v \geq (1 - \sigma^{-1})v. \tag{4.10}$$

If $(v(x_1), w(x_1)) \in S_{x_1}$ for some $x_1 > 0$, $v(x_1) \leq \frac{x_1 w(x_1)}{1 - 2w(x_1)} < 0$, and, if $(v(x), w(x))$ is a solution in S_x , $v(x) \leq v(x_1)$, since $v' = w < 0$, and from (4.10), $w' \geq (1 - \sigma^{-1})|v(x_1)|$ and w becomes positive within finite time. No escape to infinity is possible in this region since by (4.10) $w' \geq 0$, so that w , and thus v' , are bounded below by w_0 . To prove (ii), we show that $I(x, v(x), w(x))$ is non-decreasing in x as long as $(v(x), w(x)) \in S_x$:

$$\frac{d}{dx} I(x, v(x), w(x)) = \frac{2w^2}{1 - 2w} + \frac{xw'}{(1 - 2w)^2} \geq 0, \tag{4.11}$$

by (4.10).

(e) $VI \rightarrow V$. In VI, $v > 0$, $w \leq 0$. Thus, by (4.4), $v' \leq 0$, and $w' < 0$ for $x \geq x_0$, and (v, w) a solution in VI,

$$v(x) \leq v_0, \quad w(x) < w_0. \tag{4.12}$$

These bounds can be inserted again in (4.4) to yield

$$v'(x) \leq w_0, \quad w'(x) \geq \frac{x}{\sigma} w(x) - v_0(1 - 2w(x)). \tag{4.13}$$

The second (linear) bound implies that no solution can blow up in VI, and the first that $v(x)$ becomes negative within finite time, provided $w_0 < 0$. If $w_0 = 0$, we can use the bound $w' \leq -v$, to show that for $x'_0 > x_0$, $w(x'_0) < 0$, and the argument applies.

The next step is to construct two solutions, one of which blows up without crossing the positive v axis and one going once around the origin and crossing the positive v axis. These two solutions are drawn on Fig. 2.

Proposition 4.2 (Construction of w_1). *Let $(v(x), w(x))$ $x \geq 0$ be a solution of (4.4) with initial condition $(0, w_0)$ at $x = x_0 \geq 0$. There exists a constant $w_1 = w_1(\sigma) < 0$ such that if $w_0 \leq w_1$, $(v(x), w(x))$ enters region II (and thus blows up) in a finite time \bar{x}_1 going successively through regions V, III, I.*

Proof. Note first that $J(v, w) = (w - \frac{1}{2}) \exp(2(w - v^2))$ is an increasing Liapunov function. Indeed,

$$\frac{d}{dx} J(v(x), w(x)) = \frac{2x}{\sigma} w^2 e^{2(w - v^2)} \geq 0, \quad x \geq 0, \tag{4.14}$$

as can be checked by explicit calculation. Its level sets are the trajectories of the limiting equation $\sigma \rightarrow \infty$. The solution (v, w) with initial condition $(0, w_0)$ enters III

after finite time. Thus there exists an $x_1 > 0$ such that $w(x_1) = 0$ and $v(x_1) < 0$. Since J is non-decreasing,

$$J(v(x_1), 0) \geq J(0, w_0), \quad v(x_1)^2 \geq |w_0| - \frac{1}{2} \ln(1 + 2|w_0|). \quad (4.15)$$

Thus $v(x_1)$ can be made as negative as desired by choosing $|w_0|$ large enough. In region III, $v' = w \leq \frac{1}{2}$, hence, for $x \geq x_1$, $v(x) \leq v(x_1) + \frac{1}{2}(x - x_1)$. We consider times x such that

$$x - x_1 \leq |v(x_1)|. \quad (4.16)$$

For these times

$$w' \geq \frac{x}{\sigma} w - v_1 \left(\frac{1}{2} - w \right), \quad v_1 \equiv v(x_1). \quad (4.17)$$

As long as $w \leq \frac{1}{4}$, we use the second term on the right-hand side of (4.17):

$$w' \geq \frac{1}{4} |v(x_1)|, \quad w(x) \geq |v_1| \frac{1}{4} (x - x_1), \quad (4.18)$$

so that after a time at most $\frac{1}{|v_1|}$, $w(x) \geq \frac{1}{4}$. For

$$\frac{1}{2} \geq w(x) \geq \frac{1}{4} \quad \text{and} \quad x \geq \frac{1}{|v_1|} + x_1 \equiv x_2,$$

we use the first term in (4.17):

$$w' \geq \frac{x}{4\sigma}, \quad w \geq \frac{1}{4} + \frac{1}{4\sigma} (x^2 - x_2^2), \quad (4.19)$$

and $w(x) \geq \frac{1}{2}$ for some $x \leq x_3 \equiv x_1 + \frac{1}{|v_1|} + \sqrt{\sigma}$, and the solution enters I, provided the condition (4.16) is fulfilled for $x = x_3$, which is the case if $v(x_1)$ is chosen large enough. \square

Proposition 4.3 (Construction of w_2). *Let $3 < \sigma < \infty$ (i.e. $2 < d < 4$) and let $(v(x), w(x))$, $x \geq 0$ be a solution of (4.4) with initial conditions $(0, w_0)$ at $x = 0$. There exist a constant $w_2 = w_2(\sigma) < 0$, such that if $0 > w_0 \geq w_2$, (v, w) enters region VI in a finite time \bar{x}_2 , going successively through regions V, III, IV.*

Proof. Consider first the linear approximation to (4.4):

$$v'' - \frac{x}{\sigma} v' + v = 0. \quad (4.20)$$

We prove in the appendix that the odd solution of this equation has, for $\sigma > 3$ and $x > 0$, at least one zero and at least two critical points. Let x_1 be the smallest positive zero and y_1, y_2 be the smallest positive critical points, then $0 < y_1 < x_1 < y_2$. If the initial condition $w_0 = v'(0)$ is negative, the solution of (4.20) enters VI in time $y_2 + \delta$ going through the path described in the proposition. Choose then $|w_2|$ so small that the solution of the non-linear equation is so close to the solution of the linear equation (4.20) up to time $y_2 + \delta$ that it also has the property of entering VI after visiting V, III, IV. \square

The next step is the actual construction of the fixed point (v^*, w^*) . It is a solution with initial condition lying between w_2 and w_1 , which goes (for $x \geq 0$) through regions V, III to region IV, and stays there forever. In IV $v' = w \leq \frac{1}{2}$, thus no blow up in this region is possible and the solution is global. The function $v^*(x)$, $x \in \mathbb{R}$, has three zeros so that the corresponding solution of (4.1),

$$u^*(x) = \int_0^x \sqrt{d+2v^*(\sqrt{d+2y})} dy - \frac{d+2}{2d} w^*(0)$$

has three critical points. The construction of (v^*, w^*) relies on the so-called Bleher-Sinai [4] argument: to formulate it in a clear way we first modify (4.4) in the regions II and VI. Since the goal is to construct a solution that does not enter these regions, it will not matter how the equation looks like there. The modified equation is

$$v = \begin{cases} w, & \text{if } (v, w) \notin \text{VI}, \\ 0, & \text{if } (v, w) \in \text{VI}, \end{cases} \tag{4.21}$$

$$w' = \begin{cases} \frac{x}{\sigma} w - v + 2vw, & \text{if } (v, w) \notin \text{II}, \\ \frac{x}{\sigma} w, & \text{if } (v, w) \in \text{II}. \end{cases}$$

Note that the vector field (4.21) is continuous except for the negative w axis. This is not a problem since no solutions of (4.21) cross this axis. Equation (4.21) has the two following properties:

- (i) All solutions $(v(x), w(x))$ of (4.21) exist for all $x > 0$.
- (ii) For all $w_1 < w_2 < 0$ there exists an $\bar{x}_3 > 0$ such that all solutions $(v(x), w(x))$ of (4.21) with $v(0) = 0$, $w_1 < w(0) < w_2$, have the property that $v(x) > 0$ for $x > \bar{x}_3$.

Property (i) follows from the fact (Proposition 4.1) that only solutions going through II can escape to infinity in finite time. In this region (4.21) is linear and no escape is possible. Again by Proposition 4.1, the solutions considered in (ii) enter after a finite time x_1 region II or region IV. The vector field was modified in such a way that VI is invariant and no return to $v < 0$ is possible and (ii) holds. Any other modification of (4.4) in VI and II with (i), (ii) can be used for the following argument. The construction of (v^*, w^*) is contained in

Proposition 4.4. *Let $3 < \sigma < \infty$ (i.e. $2 < d < 4$). There exist a $w_0 < 0$ and times $0 < x_1 < x_2$ such that the solution $(v(x), w(x))$, $x \geq 0$ of (4.21) with initial conditions $v(0) = 0$, $w(0) = w_0$, obeys*

$$\begin{aligned} (v, w) \in \text{V}, & \quad 0 \leq x < x_1, \\ (v, w) \in \text{III}, & \quad x_1 \leq x < x_2, \\ (v, w) \in \text{IV}, & \quad x \geq x_2, \end{aligned} \tag{4.22}$$

and is thus a solution of (4.4) as well.

Proof. We consider the interval of initial conditions $I_0 = [w_1, w_2]$, where w_1, w_2 are the constants introduced in Propositions 4.2, 4.3. Let, for $x > 0$, ϕ_{x, x_0} denote the flow generated by (4.21). Choose $x > \max(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, where $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are defined in

Proposition 4.2, 4.3 and (ii), respectively. From (ii) we know that $\phi_{x,0}(0 \times I_0)$ is a continuous curve lying in $\{v > 0\}$. Moreover, by Proposition 4.2, $\phi_{x,0}(0, w_1) \in \Pi$ and, by Proposition 4.3, $\phi_{x,0}(0, w_2) \in \text{VI}$. Thus there exists a piece PQ of the curve $\phi_{x,0}(0 \times I_0)$ lying in IV such that $P \in \{(v, w) | v > 0, w = \frac{1}{2}\}$ and $Q \in \{(v, w) | v > 0, w = 0\}$ (see Fig. 2). Correspondingly there exists a closed interval $I_1 = [w_1^{(1)}, w_2^{(1)}] \subset I_0$, such that $\phi_{x,0}(0 \times I_1) = PQ$. Consider now $\phi_{x+1,0}(0 \times I_1)$:

$$\phi_{x+1,0}(0, w_1^{(1)}) = \phi_{x+1,x}(P) \in \Pi,$$

since, if $w = \frac{1}{2}$, $w' > 0$. Similarly, $\phi_{x+1,0}(0, w_2^{(1)}) \in \text{VI}$, since, if $w = 0$, $w' < 0$. Thus a piece P_1Q_1 of $\phi_{x+1,0}(0 \times I_1)$ lies in IV and

$$P_1 \in \{(v, w) | v > 0, w = \frac{1}{2}\}, \quad Q_1 \in \{(v, w) | v > 0, w = 0\}.$$

This argument can be repeated, the result being that there exist closed intervals $I_0 \supset I_1 \supset I_2 \supset \dots$, such that $\phi_{x+n,0}(0 \times I_{n+1})$ lies in $\overline{\text{IV}}$. The set

$$\bigcap_{n=0}^{\infty} I_n$$

contains (at least) an initial condition with the desired properties. \square

The proof of Theorem 3.2 is finished. In the rest of this section we prove that, as $d \uparrow d_n$, the fixed point u_{2n}^* goes to zero pointwise in x . We write the proof explicitly only for the case $n = 2$. It is sufficient to show that for $d = 4 - \varepsilon$ there exists a $\delta(\varepsilon) > 0$ such that $\delta(\varepsilon) \rightarrow 0$ as $\varepsilon \downarrow 0$ and that the solution of (4.4) with $v(0) = 0$ and $w(0) = -\delta(\varepsilon)$ blows up in finite time going successively through regions V, III, IV, II. This refinement of Proposition 4.2 can then be used with Proposition 4.3 to give (by the same Bleher-Sinai argument of Proposition 4.4) a global solution of (4.4) with

$$-\delta(\varepsilon) \leq w(0) \leq w_2(\varepsilon), \quad v(0) = 0,$$

where $w_2(\varepsilon)$ is the constant appearing in Proposition 4.3.

The idea to prove the existence of the function $\delta(\varepsilon)$ is a perturbative argument to control the solution until it enters region IV together with a monotonicity argument to compare solutions of the non-linear equation to solutions of the linearized equation. Consider first the case $d = 4$ ($\sigma = 3$). We show that all solutions with initial conditions $(0, -\delta)$ and δ sufficiently small blow up in finite time going through V, III, IV, II. To set up perturbation theory we rescale (4.4), $v \rightarrow \delta v$, $w \rightarrow \delta w$:

$$v' = w, \quad w' = \frac{x}{\sigma} w - v + \delta 2vw. \tag{4.23}$$

Let $\phi_{x_2, x_1}^{(\delta)}$ denote the flow of (4.23). For $\delta = 0$, $\phi_{x_2, x_1}^{(0)}$ is linear and given by

$$\phi_{x_2, x_1}^{(0)} = \phi_{x_2, 0}^{(0)} (\phi_{x_1, 0}^{(0)})^{-1}, \quad \phi_{x, 0}^{(0)} = \begin{pmatrix} \tilde{U}_\sigma(x) & \tilde{V}_\sigma(x) \\ \tilde{U}'_\sigma(x) & \tilde{V}'_\sigma(x) \end{pmatrix}, \tag{4.24}$$

where

$$\tilde{U}_\sigma(x) = U_\sigma(\sigma^{-1/2}x), \quad \tilde{V}_\sigma(x) = \sigma^{1/2}V_\sigma(\sigma^{-1/2}x),$$

and U_σ, V_σ are given in the appendix. First order perturbation theory gives

$$\begin{aligned} \phi_{x,0}^{(\delta)} \begin{pmatrix} 0 \\ -1 \end{pmatrix} &= \phi_{x,0}^{(0)} \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \delta \int_0^x \phi_{x,y}^{(0)} \begin{pmatrix} 0 \\ 2\tilde{V}'_\sigma(y)\tilde{V}'_\sigma(y) \end{pmatrix} dy + O(\delta^2) \\ &= \phi_{x,0}^{(0)} \begin{pmatrix} \delta \int_0^x e^{-y^2/2\sigma} (-2\tilde{V}_\sigma(y)^2 \tilde{V}'_\sigma(y)) dy \\ -1 + O(\delta) \end{pmatrix} + O(\delta^2). \end{aligned} \tag{4.25}$$

If $\sigma = 3$, $\tilde{V}_3(y) = -y^3/9 + y$ and

$$-2\delta \int_0^\infty e^{-y^2/6} \tilde{V}(y)^2 \tilde{V}'(y) dy = 12\sqrt{6\pi} > 0. \tag{4.26}$$

Choose $x_1 > 0$ so large that $-\tilde{V}_3(x_1) > 0$ and

$$a(x_1) \equiv -2 \int_0^{x_1} e^{-y^2/6} \tilde{V}^2 \tilde{V}' dy > 0. \tag{4.27}$$

Assume also $x_1 > 6$ for later purposes. Let furthermore

$$z^{(0)}(x) = \phi_{x,x_1}^{(0)} \circ \phi_{x_1,0}^{(\delta)} \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad x \geq x_1, \quad z(x) = \phi_{x,0}^{(\delta)} \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{4.28}$$

The preceding argument implies that

$$z^{(0)}(x) = (-1 + O(\delta)) \begin{pmatrix} \tilde{V}_3 \\ \tilde{V}'_3 \end{pmatrix} + (a(x_1)\delta + O(\delta^2)) \begin{pmatrix} \tilde{U}_3 \\ \tilde{U}'_3 \end{pmatrix}. \tag{4.29}$$

But, by the results in the appendix, $\tilde{V}_3, \tilde{V}'_3 \rightarrow -\infty$ and $\tilde{U}_3, \tilde{U}'_3 \rightarrow +\infty$, as $x \rightarrow \infty$ and if δ is small enough (depending on x_1)

$$z^{(0)}(x) \rightarrow \begin{pmatrix} +\infty \\ +\infty \end{pmatrix}, \quad x \rightarrow \infty.$$

Let now $\Delta z = z - z^{(0)} = \begin{pmatrix} \Delta v \\ \Delta w \end{pmatrix}$. We claim that

$$\Delta v \geq 0, \quad \Delta w \geq 0, \quad \text{for all } x \geq x_1. \tag{4.30}$$

An immediate consequence of (4.30) is that $z(x)$ blows up since it must enter the (rescaled) region II = $\{(v, w) | v \geq 0, w \geq 1/2\delta\}$. The claim is proven by noting that the set

$$S = \{(\Delta v, \Delta w) | \Delta w \geq \Delta v \geq 0\}$$

contains $0 = (\Delta v(x_1), \Delta w(x_1))$ and is invariant if $x \geq x_1$. Indeed, in S

$$\frac{d}{dx} (\Delta w - \Delta v) \geq \frac{x_1}{3} \Delta w - \Delta v - \Delta w \geq \Delta w - \Delta v, \quad \frac{d}{dx} \Delta v \geq \Delta w \geq 0,$$

where we used $x_1 \geq 6$ and the negativity of the non-linearity $-2vw$.

For each $\delta > 0$ sufficiently small we can thus choose a time $x_2(\delta)$ such that $z(x_2)$ is in the interior of region II. If $\varepsilon = 4 - d > 0$ is small depending on δ the solution of (4.4) with initial condition $(0, -\delta)$ still has the property of being in region II after

x_2 . This defines a function $\varepsilon(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ which is the inverse of the desired function $\delta(\varepsilon)$.

5. The Non-Trivial φ^{2n} Fixed Points

In this section, we prove the existence of all φ^{2n} fixed points (Theorem 3.2). In the preceding section we have proven only the part of Proposition 4.1 which is relevant to the construction of the φ^4 fixed point. We first complete the proof of Proposition 4.1, and then construct a solution which winds around the origin $n-1$ times if $2 < d < d_n$, a generalization of Proposition 4.3. We eventually construct the fixed point, replacing the Bleher-Sinai argument by a slightly more abstract one.

Proof of Proposition 4.1 (continued). What is left to prove is $V \rightarrow \text{III}$ in the case where $x_0 > 0$. It is convenient to change variables: define $s = v/x$; (4.4) becomes

$$xs' = w - s, \quad w' = x \left(\frac{1}{\sigma} w - s + 2sw \right). \quad (5.1)$$

Region V is then further partitioned in three parts V_1, V_2, V_3 [note that $(v, w) \in V \Leftrightarrow (s, w) \in V$]:

$$\begin{aligned} V_1 &= \{(s, w) \in V \mid w \geq s\}, \\ V_2 &= \left\{ (s, w) \in V \mid s > w \geq \frac{2\sigma s}{1 + 2\sigma s} \right\}, \\ V_3 &= \left\{ (s, w) \in V \mid w < \frac{2\sigma s}{1 + 2\sigma s} \right\}. \end{aligned}$$

We prove that

$$V_3 \rightarrow V_2 \rightarrow V_1 \rightarrow \text{III}, \quad (5.2)$$

in the notation of Proposition 4.1. Note first that on the boundary $V_{12} = \{s = w\} \cap V$, $s' = 0$, and $w' > 0$, so that the vector field points into region V_1 and this boundary can only be crossed in one direction. Similarly, on $V_{23} = \{w = 2\sigma s(1 + 2\sigma s)^{-1}\} \cap V$, $w' = 0$, $s' < 0$, so that also V_{23} has this property. The arguments to prove (5.2) are very similar to the ones used in Sect. 4 to prove Proposition 4.1. Therefore we will be rather sketchy. We always assume $x_0 > 0$.

(a) $V_1 \rightarrow \text{III}$: In V_1 , $w \leq 0$ implies that

$$s' \leq -\frac{s}{x} \Rightarrow s \leq s_0 \frac{x_0}{x}.$$

Using this and $w \geq s$ we conclude that in V_1 ,

$$w' \geq -(1 - \sigma^{-1})xs \geq -(1 - \sigma^{-1})x_0s_0,$$

since $\sigma > 0$. Thus w becomes positive within finite time.

(b) $V_2 \rightarrow V_1$: In V_2 , $w - s$ is negative but increasing

$$\frac{d}{dx}(w - s) \geq -\frac{(w - s)}{x}.$$

It follows that after a time $x_1 = x_1(w_0, s_0, y_0, \varepsilon)$, $w - s \geq \varepsilon$. For $x > x_1$,

$$w' \geq x \left(\frac{w}{\sigma} - s \right) \geq x_1 [(\sigma^{-1} - 1)s_0 - \varepsilon/\sigma],$$

which is bounded below by a positive constant if ε is small enough. This bound together with $s' > 0$ implies the claim.

(c) $V_3 \rightarrow V_2$: $w' \leq 0 \Rightarrow w \leq w_0$. We then use the bound defining V_3 :

$$s' \leq \frac{1}{x} \min \left(\frac{\sigma s}{1 + 2\sigma s} - s, w_0 - s \right) \leq \frac{1}{x} \min((\sigma - 1)s, w_0 - s)$$

implying that after finite time, (s, w) enters V_2 .

As in Sect. 4, we modify the equation, this time only in region II, to avoid solutions which blow up:

$$v' = w,$$

$$w' = \begin{cases} \frac{x}{\sigma} w - v + 2vw & (v, w) \notin \text{II} \\ \frac{x}{\sigma} w & (v, w) \in \text{II} \end{cases} \quad (5.3)$$

Again this modification does not affect solutions which do not enter II. We consider now solutions with initial conditions (at $x = 0$) on the w -axis and define a winding number as the number of times the solution winds around the origin until it crosses a coordinate axis for the last time. In formula, let $(v, w) \equiv (0, 0)$ be a solution of (5.3) with $v(0) = 0$, and $tg\varphi = \frac{w}{v}$. The winding number,

$$N(v, w) = \frac{1}{2} \left[-\frac{1}{\pi} \int_0^\infty \frac{d\varphi}{dx} dx \right]^2,$$

is well defined since $\frac{d\varphi}{dx} = (v^2 + w^2)^{-1}(wv' - v w')$ is continuous and finite. Since, by

Proposition 4.1, all solutions with finite winding number eventually stay in $\text{II} \cup \text{IV}$, we see that $N \in \mathbb{N}$ if $w(0) > 0$ and $N \in \mathbb{N} + \frac{1}{2}$ if $w(0) < 0$. The φ^4 fixed point constructed in Sect. 4 has $N = \frac{1}{2}$. Theorem 3.2 can be restated as follows: if $2n - 1 < \sigma < \infty$, $n = 2, 3, \dots$, there exists a solution (v_{2n}^*, w_{2n}^*) of (5.3), with $v_{2n}^*(0) = 0$, not entering region II, with winding number

$$N(v_{2n}^*, w_{2n}^*) = \frac{n - 1}{2}.$$

Assume for definiteness that n is odd. The even case can be dealt with the same way. We want to construct a solution of (5.3) with integer winding number and initial condition $(0, w_0 > 0)$ at $x = 0$. View $N(\cdot)$ as a map from the set $\mathbb{R}_+ = \{w_0 | w_0 > 0\}$, with the induced topology, to the integers. This map has the following properties:

² Here, $[\]$ means integer part

Proposition 5.1. *Let $\phi_{x,y}$ be the flow of (5.3).*

- (i) $\{w_0 \in \mathbb{R}_+ \mid N(w_0) = N \text{ and } \phi_{x,0}(0, w_0) \in \text{II for some } x\}$ is open for $N \in \mathbb{N}$.
- (ii) $\{w_0 \in \mathbb{R}_+ \mid N(w_0) \leq N\}$ is closed for all $N \in \mathbb{N}$.
- (iii) If $w_1^{(n)}, w_2^{(n)}$ are two sequences converging to w_0 , then

$$\left| \lim_{n \rightarrow \infty} N(w_1^{(n)}) - \lim_{n \rightarrow \infty} N(w_2^{(n)}) \right| \leq 1.$$

Proof. (i) II is invariant, and the vector field on its boundary does not vanish and points to the interior of II. Thus if $\phi_{x,0}(0, w_0) \in \text{II}$ for some x , then for any $x_1 > x$, $\phi_{x_1,0}(0, w_0)$ is in the interior of II. By continuity of ϕ , if $|\delta|$ is small enough, also $\phi_{x_1,0}(0, w_0 + \delta) \in \text{II}$.

(ii) Let $w_0 \in \mathbb{R}_+$ be an accumulation point of $W_N = \{w_0 \in \mathbb{R}_+ \mid N(w_0) \leq N\}$. Let us suppose that $N(w_0) \geq N + 1$ and derive a contradiction. The solution $\phi_{x,0}(0, w_0)$ winds around the origin N times ending up in region $\text{II} \cup \text{IV}$ and then crosses the v -axis to begin the next wind. At some time x_1 the solution is in the interior of region VI after N turns. But by continuity, initial conditions in a whole neighborhood of w_0 give solutions which share this property at $x = x_1$, which is a contradiction, since all these solutions have winding number $> N$.

(iii) Let $N_i = \lim_{n \rightarrow \infty} N(w_i^{(n)})$, $i = 1, 2$. We can assume that $N_1 < N_2$ and, by (ii), $N(w_0) = N_1$. First note that, if $(v(x), w(x)) = \phi_{x,0}(0, w_0)$, $\lim_{x \rightarrow \infty} v(x) = \infty$, since the only possibility for v to stay bounded in $\text{II} \cup \text{IV}$ is that $w(x) = v'(x) \rightarrow 0$ as $x \rightarrow \infty$, i.e. that (v, w) converge to a fixed point on the positive v axis, which does not exist. Let A be a large positive number to be chosen later. Let x_1 be so large that $(v(x), w(x))$ winds around the origin N_1 times for $0 \leq x \leq x_1$ and $v(x_1) \geq A$. If n_0 is large enough and $n \geq n_0$, also $(v^{(n)}(x), w^{(n)}(x)) = \phi_{x,0}(0, w_2^{(n)})$ winds N_1 times up to time x_1 and $v^{(n)}(x_1) \geq A/2$. Since (for n large) $N(w_2^{(n)}) = N_2$ there exists an $x_2 \geq x_1$ (depending on n) such that $w^{(n)}(x_2) = 0$ and $w^{(n)}(x) > 0$ for $x_1 < x < x_2$. In $\text{II} \cup \text{IV}$, $v' = w \geq 0$, so that $v^{(n)}(x_2) \geq \frac{A}{2}$. The argument in the proof of Proposition 4.2 can be used: choose A so that $J(A/2, 0) = J(0, w_1)$, where $w_1 = w_1(\sigma)$ is the constant defined in Proposition 4.2. Since J is increasing $(v^{(n)}, w^{(n)})$ will cross the negative w axis below $w_1(\sigma)$, and will enter the invariant region II doing $N_1 + 1$ turns around the origin. \square

The proof of Theorem 3.2 is now simple: fix $\sigma \in]2n - 1, \infty[$ for n odd ≥ 2 . The function $N: \mathbb{R}_+ \rightarrow \mathbb{N}$ takes the value zero [if $w_0 > 1/2, N(w_0) = 0$] and a value $\geq (n+1)/2$ [if w_0 is small enough so that the linear analysis in the appendix is applicable, one constructs as in Proposition 4.2 a solution with $N(w_0) \geq (n-1)/2$]. By (iii) in Proposition 5.1, $N(\cdot)$ has discontinuities of at most one and thus takes all integer values between 0 and $(n+1)/2$. Hence there are (at least) $(n-1)/2$ discontinuity points $w_1, \dots, w_{(n-1)/2} \in \mathbb{R}_+$ such that

$$\underline{\lim}_{w \rightarrow w_m} N(w) = \overline{\lim}_{w \rightarrow w_m} N(w) - 1 = m.$$

By the lower semicontinuity property (ii), $N(w_m) = m$. But by (i), $\phi_{x,0}(0, w_m) \notin \text{II}$, for all $x \geq 0$, and thus $\phi_{x,0}(0, w_m)$ is also a global solution of the unmodified equation (4.4) with winding number m , and the proof of Theorem 3.2 when m is odd is

complete. If n is even, note that $N : \mathbb{R}_- \rightarrow \mathbb{N} + \frac{1}{2}$ takes the value $\frac{1}{2}$ by Proposition 4.2 and a value $\geq (n + 1)/2$ by linear analysis. Then proceed exactly as in the odd case.

Appendix

In this appendix we count the zeros of the solutions of the linearization of (4.3). After changing variable $x \rightarrow x/\sqrt{\sigma}$ the equation becomes:

$$u'' - zu' + \sigma u = 0. \tag{A1}$$

Let $U_\sigma(z)$ and $V_\sigma(z)$ be the solutions of (A1) with data

$$U_\sigma(0) = 1, \quad U'_\sigma(0) = 0, \quad V_\sigma(0) = 0, \quad V'_\sigma(0) = 1. \tag{A2}$$

These functions can be represented by the integrals

$$U_\sigma(z) = \frac{2^{1+\sigma/2}}{\Gamma(-\sigma/2)} \int_0^\infty \cosh zt e^{-t^2/2} t^{-\sigma-1} dt, \quad \text{Re } \sigma < 0, \tag{A3}$$

$$V_\sigma(z) = \frac{2^{(1+\sigma)/2}}{\Gamma(-(\sigma-1)/2)} \int_0^\infty \sinh zt e^{-t^2/2} t^{-\sigma-1} dt, \quad \text{Re } \sigma < 1,$$

and can also be expressed in terms of parabolic cylinder functions. To compute U_σ, V_σ outside the range of validity of (A3) one can use the recursions,

$$U'_\sigma(z) = -\sigma V_{\sigma-1}(z), \quad V'_\sigma(z) = U_{\sigma-1}(z), \tag{A4}$$

whose proof is based on the fact that if u is a solution of (A1) then u' is a solution of (A1) with σ replaced by $\sigma - 1$.

Proposition. *Let U_σ, V_σ be the solutions of (A1) with data (A2). Then U_σ has no zero if $\sigma \leq 0$, and exactly $2n - 2$ zeros if $2n - 2 < \sigma \leq 2n, n \geq 1$; V_σ has exactly one zero if $\sigma \leq 1$ and exactly $2n - 1$ zeros if $2n - 3 < \sigma \leq 2n - 1, n \geq 1$.*

Proof. The facts that U_σ is positive if $\sigma < 0$ and that $V_\sigma(z) > 0$ for $z > 0$ if $\sigma < 1$ follow from the integral representation (A3). From (A4) we deduce that $U_0(z) = 1, V_1(z) = z$. Next we prove inductively:

- (i) If $2n - 2 < \sigma \leq 2n, n \geq 0$, then $U_\sigma(z)$ has exactly n positive zeros $0 < z_1 < \dots < z_n$; $\text{sign } U'_\sigma(z_i) = (-1)^i$ and, as $z \rightarrow \infty, U_\sigma(z) \rightarrow (-1)^n \infty$, except for $\sigma = 0$.
- (ii) If $2n - 3 < \sigma \leq 2n - 1, n \geq 0$, then $V_\sigma(z)$ has exactly n nonnegative zeros $0 = z_1 < \dots < z_n$; $\text{sign } V'_\sigma(z_i) = (-1)^i$ and, as $z \rightarrow \infty, V_\sigma(z) \rightarrow (-1)^{n-1} \infty$.

These properties are readily checked for $n = 0$. We first prove (ii) for $n = N + 1$ assuming (i) for $n = N$. Let $2(N + 1) - 3 < \sigma \leq 2(N + 1) - 1$ and let $0 < z_1 < \dots < z_N$ be the positive zeros of $U_{\sigma-1}$. For $0 < z < z_1, V'_\sigma > 0$, and, since $V'_\sigma(0) = 1, V_\sigma(z) > 0$. From $V'_\sigma(z_i) = U_{\sigma-1}(z_i) = 0$ and $\text{sign } V''_\sigma(z_i) = \text{sign } U'_{\sigma-1}(z_i) = (-1)^i$, it follows, using (A1), that $\text{sign } V_\sigma(z_i) = (-1)^{i+1}$. Thus between two consecutive zeros z_i, z_{i+1} of $U_{\sigma-1}$ there is a zero \bar{z}_{i+1} of V_σ . This zero is unique and $V'_\sigma(\bar{z}_{i+1}) \neq 0$, since $V'_\sigma(z)$ has a definite sign in $]z_i, z_{i+1}[$. Similarly, $(-1)^N U_{\sigma-1}(z) > 0$ for $z > z_N$, and $U_{\sigma-1}(z) \rightarrow (-1)^N \infty (z \rightarrow \infty)$ imply $V_\sigma(z) \rightarrow (-1)^N \infty$ monotonically for $z_N < z \rightarrow \infty$. But since $\text{sign } V_\sigma(z_N) = (-1)^{N+1}$, there must be exactly one zero of V_σ in the interval $]z_N, \infty[$

making a total of $N + 1$ zeros. In the same way one proves that (ii) for $n = N$ implies (i) for $n = N + 1$, and the proof is complete.

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