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On a Conformally Invariant Elliptic Equation on R^n

Ding Weiyue

Institute of Mathematics, Academia Sinica, Beijing, China and Nakai Institute of Mathematics, Tianjin, People's Republic of China

Abstract. For $n \geqq 3$, the equation $\Delta u + |u|^{4/(n-2)}u = 0$ on $\mathbb{R}^n$ has infinitely many distinct solutions with finite energy and which change sign.

In [4], Gidas-Ni-Nirenberg proved that any positive solution of the elliptic equation

$$\Delta u + |u|^{4/(n-2)}u = 0, \quad u \in C^2(R^n), \quad n \geqq 3, \tag{1}$$

which has finite energy, namely

$$\int_{R^n} |\nabla u|^2 dx < \infty, \tag{2}$$

is necessarily of the form

$$u(x) = \left(\frac{\sqrt{n(n-2)}a}{a^2 + |x-\xi|^2}\right)^{(n-2)/2}, \tag{3}$$

where $a > 0$, $\xi \in R^n$. Thereafter, some people tried to show without success that all the solutions of the problem (1)–(2), which are positive somewhere, are given by (3). Their efforts have to be in vain, as we will see shortly that the problem actually has a lot of solutions other than those given by (3). Our main result in this note can be stated as follows.

Theorem. *There exists a sequence of solutions* u_k *of* (1)–(2), *such that* $\int_{R^n} |\nabla u_k|^2 dx \to \infty$ *as* $k \to \infty$.

We remark that Eq. (1) is invariant under the conformal transformations of R^n. Thus, if $u(x)$ is a solution, then for any $\lambda > 0$ and $\xi \in R^n$, $\lambda^{(n-2)/2}u[(x-\xi)/\lambda]$ is also a solution. Moreover, all solutions obtained in this way have the same energy, and we will say that these solutions are equivalent. In particular, the solutions (3) are equivalent. Our theorem implies the existence of infinitely many inequivalent solutions to the problem (1)–(2).

The proof of Theorem consists of two steps. In Sect. 1 we reduce the problem to an equivalent problem on S^n, the Euclidean n-sphere. Then, in Sect. 2, we show how the latter problem can be solved by some standard variational techniques. In this process, the abundant symmetries of S^n play an important role.

Section 1

We first recall some general facts concerning certain elliptic equations on Riemannian manifolds. Let (M, g) and (N, h) be two Riemannian manifolds of dimension $n \geqq 3$. Suppose that there is a conformal diffeomorphism f from M onto N, i.e. $f^*h = \varphi^{4/(n-2)}g$ for some positive function $\varphi \in C^\infty(M)$. The scalar curvatures of (M, g) and (N, h) are R_g and R_h respectively. Consider an equation on M as follows:

$$\Delta_g u - \beta R_g(x)u + F(x, u) = 0, \qquad u \in C^2(M), \tag{4}$$

where Δ_g is the Laplacian on (M, g), $\beta = (n-2)/4(n-1)$, and $F : M \times R \to R$ is smooth. Corresponding to (4) is the equation on N:

$$\Delta_h v - \beta R_h(y)v + \tilde{\varphi}(y)^{1-q} F(f^{-1}(y), \tilde{\varphi}(y)v) = 0, \qquad v \in C^2(N), \tag{5}$$

where $q = 2n/(n-2)$ and $\tilde{\varphi} = \varphi \circ f^{-1}$.

Lemma 1. *Suppose that v is a solution of* (5). *Then $u = (v \circ f)\varphi$ is a solution of* (4) *such that* $\int_M |u|^q dV_g = \int_N |v|^q dV_h$.

Proof. Notice that f is an isometry between (M, f^*h) and (N, h). So we may assume, without loss of generality, that $(M, f^*h) = (N, h)$, i.e. $N = M$ and $f = id$. In such case we have $h = \varphi^{q-2}g$ and $\tilde{\varphi} = \varphi$, while the relation between R_g and R_h is given by

$$\Delta_g \varphi - \beta R_g \varphi + \beta R_h \varphi^{q-1} = 0. \tag{6}$$

Set $u = v\varphi$. In a local coordinate system on M we compute,

$$\begin{aligned} \Delta_h v &= |h|^{-1/2} \partial_j (|h|^{1/2} h^{ij} \partial_i (u/\varphi)) \\ &= \varphi^{-q} |g|^{-1/2} \partial_j (\varphi^2 |g|^{1/2} g^{ij} \partial_i (u/\varphi)) \\ &= \varphi^{-q} (\varphi \Delta_g u - u \Delta_g \varphi). \end{aligned} \tag{7}$$

Here, $|g| = \det(g_{ij})$ and $|h| = \det(h_{ij})$. Combining (5), (6), and (7), we see that u satisfies Eq. (4). Finally, since $dV_h = \varphi^q dV_g$ and $u = v\varphi$, we have $\int_M |u|^q dV_g = \int_M |v|^q dV_h$. Q.e.d.

Lemma 2. *Every solution v of the equation*

$$\Delta v - \tfrac{1}{4}n(n-2)v + |v|^{q-2}v = 0, \qquad v \in C^2(S^n), \tag{8}$$

where Δ is the Laplacian with respect to the standard metric on S^n, corresponds to a solution u of Eq. (1) *satisfying*

$$\int_{R^n} |\nabla u|^2 dx = \int_{S^n} |v|^q dV. \tag{9}$$

Proof. Let $\pi: S^n - \{p\} \to R^n$ be the stereographic projection, where p is the north pole of S^n. Then $f = \pi^{-1}$ is a conformal diffeomorphism from R^n onto $S^n - \{p\}$. Note that the scalar curvatures on R^n and S^n are constant 0 and $n(n-1)$ respectively. Thus, applying Lemma 1, we see that every solution v of (8) corresponds to a solution u of (1) satisfying

$$\int_{R^n} |u|^q dx = \int_{S^n} |v|^q dV < \infty . \tag{10}$$

It can be shown that (10) implies that

$$\int_{R^n} |\nabla u|^2 dx = \int_{R^n} |u|^q dx . \tag{11}$$

(Cf. the proof of Theorem 4.4 in [3].) Clearly, (9) follows from (10) and (11). Q.e.d.

Section 2

From Lemma 2 we see that the proof of Theory can be reduced to the proof of the following

Lemma 3. *There exists a sequence $\{v_k\}$ of solutions of Eq. (8), such that $\int_{S^n} |v_k|^q dV \to \infty$ as $k \to \infty$.*

Note first that solutions of Eq. (8) are in one to one correspondence with the critical points of the functional

$$J(v) = \int_{S^n} \left[\tfrac{1}{2}(|\nabla v|^2 + cv^2) - \frac{1}{q}|v|^q \right] dV$$

in $H^1(S^n)$, where $c = \frac{1}{4}n(n-2)$. Recall that $q = 2n/(n-2)$ is just the critical exponent for the embedding $H^1(S^n) \subset L^p(S^n)$, $1 \leqq p \leqq q$. Therefore, the functional J is well defined and differentiable in $H^1(S^n)$, but it fails to satisfy the Palais-Smale compactness condition in $H^1(S^n)$. However, from the analysis in Ambrosetti and Rabinowitz [1] we see that the following result holds.

Lemma 4. *Let X be a closed subspace of $H^1(S^n)$. Suppose that the embedding $X \subset L^q(S^n)$ is compact. Then the restriction of J on X, $J|X$, satisfies the Palais-Smale condition. Furthermore, if X is infinite-dimensional, then $J|X$ has a sequence of critical points v_k in X, such that $\int_{S^n} |v_k|^q dV \to \infty$ as $k \to \infty$.*

For a proof of Lemma 4, the reader is referred to the proofs of Theorems 3.13 and 3.14 in [1].

In order to find critical points of $J(v)$, we observe that S^n enjoys a lot of symmetries, namely, the compact Lie group $O(n+1)$ acts isometrically on S^n. Also, the functional J is invariant under isometries of S^n. Suppose that G is a compact subgroup of $O(n+1)$. We set

$$X_G = \{v \in H^1(S^n) : v(gx) = v(x),\ \forall g \in G \text{ and a.e. } x \in S^n\} .$$

Then, by the symmetric criticality principle [5], any critical point of the restriction $J|X_G$ is a critical point of J too. Therefore, we may apply Lemma 4 to prove Lemma 3, provided an infinite-dimensional subspace X_G can be found so that the embedding $X_G \subset L^q(S^n)$ is compact. It turns out that such X_G exists.

Let $R^{n+1}=R^k\times R^m=\{(x,y):x\in R^k,\ y\in R^m\}$, where $k+m=n+1$, $k\geqq m\geqq 2$. Then

$$S^n=\{(x,y):|x|^2+|y|^2=1\}.$$

Let $G=O(k)\times O(m)\subset O(n+1)$. For $g=(g_1,g_2)\in G$, where $g_1\in O(k)$ and $g_2\in O(m)$, the action of G on S^n is defined by $g(x,y)=(g_1x,g_2y)$. With this choice of G, we see that X_G is an infinite-dimensional closed subspace of $H^1(S^n)$. Furthermore, we have

Lemma 5. *For $r=2k/(k-2)>2n/(n-2)$, $1\leqq p\leqq r$, we have the continuous embedding $X_G\subset L^p(S^n)$. The embedding is compact if $1\leqq p<r$.*

Proof. Notice first that if $u\in X_G$, then $u=u(|x|,|y|)$, i.e. u depends only on $|x|$, or equivalently, u depends only on $|y|$, since $|x|^2+|y|^2=1$. Now, for any $\bar z=(\bar x,\bar y)\in S^n$, assume first that $\bar y\neq 0$. Then $\bar y_i\neq 0$ for some $1\leqq i\leqq m$. Set

$$h(x,y)=(x_1,\dots,x_k,y_1,\dots,y_{i-1},y_{i+1},\dots,y_m)\in R^n.$$

Then there exists a neighborhood U of $\bar z$ in S^n and $\delta>0$ such that h maps U diffeomorphically onto the open set $B^k_\delta(\bar x)\times B^{m-1}_\delta(\tilde y)$ in R^n, where

$$\tilde y=(\bar y_1,\dots,\bar y_{i-1},\bar y_{i+1},\dots,\bar y_m)\in R^{m-1}.$$

Note that in the chart (U,h), if $u\in X_G$ then u depends only on $|x|$, where $x\in B^k_\delta(\bar x)$. Next, if $\bar y=0$, then $\bar x_i\neq 0$ for some $1\leqq i\leqq k$. We can likewise take a chart in which $u\in X_G$ depends only on $|y|$, where $y\in B^m_\delta(\bar y)$.

We may assume that S^n is covered by a finite number of such charts, say (U_α,h_α), $1\leqq\alpha\leqq N$, and that the metric matrices in these charts satisfies

$$c^{-1}I\leqq(g^\alpha_{ij})\leqq cI,\qquad 1\leqq\alpha\leqq N,$$

where $c>1$ is a constant and I is the $n\times n$ identity matrix. Since the functions in X_G behave locally like functions of k or m independent variables in these charts, we have for $s=k$ or m,

$$\begin{aligned}\int_U|u|^p dV&=\int_{B^s_\delta\times B^{n-s}_\delta}|u|^p\sqrt{\det(g_{ij})}\,dx\,dy\\ &\leqq c_1\int_{B^s_\delta\times B^{n-s}_\delta}|u(x)|^p dx\,dy\\ &=c_2\int_{B^s_\delta}|u(x)|^p dx,\end{aligned}$$

for $u\in X_G$. Hence, there exists a constant $c(\alpha)$ such that

$$\|u\|_{L^p(U_\alpha)}\leqq c(\alpha)\|u\|_{L^p(B^s_\alpha)}. \tag{12}$$

Similarly, we can prove that

$$\|u\|_{H^1(U_\alpha)}\geqq d(\alpha)\|u\|_{H^1(B^s_\alpha)}, \tag{13}$$

for some $d(\alpha)>0$ and all $u\in X_G$. Combining (12), (13) and the Sobolev inequality on B^s_α yields that

$$\|u\|_{L^p(U_\alpha)}\leqq b(\alpha)\|u\|_{H^1(U_\alpha)},\qquad 1\leqq p\leqq 2s/(s-2),$$

for some $b(\alpha)>0$ and all $u\in X_G$. The global inequality now follows easily:

$$\|u\|_{L^p(S^n)} \leqq b\|u\|_{H^1(S^n)}, \qquad 1\leqq p\leqq 2k/(k-2),$$

for all $u\in X_G$. This proves $X_G \subset L^p(S^n)$ continuously, for $1\leqq p\leqq r$. The compactness of the embedding for $1\leqq p<r$ can be derived in a standard way, cf. e.g. [2]. Q.e.d.

By Lemma 5, the embedding $X_G \subset L^q(S^n)$ is compact. Therefore, as remarked before, we may apply Lemma 4 to complete the proof of Lemma 3.

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