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Hausdorff Dimension of Attractors for the Two-Dimensional Navier-Stokes Equations with Boundary Conditions

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Abstract. We consider a viscous incompressible fluid enclosed in a region of $\mathbb{R}^2$, and subject to boundary conditions. We obtain explicit bounds (depending only on the data) for the entropy (Kolmogorov-Sinaï invariant) and the Hausdorff dimension of attracting sets.

1. Introduction

We consider a viscous incompressible fluid enclosed in a region Ω of $\mathbb{R}^2$. The time evolution of the fluid is described by the Navier-Stokes equations with boundary conditions. Two kinds of boundary conditions are investigated: a given velocity and an imposed force on the boundary Γ of Ω.

Ruelle ([7]) has obtained rigourous bounds on the entropy (Kolmogorov-Sinaï invariant) and the Hausdorff dimension of attracting sets, involving the rate of energy dissipation in the fluid. These bounds have been improved by Lieb ([5]) and a complete statement of available results is given in [8] (see also [1, 2]). Using these estimates, we derive explicit bounds on these quantities (i.e., bounds depending only on the data).

2. Given Velocity on the Boundary

Let Ω a bounded open region of $\mathbb{R}^2$, with a C^3 boundary Γ. Let $\varphi \in H^{3/2}(\Gamma)^2$ [we recall $H^{3/2}(\Gamma)=\gamma_0 H^2(\Omega)$, where the linear operator γ_0 is defined on $H^1(\Omega)$ by $\gamma_0 u = u|_\Gamma$] such that $\int_\Gamma \varphi \cdot n\, d\Gamma = 0$, n being the unit outward normal on Γ. The evolution of a viscous fluid enclosed in Ω, subject to the boundary condition φ, is

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described by the following Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial v}{\partial t}+(v\cdot\nabla)v+\nabla p=\nu\Delta v \\ \nabla\cdot v=0 \\ v|_\Gamma=\varphi \\ v(0)=v_0\,, \end{cases} \tag{1}$$

where the initial condition v_0 satisfies:

$$v_0|_\Gamma=\varphi\,, \qquad \nabla\cdot v_0=0\,. \tag{2}$$

Let $\upsilon=\{w\in\mathscr{D}(\Omega)^2, \nabla\cdot w=0\}$, where $\mathscr{D}(\Omega)$ is the space of C^∞ real functions with support in Ω. Let H (respectively V) the closure of υ in $L^2(\Omega)^2$ [respectively $H^1(\Omega)^2$], endowed with the norm $|w|=\left(\sum_{i=1}^{2}\int_\Omega w_i^2(x)\,dx\right)^{1/2}$ $\left[\text{respectively } \|w\| =\left(\sum_{j=1}^{2}|D_jw|^2\right)^{1/2}\right]$. Finally, if f is a scalar function, we define $\operatorname{curl} f =(\partial_2 f, -\partial_1 f)$.

The evolution equation (1) admits a good existence and uniqueness theorem ([3]). In particular, there exists a universal attracting set $K\subset H$, compact, of finite Hausdorff dimension. In fact, from [5, 8], it follows:

$$\dim K\leqq A_2|\Omega|^{1/2}\nu^{-3/2}\sup_\mu\left\langle\int_\Omega \varepsilon_\nu(x)\,dx\right\rangle_\mu^{1/2}, \tag{3}$$

where $\varepsilon_\nu(x)=\frac{\nu}{2}\sum_{i,j}(\partial_j v_i+\partial_i v_j)^2(x)$ is the rate of energy dissipation, A_2 is an absolute constant with $A_2\leqq 0.5597$, $|\Omega|$ is the volume of Ω, $\langle\ \rangle_\mu$ denotes an average over an invariant ergodic measure μ, and $\sup_\mu$ the supremum on such measures.

From [5, 8] we have also the following estimate on the topological entropy h:

$$h\leqq A_2'\nu^{-2}\sup_\mu\left\langle\int_\Omega \varepsilon_\nu(x)\,dx\right\rangle_\mu \tag{4}$$

with $A_2'\leqq 0.1201$. Furthermore, we have

$$\int_\Omega \varepsilon_\nu\leqq 2\nu\|v\|^2\,. \tag{5}$$

Therefore, to get an explicit bound on $\dim K$ and h, we shall estimate $\langle\|v\|^2\rangle_\mu$. For this purpose, we need the following proposition:

Proposition. *There exists $G\in H^2(\Omega)^2$, such that:*

$$\begin{cases} \nabla\cdot G=0 \\ G|_\Gamma=\varphi \\ \left|\sum_{i,j}\int_\Omega w_iw_j\partial_iG_j\right|\leqq\dfrac{\nu}{2}\|w\|^2 \quad \textit{for all } w \textit{ in } V\,. \end{cases} \tag{6}$$

Proof. The idea of the proof is closely related to an argument given in [6] (pages 103–105). First, we define a stream function ψ associated to the Stokes problem:

$$\begin{cases} -\Delta F + \nabla P = 0 \\ \nabla \cdot F = 0 \\ F|_\Gamma = \varphi . \end{cases} \tag{7}$$

We know (see [9] p. 33) the existence (and unicity) of F in $H^2(\Omega)^2$; so there is a constant $c_1(\Omega)$ (depending only on Ω) such that we can choose $\psi \in H^3(\Omega)$ with $F = \operatorname{curl} \psi$ and

$$\|\psi\|_{H^3(\Omega)} \leqq c_1(\Omega) \, \|\varphi\|_{H^{3/2}(\Gamma)^2} . \tag{8}$$

Let now $G = \operatorname{curl}(\theta_\varepsilon \psi)$, where θ_ε is defined in the following lemma:

Lemma. *For all $\varepsilon > 0$, there exists a positive function $\theta_\varepsilon \in C^2(\bar{\Omega})$, with partial derivatives of degree 3 in $L^\infty(\bar{\Omega})$ and such that:*

$$\begin{cases} \theta_\varepsilon(x) = 1 \quad \textit{for} \quad \varrho(x) \leqq \frac{1}{2} \delta_\varepsilon^2 \\ \theta_\varepsilon(x) = 0 \quad \textit{for} \quad \varrho(x) \geqq 2\delta_\varepsilon \\ \sup\limits_{x \in \bar{\Omega}} \theta_\varepsilon(x) = 1 \\ |\partial_k \theta_\varepsilon(x)| \leqq c_2(\Omega) \dfrac{\varepsilon}{\varrho(x)} \\ \|\partial_{ij}^2 \theta_\varepsilon\|_{L^\infty(\bar{\Omega})} \leqq c_2(\Omega) \dfrac{\varepsilon}{\delta_\varepsilon^4} \quad \textit{for all} \quad i,j,k \in \{1,2\} \\ \|\partial_{ijk}^3 \theta_\varepsilon\|_{L^\infty(\bar{\Omega})} \leqq c_2(\Omega) \dfrac{\varepsilon}{\delta_\varepsilon^6} , \end{cases} \tag{9}$$

where $\varrho(x) =$ distance of x from Γ, $\delta_\varepsilon = \exp(-1/\varepsilon)$, $c_2(\Omega)$ being a constant depending only on Ω.

For the proof of this lemma, see the appendix.

Defining now $D\psi(x) = \left(\sum_i |\partial_i \psi(x)|^2\right)^{1/2}$ we get, for all w in $H_0^1(\Omega)$:

$$\begin{cases} |(wG_j)(x)| \leqq |(w\psi)(x)| \, c_2(\Omega) \dfrac{\varepsilon}{\varrho(x)} + |(w \, D\psi)(x)| & \text{for} \quad \varrho(x) \leqq 2\delta_\varepsilon \\ |(wG_j)(x)| = 0 & \text{for} \quad \varrho(x) > 2\delta_\varepsilon . \end{cases}$$

Therefore

$$\|wG_j\|_{L^2(\Omega)} \leqq c_2(\Omega) \, \|\psi\|_{L^\infty(\Omega)} \, \varepsilon \left\| \frac{w}{\varrho} \right\|_{L^2(\Omega)} + \left[\int_{\varrho(x) \leqq 2\delta_\varepsilon} w^2(x) \, D\psi^2(x) \right]^{1/2} . \tag{10}$$

Using $\partial_i \psi \in H^2(\Omega) \subset L^\infty(\Omega)$ (continuous embedding), hence $D\psi \in L^\infty(\Omega)$, and the Hardy inequality (see [6] p. 104), (10) yields:

$$\|wG_j\|_{L^2(\Omega)} \leqq \varepsilon c_3(\Omega) \, \|\psi\|_{L^\infty(\Omega)} \, \|w\|_{H_0^1(\Omega)} + \|D\psi\|_{L^\infty(\Omega)} \left[\int_{\varrho(x) \leqq 2\delta_\varepsilon} w^2(x) \right]^{1/2}$$

and, by Hölder and Sobolev inequalities:

$$\|wG_j\|_{L^2(\Omega)} \leqq \varepsilon c_3(\Omega)\, \|\psi\|_{L^\infty(\Omega)}\, \|w\|_{H_0^1(\Omega)} + \delta_\varepsilon^{1/3} c_4(\Omega)\, \|D\psi\|_{L^\infty(\Omega)}\, \|w\|_{H_0^1(\Omega)}.$$

Therefore, for all w in $H_0^1(\Omega)$, we get

$$\|wG_j\|_{L^2(\Omega)} \leqq c_5(\Omega, \varphi)\, \varepsilon \|w\|_{H_0^1(\Omega)}, \tag{11}$$

where $c_5(\Omega, \varphi)$ depends on Ω and $\|\varphi\|_{H^{3/2}(\Gamma)^2}$.

We conclude now the proof of the proposition. First, we notice that G satisfies $\nabla \cdot G = 0$ and $G|_\Gamma = \varphi$. Let, then, $w \in V$; (11) yields:

$$\left|\sum_{i,j} \int_\Omega w_i w_j \partial_i G_j\right| = \left|\sum_{i,j} \int_\Omega w_i G_j \partial_i w_j\right| \leqq 4c_5(\Omega, \varphi)\, \varepsilon \|w\|^2,$$

and we get (6) by the choice

$$\varepsilon = \frac{\nu}{8c_5(\Omega, \varphi)}. \tag{12}$$

Corollary. *There exist two constants $c_6(\Omega, \varphi)$, $c_7(\Omega, \varphi)$ such that G satisfies*

$$\begin{aligned} \|G\| &\leqq c_6(\Omega, \varphi) \exp[c_6(\Omega, \varphi)/\nu], \\ |f| &\leqq c_7(\Omega, \varphi) \exp[c_7(\Omega, \varphi)/\nu], \end{aligned} \tag{13}$$

where $f = \nu \Delta G - (G \cdot \nabla) G$.

Proof. (13) follows from (8), (9), (12). We have, for instance $\|G\|^2 = \sum_{i,j} \|\partial_i G_j\|^2_{L^2(\Omega)}$. Then (9) yields

$$|(\partial_i G_j)(x)| \leqq 4c_2(\Omega) \frac{\varepsilon}{\delta_\varepsilon^2} |D\psi(x)| + c_2(\Omega) \frac{\varepsilon}{\delta_\varepsilon^4} |\psi(x)| + |D^2 \psi(x)|,$$

where $D^2\psi(x) = \left[\sum_{i,j} (\partial_{ij}^2 \psi)^2 (x)\right]^{1/2}$.

By (8), (12) we get[1]

$$\|G\| \leqq c_6(\Omega, \varphi) \exp[c_6(\Omega, \varphi)/\nu].$$

In the same way, we obtain the second inequality given in (13). Now, we can state the main result of this section:

Theorem[2]**.** *For the two-dimensional Navier-Stokes equations in a bounded open region Ω, with a boundary condition φ (on the velocity) belonging to $H^{3/2}(\Gamma)^2$, there exists a universal attracting set K, of finite Hausdorff dimension with*

$$\dim K \leqq a(\Omega, \varphi) \exp[b(\Omega, \varphi)/\nu], \tag{14}$$

where $a(\Omega, \varphi)$, $b(\Omega, \varphi)$ are two finite positive constants depending only on Ω and $\|\varphi\|_{H^{3/2}(\Gamma)^2}$.

[1] We are interested in the case $\nu \ll 1$, which occurs in turbulence, so that $\nu \ll e^{1/\nu}$

[2] This result (with a sketch of the proof) was given in [4]

The topological entropy (rate of information creation on the attractor K) is bounded in the same way.

Proof. Let $u(t)=v(t)-G$, where $v(t)$ is the solution of (1). The energy equality for $u(t)$ yields:

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2+\sum_{i,j}\int_\Omega u_iu_j\partial_jG_i=-\nu\|u(t)\|^2+\sum_i\int_\Omega f_iu_i\,.$$

It follows from (6):

$$\frac{d}{dt}|u(t)|^2+\nu\|u(t)\|^2\leqq 2\sum_i\int_\Omega f_iu_i\leqq 2|f|\,|u(t)|\,.$$

Furthermore, for all $u\in V$, $|u|^2\leqq\frac{|\Omega|}{2}\|u\|^2$. So

$$\frac{d}{dt}|u(t)|^2+\frac{\nu}{2}\|u(t)\|^2\leqq\frac{|\Omega|}{\nu}|f|^2\,.$$

Averaging over an invariant ergodic measure μ, we get

$$\langle\nu\|u\|^2\rangle_\mu\leqq\frac{2|\Omega|}{\nu}|f|^2\,. \tag{15}$$

Finally (15), (13) together with (3)–(5) conclude the proof of the theorem.

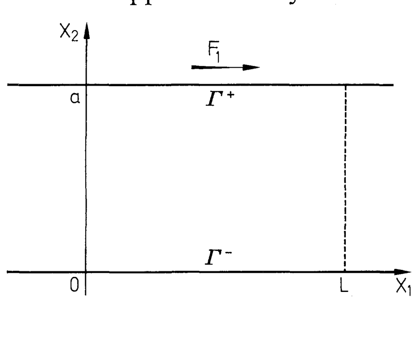
Remark. We notice that (14) displays an exponential behaviour in $1/\nu$ while for the case of homogeneous boundary conditions – with an external volumic force in the right-hand side of the Navier-Stokes equations – Ruelle [8] has obtained estimates in $1/\nu^2$ for the Hausdorff dimension and $1/\nu^3$ for the entropy (see also [2]).

3. Imposed Force on the Boundary

We consider now the evolution of a viscous incompressible fluid in a tube, $0\leqq x_2\leqq a$ with the following boundary conditions:

$$\begin{cases} v(x_1+L,x_2)=v(x_1,x_2)\\ p(x_1+L,x_2)=p(x_1,x_2)\\ v(x_1,0)=0\\ (\Sigma\cdot n)_1=F_1 \quad\text{and}\quad v\cdot n=0 \quad\text{for}\quad x_2=a\,,\end{cases} \tag{1}$$

where $\Sigma=(\sigma_{ij})$ is the stress tensor: $\sigma_{ij}=-p\delta_{ij}+\nu(\partial_jv_i+\partial_iv_j)$ and F_1 a given tangential force applied on the upper boundary Γ^+.



We write: $\Omega=[0,L]\times[0,a]$, $\Gamma=\partial\Omega$ and, as before, $\varepsilon_v(x)=\frac{\nu}{2}\sum_{i,j}(\partial_i v_j+\partial_j v_i)^2(x)$. We have

$$\int_\Omega \varepsilon_v=\nu\|v\|^2+\sum_{i,j}\nu\int_\Omega(\partial_i v_j)(\partial_j v_i)=\nu\|v\|^2+\nu\int_\Gamma v_i(\partial_i v_j)n_j\,.$$

Therefore, with the boundary conditions (1) we obtain

$$\int_\Omega \varepsilon_v=\nu\|v\|^2\,. \tag{2}$$

Furthermore, the energy equality yields

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2+\sum_{i,j}\int_\Omega v_i v_j\partial_j v_i+\sum_i\int_\Omega v_i\partial_i p=\nu\sum_i\int_\Omega v_i\Delta v_i\,. \tag{3}$$

But

$$\sum_i\int v_i\Delta v_i=-\|v\|^2+\sum_{i,j}\int_{\Gamma^+} v_i n_j\partial_j v_i\,.$$

Using $\sum_{i,j}\int_{\Gamma^+} v_i n_j(\partial_i v_j)=0$, we get

$$\sum_i\int v_i\Delta v_i=-\|v\|^2+\sum_{i,j}\frac{1}{\nu}\int_{\Gamma^+}\sigma_{ij}n_j v_i=-\|v\|^2+\frac{1}{\nu}\int_{\Gamma^+}F_1 v_1\,.$$

Returning to (3), we have finally

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2+\nu\|v(t)\|^2=\int_{\Gamma^+}F_1 v_1\,. \tag{4}$$

Let $\alpha>0$, then

$$\left|\int_{\Gamma^+}F_1 v_1\right|\leqq\frac{1}{2\alpha}|F|^2+\frac{\alpha}{2}\int_{\Gamma^+}v_1^2,\quad\text{where}\quad |F|=\left(\int_0^L F_1^2(x_1)\,dx_1\right)^{1/2}.$$

Using $v_1(x_1,0)=0$, we get

$$\int_{\Gamma^+}v_1^2=\int_0^L dx_1\int_0^a\frac{\partial}{\partial x_2}v_1^2(x_1,x_2)\,dx_2\leqq 2\left[\int_\Omega v_1^2(x)\,dx\right]^{1/2}\left[\int_\Omega\left(\frac{\partial v_1}{\partial x_2}\right)^2(x)\,dx\right]^{1/2}.$$

Therefore

$$\int_{\Gamma^+}v_1^2\leqq\frac{|v|^2}{a}+a\|v\|^2\,.$$

Furthermore

$$|v|^2=\sum_i\int_\Omega v_i^2=\sum_i\int_\Omega dx\left[\int_0^{x_2}\frac{\partial v_i}{\partial z}(x_1,z)\,dz\right]^2\leqq a\sum_i\int_\Omega dx\int_0^{x_2}\left(\frac{\partial v_i}{\partial z}\right)^2(x_1,z)\,dz\,,$$

which yields

$$|v|^2\leqq a^2\|v\|^2\,.$$

Finally, setting $\alpha=\frac{\nu}{2a}$, we get

$$\left|\int_{\Gamma^+}F_1 v_1\right|\leqq\frac{a}{\nu}|F|^2+\frac{\nu}{2}\|v\|^2\,. \tag{5}$$

Then (4), (5) yields

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2+\frac{\nu}{2}\|v(t)\|^2\leqq\frac{a}{\nu}|F|^2\,.$$

Averaging over an invariant ergodic measure μ, we obtain

$$\langle v\|v\|^2\rangle_\mu \leqq 2\frac{a}{v}|F|^2. \tag{6}$$

Let now K an attracting set for our problem. By a slight modification[3] of arguments given in [5, 7], we have

$$\dim K \leqq A_2 v^{-1}(1+2)|\Omega|^{1/2}\sup_\mu \left\langle \int_\Omega \tfrac{1}{8}\sum_{i,j}(\partial_i v_j+\partial_j v_i)^2\right\rangle_\mu^{1/2},$$

$$h \leqq L_{12} v^{-1}(1+2)\sup_\mu \left\langle \int_\Omega \tfrac{1}{8}\sum_{i,j}(\partial_i v_j+\partial_j v_i)^2\right\rangle_\mu.$$

Therefore (6) yields

$$\dim K \leqq \frac{3\sqrt{2}}{2}A_2\frac{aL^{1/2}|F|}{v^2}, \qquad h \leqq \frac{3}{2}L_{12}\frac{a|F|^2}{v^3},$$

with $A_2 \leqq 0.5597$ and $L_{12} \leqq 0.24008$.

Appendix

We give here the proof of the lemma stated in Sect. 2. This lemma is clearly a consequence of the following proposition:

Proposition. *There exists* $\lambda : \mathbb{R}_+ \to [0,1]$ *such that*

$$\begin{cases} \lambda \in C^2(\mathbb{R}_+) \\ \lambda''' \in L^\infty(\mathbb{R}_+) \\ \lambda(y)=1 \quad \text{if} \quad y \leqq \dfrac{\delta^2}{2} \\ \lambda(y)=0 \quad \text{if} \quad y \geqq 2\delta \\ |\lambda'(y)| \leqq c\dfrac{\varepsilon}{y} \\ \|\lambda''\|_{L^\infty} \leqq c\dfrac{\varepsilon}{\delta^4} \\ \|\lambda'''\|_{L^\infty} \leqq c\dfrac{\varepsilon}{\delta^6}, \end{cases} \tag{A1}$$

where $\varepsilon > 0$ *is fixed (arbitrary),* $\delta = \exp(-1/\varepsilon)$ *and* c *an absolute constant.*

[3] The estimates given in [5, 7] follow from bounds on the eigenvalues of a Schrödinger operator (which appears when one studies the growth of a perturbation ξ of a solution). In the present problem, the component ξ_1 of ξ satisfies $\xi_1=0$ on Γ^- and a Neumann condition $\partial_2\xi_1=0$ on Γ^+; we notice, now, that if ξ_1 is an eigenvector for our Schrödinger operator, then $\tilde{\xi}_1$ (defined on $\tilde{\Omega}=[0,L]\times[0,2a]$ by: $\tilde{\xi}_1(x)=\xi_1(x)$ for $x\in\Omega$ and $\tilde{\xi}_1(x_1,x_2)=\xi_1(x_1,x_2-a)$ for $a\leqq x_2\leqq 2a$) is an eigenvector – for the same eigenvalue – of a Schrödinger operator with Dirichlet boundary conditions on Γ^- and Γ^{++} (defined by: $x_2=2a$) and, for this last operator, we can use the bounds given in [5] (Ruelle, private communication)

Proof. We set

$$\lambda(y)=\begin{cases} 1 & \text{if } y\leqq\frac{\delta^2}{2} \\ \left(y-\frac{\delta^2}{2}\right)^3(a_0y+b_0)+1 & \text{if } \frac{\delta^2}{2}\leqq y\leqq y_0 \\ \varepsilon\log\frac{\delta}{y} & \text{if } y_0\leqq y\leqq y_1 \\ (y-2\delta)^3(a_1y+b_1) & \text{if } y_1\leqq y\leqq 2\delta \\ 0 & \text{if } y\geqq 2\delta \end{cases}$$

with

$$\delta^2<y_0<\delta^{3/2}<y_1<\delta. \tag{A 3}$$

A straightforward calculation shows that we can choose y_0, y_1 satisfying (A 3) such that λ defined by (A 2) ensures the first four conditions (A 1), with a_0, b_0, a_1, b_1 given by

$$\begin{cases} a_0=\dfrac{\varepsilon(3y_0-\delta^2/2)}{4y_0^2(y_0-\delta^2/2)^3} & a_1=\dfrac{\varepsilon(3y_1-2\delta)}{4y_1^2(y_1-2\delta)^3} \\ b_0=-\dfrac{\varepsilon(16y_0^2-\frac{11}{2}\delta^2y_0+\delta^4/4)}{12y_0^2(y_0-\delta^2/2)^3} & b_1=-\dfrac{\varepsilon(8y_1^2-11y_1\delta+2\delta^2)}{6y_1^2(y_1-2\delta)^3}. \end{cases} \tag{A 4}$$

Then, the three last conditions (A 1) are a consequence of (A 3), (A 4).

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