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A New Integrable Case of the Motion of the 4-Dimensional Rigid Body

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Abstract. A Lax pair for a new family of integrable systems on SO(4) is presented. The construction makes use of a twisted loop algebra of the G_2 Lie algebra. We also describe a general scheme producing integrable cases of the generalized rigid body motion in an external field which have a Lax representation with spectral parameter. Several other examples of multi-dimensional tops are discussed.

Introduction

Starting with the work of Arnold [1], the study of multi-dimensional tops has already a rather long history with the well-known paper of Manakov [2] as one of its highlights. In recent years much effort was spent on the classification of integrable cases. The case of the four-dimensional top is particularly interesting since it has direct physical significance [3]. Some of the integrable cases discovered recently are new even in the classical setting, e.g. the motion of a top in an arbitrary quadratic potential [4] (This case was considered earlier, along with several others, in [5] but passed practically unnoticed. Another interesting system indicated in [5], a pair of interacting tops, was rediscovered in [6]).

In this paper we shall describe a new integrable case of the four-dimensional top. Our construction is based on the so-called Kostant-Adler scheme and on the use of affine Lie algebras. This technique has already been applied to the study of multi-dimensional tops and related systems in [7, 5] and independently in [8, 9]. Our main technical tool consists in twisting the loop algebra of a simple Lie algebra by a Cartan automorphism which leads precisely to Hamiltonian systems of the generalized rigid body type. (This was already indicated in [7] but was missed in [8, 9].)

Our principal example is connected with the split real form of the G_2 simple Lie algebra. Recently Adler and van Moerbeke [10] announced a classification of left-invariant metrics on SO(4) that are algebraically completely integrable. Our example fits into their list, thus providing a Lax pair for the last case of the classification theorem. A comparison of our case with the results of [10] is given at the end of Sect. 2, where we also discuss the necessary conditions for complete integrability obtained by Veselov (see [13] which contains a corrected version of [14]). We shall also present a few other cases equally covered by the general construction. These include tops in quadratic potential, two interacting tops in $\mathbb{R}^n$, and systems of three and four interacting tops in $\mathbb{R}^3$. The latter systems presumably have a physical meaning in connection with the motion of a rigid body with elliptic cavities filled with ideal fluid. For the reader's convenience we also sketch the general scheme and describe how to single out the phyiscally meaningful Hamiltonians which are quadratic in momenta.

1. Integrable Systems on T^*K

We shall proceed in this section in a somewhat dogmatic way, the proofs being postponed until Sect. 4. To motivate our general theorem, recall that the natural set-up for the Hamiltonian Lax equations is provided by the affine Lie algebras. Any such algebra may be equipped with a linear Poisson structure, and integrable Lax systems are supported on its Poisson submanifolds. It is convenient to avoid a too detailed description of the corresponding stratification by adding some extra variables (the "generalized Clebsch variables," cf. [11]), i.e. by replacing Poisson subspaces by their symplectic models. For the Lax systems we have in mind, a uniform model is provided by the cotangent bundle of a compact Lie group, which allows for an easy mechanical interpretation of these systems.

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Let σ be an involution in G, $K \subset G$ its fixed subgroup. We denote the corresponding involution in $\mathfrak{g}$ by the same letter. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition, i.e. $\sigma = \mathrm{id}$ on $\mathfrak{k}$ and $\sigma = -\mathrm{id}$ on $\mathfrak{p}$, $\mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{p}^*$ the dual decomposition.

Let $\mathfrak{g}^*[\lambda, \lambda^{-1}]$ be the space of Laurent polynomials with coefficients in $\mathfrak{g}^*$. We fix a trivialization of the cotangent bundle $T^*K = K \times \mathfrak{k}^*$ by means of left translations. For any $a, f \in \mathfrak{g}^*$, we define a mapping $\mu_{a,f}: T^*K \to \mathfrak{g}^*[\lambda, \lambda^{-1}]$ by assigning to each point $\xi = (k, \varrho)$, $k \in K$, $\varrho \in \mathfrak{k}^*$, the polynomial

$$\mu_{a,f}(\xi) = a\lambda + \varrho + \mathrm{Ad}^* k^{-1} \cdot f \lambda^{-1} . \tag{1.1}$$

Let $I(\mathfrak{g})$ be the ring of Ad^*-invariant polynomials on $\mathfrak{g}^*$. For any $\varphi \in I(\mathfrak{g})$, $X \in \mathfrak{g}^*[\lambda, \lambda^{-1}]$, we can consider $\varphi(X(\lambda))$, which is a well defined element of $\mathbb{R}[\lambda, \lambda^{-1}]$.

Theorem 1. *Let $a, f \in \mathfrak{p}^*$. Functions on T^*K given by*

$$\varphi_i(\xi) = \mathrm{Res}_{\lambda=0} \lambda^i \varphi(\mu_{a,f}(\xi)) , \tag{1.2}$$

$\varphi \in I(\mathfrak{g})$, $i \in \mathbb{Z}$, are in involution with respect to the canonical Poisson bracket in T^*K.

Our next goal is to determine those combinations of the functions (1.2) that are quadratic in momenta and so may be regarded as Hamiltonians for natural dynamical systems (generalized tops in an external field). This is easily achieved. Let

$$\varphi_+(\xi) = \mathrm{Res}\, \lambda \varphi(\mu_{a,f}(\xi)\lambda^{-1}) , \tag{1.3}$$

$$\varphi_-(\xi) = \mathrm{Res}\, \lambda^{-3} \varphi(\mu_{a,f}(\xi)\lambda) . \tag{1.3'}$$

Put $b=d\varphi(a)$, $h=d\varphi(f)$, and let $\pi=-\mathrm{Ad}^*k\cdot\varrho$ be the right invariant momentum on T^*K. It is not hard to see that

$$\varphi_+(k,\varrho)=\tfrac{1}{2}\langle\Omega_{a,\varphi}\varrho,\varrho\rangle+\langle\mathrm{Ad}\,kb,f\rangle\,, \tag{1.4}$$

$$\varphi_-(k,\varrho)=\tfrac{1}{2}\langle\Omega_{f,\varphi}\pi,\pi\rangle+\langle\mathrm{Ad}\,k^{-1}h,a\rangle\,, \tag{1.4'}$$

where $\Omega_{c,\varphi}:\mathfrak{k}^*\to\mathfrak{k}$ is an angular velocity operator depending on $\varphi\in I(\mathfrak{g})$ and $c\in\mathfrak{p}^*$.

In the most interesting case when $\mathfrak{g}$ is a simple Lie algebra and σ its Cartan automorphism, there is an explicit formula for $\Omega_{a,\varphi}$. It is particularly simple when $\mathfrak{g}$ is split and $a\in\mathfrak{p}^*$ is a regular element. Let us identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of an invariant inner product which is positive definite on $\mathfrak{k}$. Then

$$\Omega_{a,\varphi}=\mathrm{ad}\,b(\mathrm{ad}\,a)^{-1}\,,\qquad b=d\varphi(a)\,. \tag{1.5}$$

Hamilton's equations of motion generated by the functions (1.2) have the Lax form $\frac{d}{dt}L=-\mathrm{ad}^*M\cdot L$, where the Lax operator L is given by $L=\mu_{a,f}(\xi)$ and M depends on the choice of φ_i. In particular, the Lax representation for the Hamiltonians $\varphi_\pm$ is $\frac{d}{dt}L=-\mathrm{ad}^*M_\pm\cdot L$ with

$$L(\lambda)=a\lambda+\varrho+s\lambda^{-1}\,, \tag{1.6}$$

where $s=\mathrm{Ad}^*k^{-1}f$, and

$$M_+=b\lambda+\Omega_{a,\varphi}\varrho\,, \tag{1.7}$$

$$M_-=-\mathrm{Ad}\,k^{-1}\cdot h\lambda^{-1}\,. \tag{1.7'}$$

Remark that under the involution $k\mapsto k^{-1}$, $\varrho\mapsto\pi$ on T^*K combined with the permutation $a\leftrightarrow f$, the functions φ_+ are carried into φ_- and vice versa. Hence the corresponding Hamiltonian systems are equivalent. However, their Lax representations are very different. We shall focus on this in Sect. 4.

We also point out that the second term (potential energy) on the right-hand side of (1.4), (1.4′) vanishes if $f=0$ or $a=0$, respectively. The equations of motion in this case reduce to equations in $\mathfrak{k}^*$.

Theorem 1 indicates that the corresponding Hamiltonian systems are completely integrable. In fact, in most cases this can easily be proven (cf. [5, 7]).

2. A New Four-Dimensional Top

In this section we apply the general construction of Sect. 1 to the Riemannian symmetric pair $(G_2,\mathrm{SO}(4))$.

Let $\mathfrak{g}$ be the real split form of the G_2 simple Lie algebra. There is a natural realization of $\mathfrak{g}$ as a subalgebra of $\mathrm{SO}(4,3)$. Specifically, let a quadratic form of signature $(4,3)$ in $\mathbb{R}^7$ be given by

$$(x,x)=\sum_{i=1}^{4}x_i^2-\sum_{i=5}^{7}x_i^2\,. \tag{2.1}$$

The subalgebra $\mathfrak{g} \subset \mathrm{so}(4,3)$ has dimension 14; it will be convenient for us to choose the following parametrization of the matrices in $\mathfrak{g}$

$$X(u,w,a,y,z)$$

$$= \begin{pmatrix} 0 & -\frac{u_3+w_3}{2} & \frac{u_2+w_2}{2} & -\frac{u_1-w_1}{2} & -y_2 & y_3 & a_1 \\ \frac{u_3+w_3}{2} & 0 & -\frac{u_1+w_1}{2} & -\frac{u_2-w_2}{2} & y_1 & a_2 & z_3 \\ -\frac{u_2+w_2}{2} & \frac{u_1+w_1}{2} & 0 & -\frac{u_3-w_3}{2} & a_3 & z_1 & -z_2 \\ \frac{u_1-w_1}{2} & \frac{u_2-w_2}{2} & \frac{u_3-w_3}{2} & 0 & y_3-z_3 & y_2-z_2 & y_1-z_1 \\ -y_2 & y_1 & a_3 & y_3-z_3 & 0 & w_1 & -w_2 \\ y_3 & a_2 & z_1 & y_2-z_2 & -w_1 & 0 & w_3 \\ a_1 & z_3 & -z_2 & y_1-z_1 & w_2 & -w_3 & 0 \end{pmatrix}, \tag{2.2}$$

where $u=(u_1,u_2,u_3)$ etc., and $\sum_{i=1}^{3} a_i=0$.

The involution $X \mapsto -X^t$ restricts to a Cartan automorphism of $\mathfrak{g}$. Its fixed subalgebra, the maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is clearly isomorphic to so(4) $=\mathrm{so}(3)\oplus\mathrm{so}(3)$. The variables u_i, w_i in (2.2) are chosen so that if we define the matrices e_i, e_i' by $\sum_{i=1}^{3} u_i e_i = X(u,0,0,0,0)$, $\sum_{i=1}^{3} w_i e_i' = X(0,w,0,0,0)$, then $\{e_i\}$ and $\{e_i'\}$ are standard bases in the so(3) components of $\mathfrak{k}$, i.e. $[e_i,e_j]=\varepsilon_{ijk}e_k$, $[e_i',e_j']=\varepsilon_{ijk}e_k'$. Symmetric matrices of the form $\mathbf{a}=X(0,0,a,0,0)$ make up a split Cartan subalgebra of $\mathfrak{g}$ (contained in the symmetric subspace $\mathfrak{p}$).

Let G be the connected subgroup of SO(4, 3) that corresponds to the Lie algebra $\mathfrak{g}$. Clearly, the maximal compact subgroup K of G with the Lie algebra $\mathfrak{k}$ is isomorphic to SO(4) and its representation on $\mathbb{R}^7$ is the direct sum of representations of dimension 4 and 3.

We are now in a position to write explicit formulae for the tops associated with the pair (G,K). As was pointed out in Sect. 1, the Hamiltonians (1.4) and (1.4′) are equivalent on T^*K, so we shall only consider (1.4).

Fix an inner product on $\mathfrak{g}$ by setting

$$(X,Y) = -\mathrm{tr}\, XY. \tag{2.3}$$

Using the expression (1.5) for the angular velocity operator, we can easily compute the kinetic energy $E(u,w)$ of the top. Now the point is that since we identify $\mathfrak{k}^*$ and $\mathfrak{k}$ by means of the inner product (2.3), the Poisson brackets of linear coordinates x_μ relative to some basis in $\mathfrak{k}$ have the form $\{x_\mu, x_\nu\} = \sum_\sigma c_{\mu\nu}^\sigma x_\sigma$, where the $c_{\mu\nu}^\sigma$ are the structure constants of $\mathfrak{k}$ in the dual basis with respect to the inner product (2.3). We therefore obtain $\{u_i,u_j\}=\varepsilon_{ijk}u_k$, $\{w_i,w_j\}=\frac{1}{3}\varepsilon_{ijk}w_k$. Taking the normalized variables u and $v=3w$ and assuming that the matrix $\mathbf{a}$ [and hence $\mathbf{b}=d\varphi(\mathbf{a})$] lies in the

Cartan subalgebra, we come down to the following expression for the kinetic energy:

$$E(u,v)=\tfrac{1}{24}\sum_{i=1}^{3}\{(9c_i+3d_i)u_i^2+6(d_i-c_i)u_iv_i+(c_i+3d_i)v_i^2\}\,, \tag{2.4}$$

where

$$c_i=\frac{b_i}{a_i}, \qquad d_i=\frac{b_j-b_k}{a_j-a_k}, \qquad \sum_{i=1}^{3}a_i=0=\sum_{i=1}^{3}b_i\,, \tag{2.5}$$

(i,j,k) being a permutation of $(1,2,3)$.

Let us first consider the case of zero potential [i.e. $f=0$ in (1.4)]. The quadratic Hamiltonians given by (1.4) form a two-dimensional linear involutive family $\left(a \text{ is fixed and } b \text{ is subject to } \sum_{i=1}^{3} b_i=0\right)$. But if b is proportional to a, the corresponding E is proportional to the Casimir function $3\sum_{i=1}^{3}u_i^2+\sum_{i=1}^{3}v_i^2$. Thus, for fixed a, (2.4) gives a unique (up to a factor and the addition of a Casimir function) quadratic Hamiltonian. A result due to A. Veselov (see below) shows that the system defined by (2.4) has no additional quadratic integral of motion. Still, there is a quartic integral,

$$I_4=\operatorname{Res}\lambda^{-3}\operatorname{tr}L(\lambda)^6\,. \tag{2.6}$$

Observe that when $\mathbf{a}$ and $\mathbf{b}$ vary in the Cartan subalgebra, (2.4) gives a three-parameter family of integrable systems. However, two of these parameters are killed by adding a Casimir function and multiplying by a number. Hence (2.4) effectively defines a one-parameter family of integrable systems.

Let us now discuss the potential energy term which in general has the form

$$V(k)=-\operatorname{tr}(kbk^tf)\,. \tag{2.7}$$

The 7×7 matrix k splits into two blocks of dimension 4 and 3; the first block is just the standard realization of SO(4) and the second one, which we denote by $\pi(k)$, is the representation of SO(4) in the space of self-dual skew-symmetric tensors in $\mathbb{R}^4$. Recall that b lies in the Cartan subalgebra. Now in terms of SO(4), i.e. of the matrix elements $k_{\mu\nu}$, $\mu,\nu=1,2,3,4$, the potential V becomes

$$V(k)=-2\sum_{i,j=1}^{3}\sum_{\mu=1}^{4}b_if_{\mu 4+j}k_{\mu i}\pi_{j4-i}(k) \tag{2.8}$$

(notice that without loss of generality f may also be taken in the Cartan subalgebra). The matrix coefficients $\pi_{ij}(k)$, $i,j=1,2,3$ are quadratic forms of the $k_{\mu\nu}$, so that (2.7) is a cubic form of the matrix elements of SO(4). Clearly, this form does not descend to the quotient group $\mathrm{SO}(3)\times\mathrm{SO}(3)=\mathrm{SO}(4)/(\pm I)$.

For the reader's convenience we shall write explicit expressions for the Lax matrices of the G_2-top given by (1.6)–(1.7). With the notation (2.2) me have

$$\begin{aligned} L(\lambda)&=X(u,\tfrac{1}{3}v,a\lambda,0,0)+s\lambda^{-1}\,,\\ M_+(\lambda)&=X(u',\tfrac{1}{3}v',b\lambda,0,0)\,, \end{aligned} \tag{2.9}$$

where u', v' are related to u, v by

$$u_i' + v_i' = d_i(u_i + v_i)\,, \qquad u_i' - \tfrac{1}{3}v_i' = c_i(u_i - \tfrac{1}{3}v_i) \tag{2.10}$$

and $s = s(k) = k^t f k$, where f is a fixed symmetric matrix in $\mathfrak{g}$.

To close the section, let us compare (2.4) with the known examples of integrable quadratic Hamiltonians on so(4). There are two explicit families, the so-called Manakov and Steklov cases, characterized by the property that there exists an additional quadratic integral. It is not hard to show that (2.4) is not contained in either of these families. To this end it is appropriate to recall a result due to A. P. Veselov: If a quadratic Hamiltonian H on so(4) admits an additional analytic integral, then (under mild nondegeneracy assumptions) it can be reduced to diagonal form

$$H(u, v) = \sum_{i=1}^{3} (\alpha_i u_i^2 + 2\beta_i u_i v_i + \gamma_i v_i^2)\,, \tag{2.11}$$

and the coefficients satisfy the relations

$$\begin{aligned}
&(\alpha_1 - \alpha_2)\beta_3^2 + (\alpha_2 - \alpha_3)\beta_1^2 + (\alpha_3 - \alpha_1)\beta_2^2 \\
&\quad + m^2(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1) = 0\,, \\
&(\gamma_1 - \gamma_2)\beta_3^2 + (\gamma_2 - \gamma_3)\beta_1^2 + (\gamma_3 - \gamma_1)\beta_2^2 \\
&\quad + n^2(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)(\gamma_3 - \gamma_1) = 0
\end{aligned} \tag{2.12}$$

for some (odd) integers m and n. For both Manakov and Steklov cases $m = n = 1$, while (2.4) satisfies (2.12) with $m = 1$, $n = 3$, so that there are no additional quadratic integrals in this case.

On the other hand, Adler and van Moerbeke [10] announced a classification of algebraically completely integrable Hamiltonians of the form (2.11). They claim that besides the Manakov and Steklov cases there is only one integrable family, for which the additional integral is quartic. In that case the coefficients of (2.11) are subject to the following system of equations

$$\beta_i^4 = (\alpha_i - \alpha_j)(\alpha_i - \alpha_k)(\gamma_i - \gamma_j)(\gamma_i - \gamma_k)\,, \tag{2.13}$$

(i, j, k) being a permutation of $(1, 2, 3)$, and, moreover, either the ratios $\delta_i^2 = \dfrac{\alpha_j - \alpha_k}{\gamma_j - \gamma_k}$ or their inverses satisfy

$$\delta_1\delta_2 + \delta_2\delta_3 + \delta_3\delta_1 = -1 \quad \text{and} \quad 3\delta_1\delta_3 + \delta_1 - \delta_3 = -1\,. \tag{2.14}$$

It can be shown that (2.4) satisfies (2.13)–(2.14), so that we get a Lie-algebraic derivation of the hitherto mysterious case of the classification.

3. Integrable Systems Connected with the Groups SL($n, \mathbb{R}$), SO(n, n), SO(n, 1)

To illustrate the general scheme, let us consider three other Lie groups. The group SL($n, \mathbb{R}$) leads to a system that describes an n-dimensional top in a quadratic potential, while SO(n, n) and SO(n, 1) give a pair of interacting tops and a symmetric heavy top (the Lagrange case), respectively.

For $G=\mathrm{SL}(n,\mathbb{R})$ the maximal compact subgroup is $\mathrm{SO}(n)$. Hence formulae (1.4)–(1.5) give rise to a system on $T^*\mathrm{SO}(n)$ with the Hamiltonian

$$H(k,\varrho)=\sum_{i<j}\frac{b_i-b_j}{a_i-a_j}\varrho_{ij}^2+\sum_{i,j,l}b_i f_{jl}k_{ji}k_{li}, \tag{3.1}$$

describing the motion of an n-dimensional top in an arbitrary quadratic potential.

For $G=\mathrm{SO}(n,n)$ the maximal compact subgroup is $\mathrm{SO}(n)\times\mathrm{SO}(n)$. Hence we get a system on $T^*\mathrm{SO}(n)\times T^*\mathrm{SO}(n)$ described by the Hamiltonian

$$\begin{aligned}H(k,\varrho;k',\sigma)=&\sum_{i<j}\frac{a_ib_i-a_jb_j}{a_i^2-a_j^2}(\varrho_{ij}^2+\sigma_{ij}^2)\\&-2\sum_{i<j}\frac{a_ib_j-a_jb_i}{a_i^2-a_j^2}\varrho_{ij}\sigma_{ij}-2\sum_{i,j,l}b_if_{jl}k_{ji}k'_{li}.\end{aligned} \tag{3.2}$$

This Hamiltonian includes interaction of two tops by means of their kinetic momenta and also by means of a bilinear potential. In a special case when $b=a$ this amounts to a pure potential interaction of two spherical tops. If, moreover, all a_i are equal to 1, our system reduces to a spherical top in a linear potential field which may be regarded as a superposition of n homogeneous fields of different nature.

For $G=\mathrm{SO}(n,1)$, the maximal compact subgroup is $\mathrm{SO}(n)$, so our phase space is again the same as for the n-dimensional top. Since $\mathrm{so}(n,1)$ is not split, the simple formula (1.6) is no longer valid. The general formula given in the lemma of Sect. 4 easily yields

$$H(k,\varrho)=\alpha\,\mathrm{tr}\,\varrho^2+\beta\,\mathrm{tr}\,\varrho_a^2+2\alpha(f,ka). \tag{3.3}$$

Here $a,f\in\mathbb{R}^n$, $\varrho_a=P_a\varrho P_a$, P_a is the orthogonal projection operator onto the hyperplane in $\mathbb{R}^n$ orthogonal to a. The Hamiltonian (3.3) describes the motion of a heavy top in a homogeneous gravitational field. It corresponds to the so-called Lagrange case, the inertia tensor of the top being symmetric with respect to the a axis. Here f is the gravity strength, a is the center-of-mass vector in the moving frame, hence ka is the center-of-mass vector in the rest frame.

There are also some interesting cases of interacting so(3)-tops which are connected with the splitting $\mathrm{so}(4)=\mathrm{so}(3)+\mathrm{so}(3)$. For example, a pair of interacting so(4)-tops connected with so(4, 4) may be regarded as a system of four so(3)-tops. In the normalized variables u, v, w, z on $\bigoplus_1^4\mathrm{so}(3)$, the kinetic energy term has the form

$$\begin{aligned}E(u,v,w,z)=\sum_{i=1}^{3}\{&(c_{jk}+c_{4i})(u_i^2+v_i^2+w_i^2+z_i^2)\\&+2(c_{jk}-c_{4i})(u_iv_i+w_iz_i)\\&-2(d_{jk}+d_{4i})(u_iw_i+v_iz_i)\\&-2(d_{jk}-d_{4i})(u_iz_i+v_iw_i)\,,\end{aligned} \tag{3.4}$$

where $c_{\mu\nu}=\dfrac{a_\mu b_\mu-a_\nu b_\nu}{a_\mu^2-a_\nu^2}$, $d_{\mu\nu}=\dfrac{a_\mu b_\nu-a_\nu b_\mu}{a_\mu^2-a_\nu^2}$, and the subscripts (i,j,k) form a permutation of $(1,2,3)$. The potential energy

$$V(k,k')=\sum f_{\mu\nu}b_\sigma k_{\mu\sigma}k'_{\nu\sigma}\,,$$

$k, k' \in SO(4)$ may be regarded as a function on the product $\prod_1^4 SU(2)$ covering the group SO(4) covering the group SO(4) × SO(4) and is a 4-linear form in its entries.

Another case is connected with SO(4, 3). Its maximal compact subgroup is SO(4) × SO(3) and is covered by $\prod_1^3 SU(2)$. This gives a system of three interacting tops. In the variables u, v, w on $\bigoplus_1^3 so(3)$, the kinetic term is given by

$$E(u, v, w) = \sum_{i=1}^{3} \left\{ \left(c_{jk} + \frac{b_i}{a_i} \right) (u_i^2 + v_i^2) + c_{jk} w_i^2 \right.$$
$$\left. + 2\left(c_{jk} - \frac{b_i}{a_i} \right) u_i v_i - 2d_{jk}(u_i w_i + v_i w_i) \right., \tag{3.5}$$

where c_{ij}, d_{ij} are the same as above, (i, j, k) is a permutation of $(1, 2, 3)$. The potential energy is now a tri-linear function on SU(2) × SU(2) × SO(3).

4. Affine Lie Algebras and the Proof of Theorem 1

We recall briefly a general group-theoretic construction of integrable systems [7, 5]. Theorem 2 below generalizes the Kostant-Adler-Symes commutativity theorem (cf. [12]).

Let $\mathfrak{g}$ be a Lie algebra, R a linear operator on $\mathfrak{g}$. We say that R defines the structure of a double Lie algebra on $\mathfrak{g}$ if

$$[x, y]_R = \tfrac{1}{2}([RX, Y] + [X, RY]) \tag{4.1}$$

is a Lie bracket, i.e. it satisfies the Jacobi identity. In that case, the space $\mathfrak{g}^*$ is equipped with a second Lie-Poisson bracket which will be referred to as the R-bracket. The operator R is called the classical R-matrix. Let $I(\mathfrak{g})$ be the algebra of Ad*-invariant functions on $\mathfrak{g}^*$ (here Ad* denotes the coadjoint representation with respect to the original Lie bracket).

A typical example of the R-matrix is constructed as follows. Let $\mathfrak{g}_+, \mathfrak{g}_- \subset \mathfrak{g}$ be Lie subalgebras such that $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ as a linear space. Let $P_\pm$ be the projection operators onto $\mathfrak{g}_\pm$ parallel to the complementary subalgebra. Put

$$R = P_+ - P_- . \tag{4.2}$$

In that case (which is the most important one for applications) the R-bracket is the difference of the Lie-Poisson brackets for $\mathfrak{g}_+$ and $\mathfrak{g}_-$.

Theorem 2. (i) *Functions from $I(\mathfrak{g})$ Poisson commute with respect to the R-bracket on $\mathfrak{g}^*$.* (ii) *Hamilton's equations of motion defined by $\varphi \in I(\mathfrak{g})$ with respect to the R-bracket have the form*

$$\frac{d}{dt}\xi = -\mathrm{ad}^* M \cdot \xi, \quad M = \tfrac{1}{2} R d\varphi(\xi) .$$

If R is given by (4.2), *the equations of motion can be written as*

$$\frac{d}{dt}\xi = -\mathrm{ad}^* M_\pm \cdot \xi, \tag{4.3}$$

where $M_\pm = \pm P_\pm d\varphi(\xi)$.

Proof. The R-bracket of two functions φ, ψ on $\mathfrak{g}^*$ is

$$\{\varphi, \psi\}_R(\xi) = \tfrac{1}{2}\langle [Rd\varphi(\xi), d\psi(\xi)], \xi\rangle + \tfrac{1}{2}\langle [d\varphi(\xi), Rd\psi(\xi)], \xi\rangle .$$

The invariance of φ is equivalent to $\langle [d\varphi(\xi), X], \xi\rangle = 0$ for all $X \in \mathfrak{g}$. This implies (i). Now, if φ is an invariant Hamiltonian, the equations of motion $\frac{d}{dt}\psi = \{\varphi, \psi\}_R$ reduce to

$$\frac{d}{dt}\psi(\xi) = \tfrac{1}{2}\langle [Rd\varphi(\xi), d\psi(\xi)], \xi\rangle = -\tfrac{1}{2}\langle d\psi(\xi), \mathrm{ad}^* Rd\varphi(\xi)\cdot \xi\rangle .$$

Hence $\frac{d}{dt}\xi = -\frac{1}{2}\mathrm{ad}^* Rd\varphi(\xi)\cdot\xi$. Since $\mathrm{ad}^* d\varphi(\xi)\cdot\xi = 0$, the right-hand side can be replaced by $-\frac{1}{2}\mathrm{ad}^*(R\pm 1)d\varphi(\xi)\cdot\xi = -\mathrm{ad}^* M_\pm\cdot\xi$. □

If there is a non-degenerate invariant inner product on $\mathfrak{g}$, then $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$ and Eq. (4.3) takes the Lax form

$$\frac{d}{dt}L = [L, M_\pm] . \tag{4.4}$$

To demonstrate the assertions of the previous sections, we apply Theorem 2 to the Lie algebras defined as follows. Let σ be an involution in $\mathfrak{g}$. Let $\mathfrak{L}(\mathfrak{g}) = \mathfrak{g}\otimes\mathbb{R}[\lambda, \lambda^{-1}]$ be the loop algebra of $\mathfrak{g}$, i.e. the algebra of Laurent polynomials with values in $\mathfrak{g}$. We extend the involution σ to $\mathfrak{L}(\mathfrak{g})$ by setting $(\tilde{\sigma}X)(\lambda) = \sigma(X(-\lambda))$. By definition, the twisted loop algebra $\mathfrak{L}(\mathfrak{g}, \sigma)$ is the fixed subalgebra of $\tilde{\sigma}$. In other words, $\mathfrak{L}(\mathfrak{g}, \sigma)$ consists of Laurent polynomials $X(\lambda) = \Sigma x_i\lambda^i$ such that $\sigma x_i = (-1)^i x_i$. Recall the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the fixed subalgebra of σ, and the dual decomposition $\mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{p}^*$.

If $\mathfrak{g}$ is a simple Lie algebra, then the twisted loop algebra $\mathfrak{L}(\mathfrak{g}, \sigma)$ is an example of an affine Lie algebra.

We identify the dual space $\mathfrak{L}(\mathfrak{g}, \sigma)^*$ with the subspace of Laurent polynomials $\xi(\lambda) = \Sigma\xi_i\lambda^i$ with coefficients in $\mathfrak{g}^*$ such that $\sigma^*\xi_i = (-1)^i\xi_i$. The pairing between $\mathfrak{L}(\mathfrak{g}, \sigma)$ and $\mathfrak{L}(\mathfrak{g}, \sigma)^*$ is given by

$$\langle X, \xi\rangle = \mathrm{Res}\,\lambda^{-1}\langle X(\lambda), \xi(\lambda)\rangle . \tag{4.5}$$

The standard decomposition $\mathfrak{L}(\mathfrak{g}, \sigma) = \mathfrak{L}_+ + \mathfrak{L}_-$ is defined by the natural $\mathbb{Z}$-grading in powers of λ,

$$\mathfrak{L}_+ = \left\{\sum_{i\geqq 0} x_i\lambda^i\right\}, \qquad \mathfrak{L}_- = \left\{\sum_{i<0} x_i\lambda^i\right\}. \tag{4.6}$$

It is easy to see that the polynomial functions

$$\varphi_k(X) = \mathrm{Res}\,\lambda^{-k}\varphi(X(\lambda)), \qquad \varphi \in I(\mathfrak{g}) \tag{4.7}$$

are invariant functions on $\mathfrak{L}(\mathfrak{g}, \sigma)^*$.

In order to derive Theorem 1 from Theorem 2 it remains to prove the following assertion (cf. [7, 11]).

Theorem 3. *Let $a, f \in \mathfrak{p}^*$. The mapping $\mu: T^*K \to \mathfrak{L}(\mathfrak{g}, \sigma)^*$,*

$$\mu(k, \varrho) = a\lambda + \varrho + \mathrm{Ad}^* k^{-1} f \lambda^{-1}$$

is a Poisson mapping with respect to the R-bracket in $\mathfrak{L}(\mathfrak{g}, \sigma)^*$ defined by the splitting (4.6), *with R given by* (4.2).

Proof. Let $X, Y \in \mathfrak{L}(\mathfrak{g}, \sigma)$ be linear functions on $\mathfrak{L}(\mathfrak{g}, \sigma)^*$. We must check that

$$\{X \circ \mu, Y \circ \mu\} = [X, Y]_R \circ \mu. \tag{4.8}$$

It is sufficient to consider three cases, $X = s\lambda^{-1}$, $X = \pi$, and $X = s\lambda$, where $s \in \mathfrak{p}$, $\pi \in \mathfrak{k}$, and similar alternatives for Y. It is easy to see that the function $s\lambda^{-1}$ commutes with all the others. Moreover, for $X = s\lambda$, $Y = s'\lambda$, both sides of (4.8) are zero. Now, observe that the mapping $T^*K \to \mathfrak{k}$, $(k, \varrho) \mapsto \varrho$ preserves the Poisson brackets, since it is the moment map for the natural action of K on T^*K by right translations. This implies (4.8) for the case $X = \pi$, $Y = \pi'$, $\pi, \pi' \in \mathfrak{k}$. Finally, let $X = \pi$, $Y = s\lambda$. In that case, $X \circ \mu$ is the Hamiltonian of right translations from the subgroup $\exp t\pi$. Hence we get

$$\begin{aligned}\{X \circ \mu, Y \circ \mu\}(k, \varrho) &= \left(\frac{d}{dt}\right)_{t=0} Y \circ \mu(k e^{t\pi}, \mathrm{Ad}^* e^{-t\pi} \varrho) \\ &= \left(\frac{d}{dt}\right)_{t=0} \langle Y, \mathrm{Ad}^*(e^{-t\pi} k^{-1}) \cdot f \rangle = -\langle Y, \mathrm{ad}^* \pi \, \mathrm{Ad}^* k^{-1} \cdot f \rangle \\ &= \langle [\pi, Y], \mathrm{Ad}^* k^{-1} f \rangle = [\pi, Y] \circ \mu(k, \varrho).\end{aligned}$$

To complete our argument, notice that in this case $[X, Y]_R = [X, Y]$. □

Remark. The range of μ may be naturally identified with (a Poisson submanifold of) the dual space of the semidirect product $\mathfrak{g}_0 = \mathfrak{k} \propto \mathfrak{p}$ equipped with its natural Lie-Poisson bracket. It is easy to include μ in the dual pair in the sense of [11]. Namely, let K_f be the stationary subgroup of f, $\mathfrak{k}_f$ its Lie algebra and let $\nu: T^*K \to \mathfrak{k}_f^*$ be the moment map corresponding to the natural action of K_f on T^*K by left translations. One checks without difficulty that $\mathfrak{k}_f^* \xleftarrow{\nu} T^*K \xrightarrow{\mu} \mathfrak{g}_0^*$ is a dual pair. In particular, symplectic leaves in the range of μ are obtained by Hamiltonian reduction with respect to the action of K_f. Note that the Hamiltonians (1.4) are clearly right K_a-invariant and left K_f-invariant. Under the involution $k \mapsto k^{-1}$, $\varrho \mapsto \pi = -\mathrm{Ad}^* k \cdot \varrho$, $a \leftrightarrow f$ the functions φ_+ go into φ_-. The difference between the two Lax representations for these equivalent systems stems from the fact that the associated mappings μ are different and correspond to reductions with respect to the action of different groups K_a, K_f.

Let us now derive (1.5) along with a more general expression for $\Omega_{a,\varphi}$ valid for any real simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition, let $\mathfrak{k}_a$ be the stationary subalgebra of a point $a \in \mathfrak{p}$, $\mathfrak{k}_a^\perp$ its orthogonal complement. Consider the function $\varphi_a(\varrho) = \varphi(\varrho + a)$ on $\mathfrak{k}_a$ and let $d^2\varphi_a \in \mathrm{End}\, \mathfrak{k}_a$ be its second derivative at $\varrho = 0$. Put $b = d\varphi(a)$.

Lemma. *$\Omega_{a,\varphi}=d^2\varphi_a$ on $\mathfrak{k}_a$ and $\Omega_{a,\varphi}=\operatorname{ad}b(\operatorname{ad}a)^{-1}$ on $\mathfrak{k}_a^{\perp}$.*

Proof. The expression above is unambiguous since $\ker\operatorname{ad}a\subset\ker\operatorname{ad}b$. Let us consider the function φ_+. Since it is defined as the coefficient of λ^{-2} in the polynomial $\varphi(a+\varrho\lambda^{-1}+s\lambda^{-2})$, $s=\operatorname{Ad}k^{-1}f$, we have $\varphi_+(L)=\frac{1}{2}d^2\varphi(a)(\varrho)+(d\varphi(a),s)$. To compute the quadratic form $d^2\varphi(a)(\varrho)$, put $\varrho=\varrho_1+\varrho_2$, $\varrho_1\in\mathfrak{k}_a$, $\varrho_2\in\mathfrak{k}_a^{\perp}$ (note that $\operatorname{ad}a\cdot\mathfrak{p}=\mathfrak{k}_a^{\perp}$), and let $X=(\operatorname{ad}a)^{-1}\varrho_2$, i.e. $X\in\mathfrak{p}$, $\operatorname{ad}a\cdot X=\varrho_2$. Then for $g=\exp\varepsilon X$, we have

$$\operatorname{Ad}g(a+\varepsilon\varrho)=a+\varepsilon\varrho_1+\frac{\varepsilon^2}{2}[X,\varrho]+\varepsilon^2[a,v]$$

up to terms of higher degree in ε. Here v is determined by the ε^2-term in the expansion of g. Since φ is invariant, we have $\varphi(a+\varepsilon\varrho)=\varphi(\operatorname{Ad}g(a+\varepsilon\varrho))$. Computing the coefficient of ε^2, and taking into account that $(d\varphi(a),[a,v])=0$, we get

$$\begin{aligned}d^2\varphi(a)(\varrho)&=(d^2\varphi_a\varrho_1,\varrho_1)+(d\varphi(a),[X,\varrho])\\&=(d^2\varphi_a\varrho_1,\varrho_1)+([d\varphi(a),X],\varrho)=(\Omega_{a,\varphi}\varrho,\varrho)\,. \quad\square\end{aligned}$$

We now give the proof of (1.7). Consider the function $\varphi_+(L)=\operatorname{Res}\lambda\varphi(L(\lambda)\lambda^{-1})$ on the whole space $\mathfrak{L}(\mathfrak{g},\sigma)$. Clearly, its gradient is $d\varphi_+(L)=\lambda d\varphi(L(\lambda)\lambda^{-1})$, where $d\varphi$ is the gradient of φ on $\mathfrak{g}$. For $L(\lambda)=a\lambda+\varrho+s\lambda^{-1}$, we have $d\varphi(L(\lambda)\cdot\lambda^{-1})=d\varphi(a)+\Omega_{a,\varphi}\varrho\lambda^{-1}+\ldots$ plus terms of lower degree in λ. Since M_+ is the projection of $d\varphi_+(L)$ to $\mathfrak{L}_+$, we get $M_+=d\varphi(a)\lambda+\Omega_{a,\varphi}\varrho$. The proof of (1.7′) is quite similar. $\square$

There is a criterion of when the kinetic energy in (1.4) is positive. Let $\mathfrak{a}\subset\mathfrak{p}$ be the split part of the Cartan subalgebra containing the point $a\in\mathfrak{p}$, and let $\mathfrak{a}_+$ be the closure of the Weyl chamber containing a. If $d\varphi(a)\in\mathfrak{a}_+$ and the quadratic form $d^2\varphi_a$ is positive definite, the form $(\Omega_{a,\varphi}\varrho,\varrho)$ is positive. It is positive definite if the centralizers of a and $d\varphi(a)$ in $\mathfrak{k}$ coincide.

Finally, let us point out that the problem of solving equations (4.3) can be reduced to the following factorization problem. Let $G_\pm$ be the subgroups that correspond to the subalgebras $\mathfrak{g}_\pm$, and let $g_\pm(t)$ be the solution of the factorization problem $\exp tM=g_+(t)^{-1}g_-(t)$, $M=d\varphi(\xi)$, where $g_\pm(t)$ are smooth functions with values in the subgroups $G_\pm, g_\pm(0)=I$. Then the solution of (4.3) is given by $\xi(t)=\operatorname{Ad}^*g_\pm(t)\cdot\xi$. In the context of affine Lie algebras this immediately implies that the equations of motion linearize on the Jacobian of the spectral curve of the Lax matrix (cf. [7]). Therefore their solutions (in particular, the solutions of the G_2-top equations) can be expressed in terms of the associated theta functions and are meromorphic functions of the time variable.

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