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Map Dependence of the Fractal Dimension Deduced from Iterations of Circle Maps

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Abstract. Every orientation preserving circle map g with inflection points, including the maps proposed to describe the transition to chaos in phase-locking systems, gives occasion for a canonical fractal dimension D, namely that of the associated set of μ for which $f_\mu = \mu + g$ has irrational rotation number. We discuss how this dimension depends on the order r of the inflection points. In particular, in the smooth case we find numerically that $D(r) = D(r^{-1}) = r^{-1/8}$.

1. Introduction

Mathematical models for periodically stimulated oscillators are usually formulated as a system of coupled differential equations [1–5]. The associated Poincaré map gives the oscillator state at time n/v as a function of the state at time $(n-1)/v$, where v is the external frequency. In appropriate limits it has often been possible to reduce this map to a one-dimensional map of the form of those we consider here [2], [6–8].

The investigation of these circle maps has been particularly useful in studying the transition to chaos [9–11]. The fractal obtained along the critical line defined by the points where chaos sets in, is described by the fractal dimension [12] obtained from iterations of a circle map [5], and this dimension seems to be universal [9, 10]. As an example, the transition to hysteresis and chaos of the resistively shunted Josephson junction modulated by an rf microwave signal [13] can be modelled by the behavior of a circle map which passes from invertibility to non-invertibility through development of an inflection point of order three [8, 14]. This transition gives occasion for a complete devil's staircase structure [12], where the fractal dimension of the associated Cantor set is $D = 0.87$ [10].

In this paper we study numerically maps with inflection points with orders other than three. In particular we find that the related fractal dimension varies like the $1/8^{\text{th}}$ power of the order. In Sect. 2 we define a set G of circle maps, and in Sect. 3 we report the results of a numerical investigation of the fractal dimension of the non-phaselocking Cantor set for these maps. In Sect. 4 the group G is extended, and finally in Sect. 5 we discuss some cases where the map does not belong to G.

2. A Group G of Circle Maps

We start with a definition: Let G be the set of real functions $g: \mathbb{R} \to \mathbb{R}$ which satisfy the following conditions[1]:

(i) $g(0)=0$,

(ii) $g(x+1)=g(x)+1$,

(iii) g is bijective,

(iv) Both g and g^{-1} are continuously differentiable on $\mathbb{R}\backslash\mathbb{Z}$, and one of them is continuously differentiable at the integers $\mathbb{Z}$ as well.

(v) There exists a positive real number r such that

$$\lim_{x \to 0^+} \frac{g(x)}{x^r} \quad \text{and} \quad \lim_{x \to 0^-} \frac{g(x)}{(-x)^r}$$

exist in $\mathbb{R}$ and are non-zero.

In this case, we say that g has order r.

For clarity we have not included functions with the inflection point not placed at the origin, or functions which have more than one inflection point. We return to these types later.

It is easy to see that G forms a group under composition; that if $g \in G$ has order r, g^{-1} has order r^{-1}; and that if $f \in G$ has order r and $g \in G$ has order s, then $f \circ g$ has order $r \cdot s$. Also if we define $i_g(x) = -g(-x)$ for $g \in G$, then $i_g \in G$ and has the same order as g.

3. Two Questions

We now define $f_{\mu,g}: \mathbb{R} \to \mathbb{R}$ for $\mu \in [0;1]$ and $g \in G$ to be given by

$$f_{\mu,g}(x) = \mu + g(x). \tag{3.1}$$

The rotation number

$$W_g(\mu) = \lim_{n \to \infty} \frac{f^n_{\mu,g}(x_0) - x_0}{n} \tag{3.2}$$

is a continuous function of μ and g (in the C^0-topology) and does not depend on x_0 [15, 16]. As a function of μ, $W_g(\mu)$ creates a devil's staircase with a fractal dimension $D(g) \leq 1$ [12].

In this connection a question arises:

I: *Does $D(g)$ only depend on the order?*

If $g \in C^3(\mathbb{R})$ and has the order $r=1$, Herman has solved the problem affirmatively [17]. The points μ with an irrational rotation number have a positive Lebesgue

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[1] From (ii) and (iv) we get that both g and g^{-1} are continuous, and by (ii) we conclude that g induces an orientation preserving homeomorphism of the circle onto itself. The condition (v) contains the Hölder condition of degree $r-[r]$ at zero, see e.g. [15]

measure and that means $D(g)=1$. In the special case where

$$g(x)=x-\frac{1}{2\pi}(\sin 2\pi x+a\sin^3 2\pi x), \frac{4}{3}<a<\frac{1}{6}, \tag{3.3}$$

i.e. the order $r=3$, the question is analyzed in [10], and it seems that $D(g)=0.87$ independent of a.[2]

We assert the following:

Theorem 3.1. *If $g\in G$, then $D(g^{-1})=D(g)=D(i_g)$.*

Proof. Call $(\mu, \theta_1, \ldots, \theta_n)$ a $\frac{m}{n}$ cycle for g if $f_{\mu,g}(\theta_i)=\theta_{i+1}$ for $1\leqq i<n$ and $f_{\mu,g}(\theta_n)=\theta_1+m$.

The theorem follows then from the next two lemmas, and the fact that a map has rotation number $\frac{m}{n}$ if and only if it has a $\frac{m}{n}$ cycle [16].

Lemma 3.2. *If $(\mu, \theta_1, \ldots, \theta_n)$ is a $\frac{m}{n}$ cycle for $g\in G$, then $(1-\mu, g(\theta_1), g(\theta_n)-m+1, g(\theta_{n-1})-m+2, \ldots, g(\theta_2)-m+n-1)$ is a $\left(1-\frac{m}{n}\right)$ cycle for g^{-1}.*

Lemma 3.3. *If $(\mu, \theta_1, \ldots, \theta_n)$ is a $\frac{m}{n}$ cycle for $g\in G$, then $(1-\mu, 1-\theta_1, 2-\theta_2, \ldots, n-\theta_n)$ is a $\left(1-\frac{m}{n}\right)$ cycle for i_g.*

The proofs of the two lemmas are left for the reader.

If we accept the answer to question I to be affirmative, we have

$$D(g)=D(|\ln r|),$$

where r belongs to g. Under these circumstances we ask another question.

II: *Let $z_f=|\ln r|$, where r belongs to $f\in G$ and $z_g=|\ln s|$, where s belongs to $g\in G$. Is $D(z_f+z_g)$ equal to $D(z_f)\cdot D(z_g)$?*

If f and g both belong to one of the semigroups $\{g\in G|\text{order}\geqq 1\}$ or $\{g\in G|\text{order}\leqq 1\}$, we can reformulate the question:

II′: *Is $D(f\circ g)=D(f)\cdot D(g)$?*

Question I is then the special case where $f=h\circ g^{-1}$ has $z_f=0$.

To treat question II we define

$$g_r(x)=\begin{cases}\frac{1}{2}(2x)^r & 0\leqq x<\frac{1}{2}\\ 1-\frac{1}{2}(2(1-x))^r & \frac{1}{2}\leqq x<1,\end{cases} \tag{3.4}$$

g_r is extended to $\mathbb{R}$ by the condition (ii). Then $g_r\in G$, and g_r has the order r.

[2] If a is chosen to be 1/6, the related number will be $r=5$, and the fractal dimension has been calculated to be $D\simeq 0.81$ [11]

To calculate the fractal dimension, we put down a grid of points $\mu_i = i/J$, $0 \leqq i \leqq J$ and calculate for each μ_i an approximation $W(\mu_i)$ to the rotation number of f_{μ_i, g_r} as follows: It is easy to see that f_{μ, g_r} has a $\frac{0}{1}$ cycle if $0 \leqq \mu \leqq \omega_{0.5}$ and a $\frac{1}{1}$ cycle if $1-\omega_{0.5} \leqq \mu \leqq 1$, where

$$\omega_{0.5} = \frac{|r-1|}{2r^{r/(r-1)}}. \tag{3.5}$$

Thus, we put $W(\mu_i)=0$ for $0 \leqq \mu_i \leqq \omega_{0.5}$ and $W(\mu_i)=1$ for $1-\omega_{0.5} \leqq \mu_i \leqq 1$. For the remaining μ_i's, we put $W(\mu_i)$ equal to $\frac{1}{M}$ times the integer part of $f^M_{\mu_i, g_r}(0)$, with M chosen substantially larger than J (to reduce the round-off error in computing $f^M_{\mu_i, g_r}(0)$, we kept successive iterates $f^j_{\mu_i, g_r}(0)$ in [0, 1] by reducing by 1 whenever the value exceeded or reached 1 and keeping track of the number of times we did so).

Now let $S(J)$ denote J times the sum of the lengths of all stability intervals of length greater than $1/J$, and put $N(J) = J - S(J)$. Our procedure for estimating the fractal dimension D is based on the fact that

$$N(J) \sim J^D. \tag{3.6}$$

What we actually compute, however, is $S'(J)$, the number of i's such that $W(\mu_{i+1}) = W(\mu_i)$, which approximate $S(J)$. More precisely: The difference $S(J) - S'(J)$ will at most go like $A(J)$, the number of stability intervals of length greater than $1/J$. However, it is easily seen that

$$A(J) = S(J) - \int \frac{S(J)}{J} dJ \sim J^D. \tag{3.7}$$

Thus,

$$N'(J) = J - S'(J) \sim J^D. \tag{3.8}$$

In Fig. 1a $N'(J)$ is plotted versus J in a double-logarithmic plot for several selected values of the order r. The number of iterations M is consequently chosen to be ten times the value of J, but this is not essential [in fact, even though M in some cases has been varied up to 10^5, a change in $N'(J)$ is never observed]. The slopes determined from Fig. 1a are listed in Table 1 together with the fractal dimensions found for other values of r. It is noticed that the results for $r=3$ and $r=5$ are in accordance with [10] and footnote 2.

In this connection we have in continuation of Eq. (3.3) considered the map

$$g_{a,b,c}(x) = x - \frac{1}{2\pi}(\sin 2\pi x + a \sin^3 2\pi x + b \sin^5 2\pi x + c \sin^7 2\pi x) \tag{3.9}$$

in the four cases $(a,b,c) = (0,0,0)$, $(1/6,0,0)$, $(1/6,3/40,0)$ $(1/6,3/40,5/112)$, where $g_{a,b,c}$ has the order $r=3,5,7,9$ respectively. Figure 1b shows $N'(J)$ versus J for these maps (in the calculation $\omega_{0.5}$ is replaced by the first instability points, i.e. $1/2\pi$ for $r=3$, $7/12\pi$ for $r=5$, $149/240\pi$ for $r=7$, and $2161/3360\pi$ for $r=9$). The slopes found are in agreement with the slopes found for g_r in Fig. 1a.

We point out that the branches of g_r are connected at $x = x_c = \frac{1}{2}$, thereby making $\frac{1}{2}$ like a symmetry point, as is the case for the sine map (3.9). We have

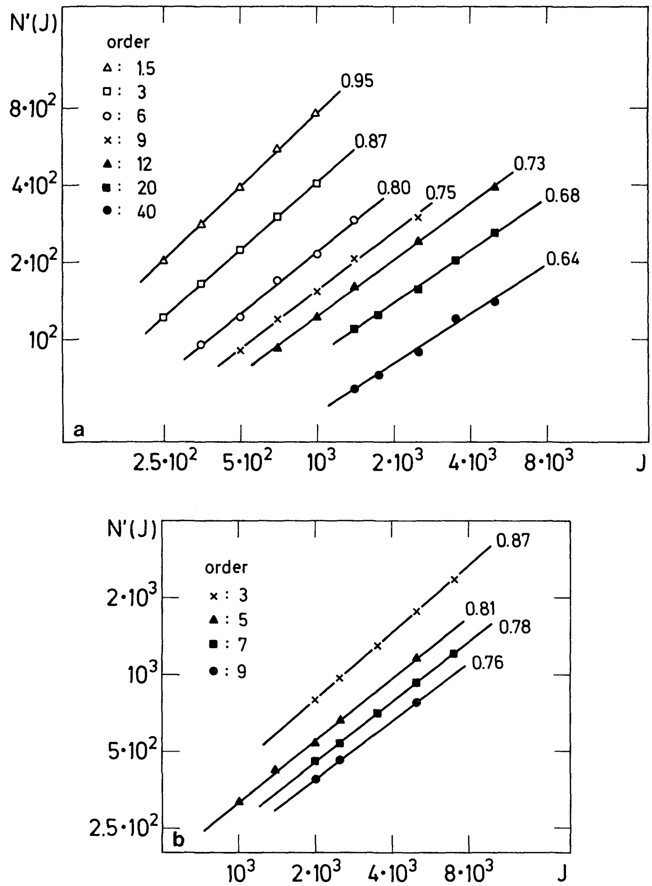

Fig. 1a and b. $N'(J)$ plotted versus J in a double-logarithmic plot for different values of the order. The fractal dimensions are obtained as the slopes of the lines found from a least square fit. **a** $g = g_r$ given by (3.4). **b** $g = g_{a,b,c}$ given by (3.9)

Table 1. The fractal dimension D for several values of the order r

r	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0
D	1.00	0.950±0.005	0.915	0.87	0.83±0.01	0.81	0.80	0.78
r	8.0	9.0	12	14	16	20	26	40
D	0.77	0.75	0.73	0.72	0.70	0.68±0.015	0.66	0.64

therefore in continuation of question I also calculated the fractal dimension for the functions

$$h_{x_c}(x) = \begin{cases} x_c(x/x_c)^r & 0 \leqq x < x_c \\ 1-(1-x_c)\,((1-x)/(1-x_c))^r & x_c \leqq x < 1 \end{cases} \tag{3.10}$$

[h_{x_c} is extended to $\mathbb{R}$ by (ii)]; $h_{x_c} \in G$ with order r and with connection point x_c. The fractal dimension was obtained for $r = 3, 5$ and for $x_c = 0.1, 0.2, 0.3, 0.4$, and 0.5 (by

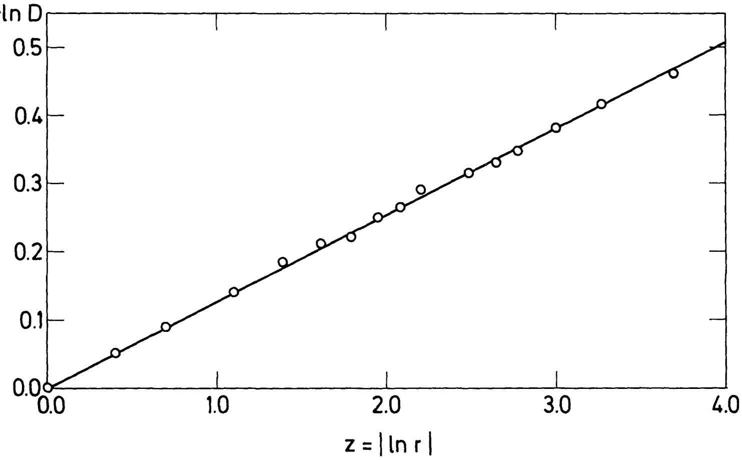

Fig. 2. The fractal dimension D as a function of the number $z=|\ln r|$, where r is the order. The vertical axis is logarithmic. The slope is determined to be 1/8

Theorem 3.1 we can take $x_c \leqq 0.5$) and the results were found to be the same, namely $D(h_{x_c})=0.87$ for $r=3$ and $D(h_{x_c})=0.81$ for $r=5$. [The computation is performed as before, one just has to replace $\omega_{0.5}$ by $\omega_{x_c}=2x_c\omega_{0.5}$ and $1-\omega_{0.5}$ by $1-\omega_{1-x_c}=1-2(1-x_c)\omega_{0.5}$.]

In Fig. 2 the set of points $(r, D(z))$ given in Table 1 is plotted on double-logarithmic paper. Within the computational uncertaincy we find that the points quite nicely follow a straight line with slope $\alpha \simeq 1/8$. This means that $D(z) \simeq e^{-\alpha z}$, and question II is confirmed.

4. The Extended Group G_∞

As mentioned earlier there are of course still "relevant" functions which do not belong to G, e.g. it is evident that the considerations above can be generalized by allowing the inflection point to be somewhere other than at the origin, the fractal dimension remains independent of such a removal. This is also the case if the map is contracted by some integer, e.g. the sine map (3.3) with $a=-\frac{4}{3}$ is just the sine map with $a=0$ contracted by a factor of three.

In the light of this we can extend G to a larger group of functions G_∞ by making the conditions (iv)–(v) weaker; namely by considering maps g with any finite number of inflection points in $[0;1]$, each of these inflection points satisfying the appropriate modification of condition (iv) and (v) but with different orders r_i allowed at the different inflection points. We speculate that, in this case, the fractal dimension will depend only on the maximum of $z=|\ln(r_i)|$.

5. Circle Maps not Belonging to G_∞

Finally we mention some examples of maps not belonging to G_∞.

1) $f(x)=1-\alpha(1-x)$ for $x \in [0;1[$, $\alpha \in [0;1[$ and f is extended to $\mathbb{R}$ by the condition (ii).

This case has been investigated by Söderberg [18]. The stability intervals $\Delta\left(\frac{m}{n}\right)$ can be calculated analytically and turn out to be independent of m:

$$\Delta\left(\frac{m}{n}\right)=(1-\alpha)^2\frac{\alpha^{n-1}}{1-\alpha^n}. \tag{5.1}$$

The sum S of all stability intervals is given by a Lambert series [19]:

$$S=\sum_{n=1}^{\infty}\phi(n)\,\Delta\left(\frac{m}{n}\right)=\frac{(1-\alpha)^2}{\alpha}\Phi(\alpha)$$

with

$$\Phi(\alpha)=\sum_{n=1}^{\infty}\phi(n)\frac{\alpha^n}{1-\alpha^n}. \tag{5.2}$$

$\Phi(\alpha)$ is the generating function of the Euler function $\phi(n)$, $g(n)=n$ is the Möbius transform $n\;\phi(n)$ and

$$\sum_{n=1}^{\infty}n\alpha^n=\alpha/(1-\alpha)^2. \tag{5.3}$$

Hence,

$$\Phi(\alpha)=\alpha/(1-\alpha)^2 \quad \text{and} \quad S=1, \tag{5.4}$$

saying that the staircase is complete. Then because $\Delta\left(\frac{m}{n}\right)$ decreases exponentially, the fractal dimension is zero.

Once case 1) is solved, case

2) $h(x)=\begin{cases}0 & \text{for} \quad 0\leqq x<1-\alpha\\ (x+\alpha-1)/\alpha & \text{for} \quad 1-\alpha\leqq x<1\end{cases},$

where $h(x)$ is extended to $\mathbb{R}$ as before, is too, because $h\circ f=\mathrm{Id}$, and Lemma 3.2 will do.

3) f satisfies (i)–(iii), and is a continuous piecewise linear function with a finite number of elbows in [0; 1]. Recently, Henley has considered this case [20]. In particular the situation where the product of the slopes is one was investigated. Even though f is a limit of a sequence of functions $f_n\in G$ all with order $r=1$, the staircase $W_f(\mu)$ will be complete and the fractal dimension will depend on the slopes. If we, for example, in the unit interval [0; 1[take

$$f(x)=\begin{cases}\alpha x & 0\leqq x<x_0\\ 1-\beta(1-x) & x_0\leqq x<1\end{cases}, \tag{5.5}$$

where $\alpha<1<\beta$ and x_0 is the connection point, $x_0=(\beta-1)/(\beta-\alpha)$, we get by the same procedure as used in Sect. 3 that $D(f)\simeq 0.92$ for $(\alpha,\beta)=(\frac{1}{2},2)$ and $D(f)\simeq 0.81$ for $(\alpha,\beta)=(1/8,2)$. This shows the importance of the condition (iv).

In conclusion, the scaling behaviour at the transition to chaos in phase-locking systems described by one dimensional circle maps is believed to be universal for a large group of functions. For maps belonging to this group there is, in the light of numerical calculations, suggested a value of the fractal dimension of the devil's staircase determined by the rotation number.

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