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Existence and Uniqueness of Positive Solutions of Semilinear Elliptic Equations with Coulomb Potentials on $\mathbb{R}^3$

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Dedicated to the memory of Professor Mark Kac

Abstract. We study the positive solutions of a semilinear equation with a Coulomb potential on $\mathbb{R}^3$. We give a new uniqueness theorem for the positive radial solutions of such an equation and we apply these results to the Thomas-Fermi-Dirac-von Weizsäcker equation without electrostatic repulsion.

1. Introduction

In this article we shall discuss the existence and uniqueness of positive classical solutions $u(x)$ of the problem

$$-\Delta u - Z|x|^{-1}u + a(u) = 0 \quad \text{in} \quad \mathbb{R}^3, \qquad u(x) \to 0 \quad \text{as} \quad x \to \infty. \tag{1}$$

In Eq. (1), Z is a positive real constant and the function $a(\cdot)$ satisfies the following hypothesis:

A1. $a(t)$ is locally Lipschitz continuous for $t \geqq 0$, $a(0) = 0$ and

$$\lim_{t \downarrow 0} (a(t)/t) \equiv \lambda > 0. \tag{2}$$

A2. There are positive constants δ, C_+ and C_- such that

$$C_- t^p \leqq a(t) \leqq C_+ t^p \tag{3}$$

for all $t \geqq \delta$. Here, $1 < p < \infty$.

A3. For all $t > 0$,

$$F(t) \equiv 2 \int_0^t a(s)\,ds > 0. \tag{4}$$

Let us also define

$$\alpha \equiv \inf_{t > 0} (F(t)/t^2); \tag{5}$$

the hypothesis A1, A2, and A3 ensure that $0 < \alpha \leqq \lambda$.

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Problem (1) arises in the Thomas-Fermi-von Weizsäcker (TFW) theory of atoms and molecules [1, 2] without electronic repulsion. There, $Z|x|^{-1}$ is the electric potential due to a fixed nucleus of atomic number Z located at the origin, $u(x)^2$ stands for the electronic density and $\int u(x)^2 dx$ is the total number of electrons. In the usual TFW model,

$$a(u) = a \cdot u^{7/3} + \lambda u, \tag{6}$$

where a and λ are positive constants. Here λ is the chemical potential. On the other hand, in the TFW model with exchange correction [1, 3, 4],

$$a(u) = a \cdot u^{7/3} - bu^{5/3} + \lambda u, \tag{7}$$

where a and λ are the same as above and b is a positive constant. The TFW equation for an infinite atom, i.e. Eq. (1), with $a(u)$ given by (6) and $\lambda = 0$, has been studied in [5, 6].

The proof of the existence of solutions to Problem (1) is achieved in Sect. 2 (Lemma 3) by solving the related problem

$$\operatorname{Min}\{S(u) | u \in E\}, \tag{8}$$

where

$$S(u) = \int (\nabla u)^2 dx + \int F(u) dx - \int Z|x|^{-1} u^2 dx, \tag{9}$$

and

$$E = \{u | u \geqq 0, u \in L^2 \cap L^6 \cap L^{p+1}, \nabla u \in L^2, 1 < p < \infty\}. \tag{10}$$

In Eq. (9), dx denotes the usual Lebesgue measure in $\mathbb{R}^3$ and $F(u)$ is defined in terms of $a(u)$ by Eq. (4).

In Sect. 3 we give a new uniqueness result for the positive radial solutions of Eq. (1). It is well known that if $a(u)/u$ is increasing in u [or, equivalently $F(u)$ convex in u^2], (1) has a unique positive radial solution. Here, we prove a stronger result namely, if $\lambda < Z^2/4$ and $G(u) \equiv a(u)/u - \frac{1}{4}Z^2$ has only one zero on the interval $(0, \infty)$ then there is at most one positive radial solution to Eq. (1) (see Theorem 17 below). Here, $-Z^2/4$ is the lowest eigenvalue of the linear part of (1). Note that this new hypothesis on $a(\cdot)$ is obviously satisfied if $a(u)/u$ is increasing in u.

Our proof of uniqueness is performed in two steps. We first introduce the wronskian between the ground state of the linear part of (1) and a solution to Eq. (1) [see Eqs. (42) and (43) below] and we prove that under the hypothesis on $a(\cdot)$ just introduced, this wronskian is positive in $(0, \infty)$. We then use this result and the method of separation of solutions introduced by Peletier and Serrin [7] to conclude the proof of uniqueness. It is worth mentioning here that if we relax the last hypothesis on $a(\cdot)$ so that $G(u)$ has two or more zeros in $(0, \infty)$ we can prove an oscillation theorem for the wronskian (see Lemma 13 below). We believe that in the case Eq. (1) has more than one positive radial solution, the different solutions are characterized by a different oscillatory behavior of their wronskian (with the ground state of the linear part).

Finally in Sect. 4 we apply the existence and uniqueness results to the TFDW equation without electrostatic repulsion, that is, to Eq. (1) with $a(u)$ given by (7). These results on TFDW are summarized in Lemma 22 below.

2. Existence of Solutions

In this section we will discuss the existence of positive classical solutions to Eq. (1). The proof of existence, which is rather standard, is achieved by solving the related minimization problem (8). In fact, Eq. (1) is the Euler equation corresponding to the variational problem. We then use standard elliptic regularity theorems to show that these solutions satisfy Eq. (1) in the classical sense.

We start by summarizing some elementary properties of $S[u]$:

Lemma 1. *For every positive ε, there is a positive constant $C_{\varepsilon,q}$ such that*

$$\int u(x)^2 Z|x|^{-1} dx \leqq \varepsilon \|u\|_6^2 + C_{\varepsilon,q}\|u\|_q^2, \tag{11}$$

where $2 \leqq q < 3$.

Proof. Decompose $V(x) \equiv Z|x|^{-1}$ as in [2, Lemma 2] and use Hölder's inequality. □

Remark. If $u \in E$, $S[u] < \infty$. In fact, if $u \in E$ the first term in Eq. (9) is finite and, because of the previous lemma, the third term is also finite. As for the second term we note that hypothesis A 1 implies that $F(u)/u^2$ is finite in a neighborhood of $u=0$. Moreover, since $F(u)$ is continuous, $\theta \equiv \max\{F(u)/u^2 | 0 \leqq u \leqq \delta\}$ is finite. From here and hypothesis A 2 we get

$$F(u) \leqq \theta u^2 + 2C_+ u^{p+1}/(p+1).$$

Since $u \in E$, $u \in L^2 \cap L^{p+1}$ and thus, the second term in Eq. (9) is also finite.

Lemma 2. *There are positive constants β and γ such that*

$$S[u] \geqq \beta\{\|\nabla u\|_2^2 + \|u\|_6^2 + \|u\|_2^2 + \|u\|_{p+1}^{p+1}\} - \gamma. \tag{12}$$

for all $u \in E$.

Proof. Because of Lemma 1,

$$S[u] \geqq \int (\nabla u)^2 dx + \int F(u) dx - \varepsilon \|u\|_6^2 - C_{\varepsilon,q}\|u\|_q^2. \tag{13}$$

Since for all $\eta > 0$ and $q > 2$ there exists $K_\eta > 0$ such that $z \leqq \eta z^{q/2} + K_\eta$, all $z \geqq 0$, we conclude from (13)

$$S[u] \geqq \int (\nabla u)^2 dx + \int (F(u) - Au^q) dx - \varepsilon \|u\|_6^2 - C_{\varepsilon,q} K_\eta, \tag{14}$$

with $A = C_{\varepsilon,q} \cdot \eta$. We now show that there are positive constants B and D such that

$$F(u) - Au^q \geqq Bu^2 + Du^{p+1} \tag{15}$$

for all $u \geqq 0$ and $2 < q < p+1$. Equation (5) implies

$$F(u) - Au^q \geqq (\alpha - A\delta^{q-2})u^2$$

for $u \leqq \delta$. Choose $\eta = B\delta^{2-q}/C_{\varepsilon,q}$ in (14), with

$$B = \min(\alpha/4, C_- \delta^{p-1}/2(p+1)). \tag{16}$$

Hence, $F(u) - Au^q \geqq 3Bu^2 \geqq Bu^2 + Du^{p+1}$ for $u \leqq \delta$, where

$$D = 2B\delta^{1-p}. \tag{17}$$

For $u \geqq \delta$, Eqs. (3) and (4) imply

$$F(u) - Au^q \geqq F(\delta) + \frac{2C_-}{p+1}[u^{p+1} - \delta^{p+1}] - Au^q \geqq \alpha\delta^2 - \sigma\frac{2C_-}{p+1}\delta^{p+1}$$
$$+\sigma\frac{2C_-}{p+1}u^{p+1} - Au^q$$

for all $0 \leqq \sigma \leqq 1$. Choosing $\sigma = \min\left[1, \frac{\alpha}{2C_-}(p+1)\delta^{1-p}\right]$ and $q > p+1$, we get

$$F(u) - Au^q \geqq \sigma\frac{C_-}{p+1}u^{p+1} + \left[\sigma\frac{C_-}{p+1}\delta^{p-1} - A\delta^{q-2}\right]u^2 = Bu^2 + Du^{p+1}.$$

Finally, the lemma follows from (14) and (15) by using Sobolev's inequality (see [1]). □

After these preliminary lemmas we are in position to prove the existence of a minimizing solution (*s*) for $S[u]$ in E.

Lemma 3. *Min* $\{S[u] \,|\, u \in E\}$ *is achieved at some* $u_0 \in E$.

Proof. Let $\{u_n\} \subset E$ be a minimizing sequence. By Lemma 2 there is a positive constant (independent of n) such that

$$\|\nabla u_n\|_2 < C, \quad \|u_n\|_6 < C, \quad \|u_n\|_{p+1} < C, \quad \|u_n\|_2 < C.$$

Therefore we may extract a subsequence, still denoted by u_n, such that

$$u_n \to u_0 \text{ weakly in } L^2,\ L^6, \text{ and } L^{p+1}, \tag{18}$$

$$\nabla u_n \to \nabla u_0 \text{ weakly in } L^2, \tag{19}$$

$$u_n \to u_0 \quad \text{a.e.} \tag{20}$$

[(20) relies on the fact that if Ω is a bounded smooth domain then $H^1(\Omega)$ is relatively compact in $L^2(\Omega)$. (18) and (19) imply that $\{u_n\}$ is bounded in $H^1(\Omega)$. Hence $\{u_n\}$ has a subsequence converging in $L^2(\Omega)$ and a.e.]. Hence,

$$\liminf \int (\nabla u_n)^2 dx \geqq \int (\nabla u_0)^2 dx. \tag{21}$$

Since $F(t)$ is continuous, $F(u_n) \to F(u_0)$ a.e. Thus,

$$\liminf \int F(u)dx \geqq \int F(u_0)dx \tag{22}$$

by Fatou's lemma. Finally since $V = Z|x|^{-1} \in L^{3/2} + L^4$, (18) implies that $\int Vu_n^2 dx \to \int Vu_0^2 dx$. Therefore

$$S[u_0] \leqq \liminf S[u_n] = \operatorname{Inf} S[u_n]. \quad \square$$

We now derive the Euler equation satisfied by u_0.

Lemma 4. *The minimizing* u_0 *('s) satisfies (satisfy)*

$$-\Delta u_0 + a(u_0) - Z|x|^{-1}u_0 = 0 \tag{23}$$

in the sense of distributions.

Proof. Let us first verify that (23) has a meaning in the sense of distributions. Since $\nabla u_0 \in L^2$, it is enough to check that $a(u_0) - Z|x|^{-1}u_0 \in L^1_{\text{loc}}$. Here $u_0 \in L^2$ and $Z|x|^{-1} \in L^2_{\text{loc}}$, thus $Z|x|^{-1}u_0 \in L^1_{\text{loc}}$. (3) and A1 imply

$$|a(u)| \leqq \max\left[\max_{t\in[0,\delta]} |a(t)|, C_+ u^p\right]. \tag{24}$$

Since $u_0 \in L^{p+1}$, $u_0 \in L^p_{\text{loc}}$ and therefore $a(u) \in L^1_{\text{loc}}$. Consider the set $E = \{v | v \in L^2 \cap L^6 \cap L^{p+1}, p > 1, \nabla v \in L^2\}$. Note that $v \geqq 0$ is not assumed. If $v \in E$, then $|v| \in E$ and $S[v] = S[|v|]$. Here, $F(u)$ has been extended to $\mathbb{R}$ by setting $F(-t) = F(t)$ for all $t \geqq 0$. Indeed it suffices to recall that $\nabla |v| = \nabla v$ (sign (v)) (see [8]). Let $\eta \in C_0^\infty$. The lemma follows by taking $\frac{d}{dt} S[u + t\eta]|_{t=0} = 0$. □

We end this section by proving some properties of the minimum (or minima) of $S[u]$ in E. We first show that the minima of $S[u]$ are symmetric decreasing (see [9] and [10] for definitions and details). Let

$$SD = \{f : \mathbb{R}^3 \to [0, \infty) | f(x) \leqq f(y) \quad \text{if} \quad |x| \geqq |y|\} \tag{25}$$

be the symmetric decreasing functions. Then we have

Lemma 5. *The minima of $S[u]$ in E belong to SD.*

Proof. This follows by Lemma 5 in [10] and standard results on SD. □

In the next lemma we summarize the regularity properties of the solutions to Eq. (1) which belong to E.

Lemma 6. *If $u \in E$ is a solution of Eq. (1), then*

i) *u is continuous in $\mathbb{R}^3$, more precisely $u \in C^{0,\alpha}$ all $\alpha < 1$.*

ii) *u is bounded in $\mathbb{R}^3$ and $u(x) \to 0$ as $|x| \to \infty$.*

iii) *$u \in H^2$.*

iv) *Either $u \equiv 0$ or $u > 0$ everywhere and $u \in C^2$ except at $x = 0$.*

v) *If u has radial symmetry, it satisfies the following cusp condition at the origin:*

$$2\frac{du}{dr}(0) + Zu(0) = 0. \tag{26}$$

Proof. i) The hypothesis A1 and A2 ensure that $a(u)$ is uniformly bounded from below, i.e. there is a (nonpositive) constant M such that $a(u) \geqq M$ for all $u > 0$. Thus, from Eq. (1) we have

$$-\Delta u \leqq |M| + Z|x|^{-1}u \equiv f.$$

Since $Z|x|^{-1} \in L^{3-\varepsilon}_{\text{loc}}$ for all $\varepsilon > 0$ and $u \in L^6$, $Z|x|^{-1} \cdot u \in L^{2-\varepsilon}_{\text{loc}}$ for all $\varepsilon > 0$. Hence, $f \in L^q_{\text{loc}}$ for some $q > 3/2$. We may, therefore apply a result of Stampacchia (see [8, Remarque 5.2]) to conclude that $u \in L^\infty_{\text{loc}}$. Going back to (1) and using the fact that $u \in L^\infty_{\text{loc}}$, we now see that $\Delta u \in L^{3-\varepsilon}_{\text{loc}}$ for all $\varepsilon > 0$. The standard elliptic regularity theory [11] implies that $u \in C^{0,\alpha}$ for all $\alpha < 1$.

ii) The hypothesis A1 and A2 also imply that $a(u)/u$ is uniformly bounded from below, i.e. there is a (nonpositive) constant N such that $a(u)/u > N$, for all

$u \geqq 0$. Thus, from Eq. (1) we have

$$-\Delta u + u \leqq (Z|x|^{-1} + |N| + 1)u.$$

Clearly, $(Z|x|^{-1} + |N| + 1)u \in L^2$, and so

$$u \leqq (-\Delta + I)^{-1}[(Z|x|^{-1} + |N| + 1)u]. \tag{27}$$

As is well known, the right side in (27) is bound and tends to zero as $|x| \to \infty$.

iii) Since u is bounded, $a(u) \leqq Cu$ for some positive constant C. Also, $Z|x|^{-1}u \in L^2$. Therefore, from Eq. (1) we have that $\Delta u \in L^2$ and so $u \in H^2$.

iv) From Eq. (1) we have $-\Delta u + bu = 0$, with $b \in L^q_{\text{loc}}$, $q > 3/2$. It follows from Harnack's inequality (see e.g. [12, Corollary 5.3]) that either $u > 0$ everywhere or $u \equiv 0$. We shall show (see Lemma 20 below) that $u \not\equiv 0$ for some given interval in λ. Finally, the fact that $u \in C^2(\mathbb{R}^3 \setminus \{0\})$ follows easily from A1, the part i) of this lemma and Lemma 4.2 in [11].

v) Multiply Eq. (1) by $|x|$, then take the limit $|x| \to 0$ and use the regularity properties of u. $\square$

Remark. If $a(u)$ is C^∞, then $u \in C^\infty$ away from the origin; this follows easily from (1) by a standard bootstrap argument.

3. Uniqueness of Positive Radial Solutions

In this section we show that under an extra hypothesis on $a(u)$ (namely, hypothesis A4 below) there is at most one positive radial solution of Eq. (1). This in turn implies that for those $a(u)$ that satisfy A1 through A4, the minimum of $S[u]$ in E is unique. We suspect (after the results on [13, 14]) that all positive solutions to Eq. (1) with $u(x) = O(|x|^{-m})$, $m > 0$ near infinity are necessarily spherically symmetric about the origin. If this were the case the uniqueness of positive radial solutions would imply the uniqueness of positive solutions. However, the results of [14] (in particular Theorem 1′) do not apply directly to our case.

The hypothesis A4 is weaker than requiring $F(u)$ to be convex in u^2. If $F(u)$ is strictly convex in u^2 then the proof of the uniqueness of a minimum of $S[u]$ is easy:

Lemma 7. *If $F(t)$ is strictly convex in t^2 then the minimum of $S[u]$ in E is unique.*

Proof. It is enough to show that $S[u]$ is strictly convex in u^2. It is clear that the last two terms in Eq. (9) are strictly convex in u^2. As for the first term, its convexity in u^2 follows by integration by parts and Schwarz inequality (see [2] and [15]). $\square$

Remark. As it is pointed out in [15], the fact that $F(u)$ is strictly convex in u^2 implies that $a(u)/u$ is increasing in u. We relax here this last property of $a(\cdot)$ and we introduce the following hypothesis on $a(\cdot)$:

A4. For $\lambda < Z^2/4$,

$$G(u) \equiv a(u)/u - Z^2/4 \tag{28}$$

has only one zero in the interval $[0, \infty)$.

Remark. It is clear from A 1 and A 2 that if we restrict λ to be less than $Z^2/4$, $G(0)<0$ and $G(u)\to\infty$ as $u\to\infty$. Hence, A 4 is certainly satisfied if $a(u)/u$ is increasing in u.

Our main result in this section is Theorem 17 below. We need some preliminary results. We start proving that all positive radial solutions of Eq. (1) are strictly decreasing functions of r, a result closely connected with Lemma 5 and also with the results of [13, 14].

In what follows we will only consider the radial equation

$$-u''-\frac{2}{r}u'-\frac{Z}{r}\cdot u+a(u)=0\,. \tag{29}$$

Here $u'=\dfrac{du}{dr}$, etc. We have,

Lemma 8. *Let $u(r)\in C^2(0,\infty)$ be a positive solution of Eq.* (29) *which goes to zero at infinity and satisfies the cusp condition* (26). *Then*

$$u'(r)<0\,,\quad \text{all}\quad r\geqq 0\,. \tag{30}$$

Proof. Here, we will only show that $u'(r)\leqq 0$, all $r\geqq 0$, leaving the proof of the strict inequality for the Appendix (see Lemma A.2 below). It is clear from (26) that $u'(0)<0$ [otherwise $u\equiv 0$ in $(0,\infty)$; see Lemma A.1 in the Appendix]. Now, suppose for contradiction that there exists $\xi\in(0,\infty)$ such that $u'(\xi)>0$. Since $u'(0)<0$ and $u\in C^1(0,\infty)$, there is at least a point in the interval $(0,\xi)$ such that $u'=0$. Let $r_0\in(0,\xi)$ be the point closest to ξ with $u'(r_0)=0$. Since $u(r)$ goes to zero as r goes to infinity there is a point $r_1\in(\xi,\infty)$ such that $u(r_1)=u(r_0)$ and $u(r)\geqq u(r_0)$, all $r\in[r_0,r_1]$. Now, multiplying (29) by u' and integrating over (r_0,r_1), we obtain

$$\tfrac{1}{2}u'(r_1)^2+\int_{r_0}^{r_1}2(u')^2r^{-1}dr+\int_{r_0}^{r_1}Zr^{-1}uu'dr-\int_{r_0}^{r_1}a(u)u'dr=0\,. \tag{31}$$

Note that

$$\int_{r_0}^{r_1}a(u)u'dr=\int_{u(r_0)}^{u(r_1)}a(u)\,du=0\,, \tag{32}$$

since $u(r_0)=u(r_1)$. Also,

$$\int_{r_0}^{r_1}Zr^{-1}uu'dr=\frac{Z}{2}u(r_0)^2(r_1^{-1}-r_0^{-1})+\frac{Z}{2}\int_{r_0}^{r_1}(u(r)/r)^2dr\geqq 0\,, \tag{33}$$

since $u(r)\geqq u(r_0)$ all $r\in[r_0,r_1]$. Moreover, since $u'(\xi)>0$ and u' is continuous,

$$\int_{r_0}^{r_1}(Z/r)(u')^2dr>0\,. \tag{34}$$

Introducing (32), (33), and (34) in Eq. (31) we conclude $u'(r_1)^2<0$, which is impossible. □

Remarks. i) The idea of the proof of this lemma is taken from [7], (Lemma 3).

ii) The Coulomb potential Z/r is not crucial here. If we change it by any other decreasing potential $V(r)$, the lemma still holds.

In the next two lemmas we establish some facts about the asymptotic behavior of $u(r)$ which we require in the sequel.

Lemma 9. *Let $u(r) \in C^2(0, \infty)$ be a positive solution of Eq.* (29) *which goes to zero at infinity and satisfies the cusp condition* (26). *Then*

$$\lim_{r \to \infty} u'(r) = 0 . \tag{35}$$

Proof. The hypothesis A1 implies that for r large enough

$$(Z/r) - (a(u)/u) < 0 . \tag{36}$$

Introducing (30) and (36) in Eq. (29) we conclude that $u''(r) > 0$ for r large enough. Hence $\lim_{r \to \infty} u'(r)$ exists and it is non-positive. Since $u(r) > 0$ for all $r \geqq 0$, it follows that $\lim_{r \to \infty} u'(r) = 0$. □

Lemma 10. *Let $u(r)$ be as in Lemma 9. Then*

$$\lim_{r \to \infty} \frac{u'(r)}{u(r)} = -\sqrt{\lambda} . \tag{37}$$

Proof. Let $f = -u'/u$. Since $u > 0$ and $u' < 0$, $f > 0$, all r. From Eq. (29) we get

$$f' = f^2 - 2r^{-1} f + r^{-1} Z - (a(u)/u) . \tag{38}$$

Since $u \to 0$ as $r \to \infty$ and because of A1, we can choose $\bar{r}$ large enough such that for all $r \geqq \bar{r}$ we have

$$2/r \leqq \tfrac{1}{2}(1+\lambda)^{1/2} \quad \text{and} \quad a(u)/u \leqq 1 + \lambda . \tag{39}$$

Let $D = \{(r, f) | r \geqq \bar{r}, \ f \geqq 2(1+\lambda)^{1/2}\}$. On D,

$$f' \geqq \tfrac{1}{2} f^2 + (\tfrac{1}{2} f^2 - \tfrac{1}{2}(1+\lambda)^{1/2} f - (1+\lambda)) \geqq \tfrac{1}{2} f^2 . \tag{40}$$

From this last inequality we conclude that if the integral curve $f = f(r)$ for Eq. (38) enters the region D, it will eventually blow up at a finite value of r. Since this is impossible, we conclude that

$$\limsup_{r \to \infty} f(r) \leqq 2(1+\lambda)^{1/2} , \tag{41}$$

i.e. f is bounded uniformly in $(0, \infty)$. We now evaluate $\lim (u'/u)^2$ by L'Hôpital's rule. We get

$$\lim_{r \to \infty} \left(\frac{u'}{u} \right)^2 = \lim_{r \to \infty} \frac{u''}{u} = \lim_{r \to \infty} \left(\frac{2}{r} f(r) - \frac{Z}{r} + \frac{a(u)}{u} \right) = \lambda ,$$

where we have used (41) and A1. From here and (30) the lemma follows. □

Remark. The proof of the previous lemma is taken from the proof of Lemma 5 in [7].

In what follows we prove some oscillation properties of the function

$$h(r) \equiv 2u'(r) + Zu(r) , \tag{42}$$

in $(0, \infty)$, where u is defined as in Lemma 9. These results do not depend crucially on the fact that we have a Coulomb potential in Eq. (1). The same results can be obtained by replacing Z/r in Eq. (1) by any decreasing potential $V(r)$ as long as one changes $h(r)$ by the wronskian

$$h(r) \equiv u_0(r)u'(r) - u_0'(r)u(r), \tag{43}$$

and $G(u)$ by

$$G(u) = [a(u)/u] - e_0. \tag{44}$$

Here, u_0 and e_0 are, respectively, the ground state and the lowest eigenvalue of the hamiltonian $-\Delta + V$, on $\mathbb{R}^3$.

In the Lemmas 11 and 12 below we consider $\lambda < Z^2/4$ and we assume $a(s)$ satisfies A1 through A4. We denote by w the only zero of $G(u)$ in $(0, \infty)$ and $u(r)$ is defined as in Lemma 9.

Lemma 11. $u(0) > w$.

Proof. From (29) we get the following equation for h,

$$h' + (2r^{-1} - \tfrac{1}{2}Z)h = 2uG(u), \quad \text{all} \quad r > 0 \tag{45}$$

with $G(u)$ given by (28). From (45), it follows that

$$r^2 h(r) = \int_0^r 2\exp\left[\frac{Z}{2}(r-s)\right] s^2 u(s) G(u(s))\, ds, \tag{46}$$

all $0 < r < \infty$. Suppose, for contradiction, that $u(0) \leqq w$. Since $u'(r) < 0$, all r, this implies that $u(r) < w$, all $r > 0$. Since $u(r)$ goes to zero as r goes to infinity, $G(u(r)) \to \lambda - (Z^2/4) < 0$ as $r \to \infty$. Now, w is the only zero of $G(u)$ and $u(r) < w$, all r; hence $G(u(r)) < 0$, all $r > 0$. In particular, (46) implies that $h(r) < 0$, all $r < \infty$. On the other hand, Lemma 10 implies that $h(r) > 0$ for r large enough, which is a contradiction. □

Lemma 12. $h(r) > 0$, *all* $r > 0$.

Proof. By the cusp condition (26), $h(0) = 0$. Moreover, taking the limit $r \to 0$ in (45) we get $h'(0) = (2/3)u(0)G(u(0)) > 0$; this last inequality follows from Lemma 11. Also, from Lemma 10, $h(r) > 0$ for r large enough. Now, assume there is $0 < \xi < \infty$ such that $h(\xi) < 0$. Since h is continuous, there must exist r_1, r_2 with $r_1 < \xi < r_2$, so that $h(r_1) = h(r_2) = 0$, with $h'(r_1) \leqq 0$, and $h'(r_2) \geqq 0$. From Eq. (45) it then follows $G(u(r_1)) \leqq 0$ and $G(u(r_2)) \geqq 0$. $G(u(r_1)) \leqq 0$ implies $u(r_1) \leqq w$. However, $u(r_2) < u(r_1)$ because $u(r)$ is strictly dereasing (Lemma 8). Hence, $u(r_2) < w$ and therefore $G(u(r_2)) < 0$, which is a contradiction. Thus $h(r) \geqq 0$, all $r > \infty$. Finally, we prove that $h(r)$ does not vanish in $(0, \infty)$. In fact, assume there is $0 < \xi < \infty$ so that $h(\xi) = 0$. From the previous remarks (since h is $C^1(0, \infty)$), $h'(\xi) = 0$. Therefore, from Eq. (45) we get $u(\xi) = w$. Hence, $G(u(r)) < 0$, all $r > \xi$. Now, from (45) and since $h(\xi) = 0$, it follows that

$$r^2 h(r) = \int_\xi^r 2\exp\left[\frac{Z}{2}(r-s)\right] s^2 u(s) G(u(s))\, ds < 0,$$

for all $r > \xi$, which is impossible. □

In the next lemma we relax the hypothesis A4 and we allow $G(u)$ to have any (odd) number of simple zeros in $(0, \infty)$. In this case we give a bound on the number of zeros of the "wronskian" h in the interval $(0, \infty)$.

Lemma 13. *Let $\lambda < Z^2/4$ and suppose $G(u)$ has n (odd) simple zeros in $(0, \infty)$. We label these zeros by $0 < w_1 < w_2 < \ldots < w_n$. Let $u(r)$ be as in Lemma 9. Then,*

i) $u(0) > w_1$.

ii) *If $w_i < u(0) < w_{i+1}$, $1 \leqq i \leqq n-1$, then h has at most $i-1$ zeros in $(0, \infty)$. If $u(0) > w_n$, h has at most $n-1$ zeros in $(0, \infty)$.*

iii) *Let R be such that $u(R) = w_1$. Then, for all positive solutions of (29), satisfying (26), which go to zero at infinity, we have $h(r) \geqq 0$, all $r \geqq R$.*

Proof. The proof of this lemma is completely similar to the proof of Lemmas 11 and 12 and we omit it here. □

Having shown the positivity of the "wronskian" h under the hypothesis A4 on $a(\cdot)$, we proceed to use the method of separation of solutions introduced by Peletier and Serrin in [7]. We start by assuming there are two solutions of Eq. (29), which go to zero at infinity and satisfy the cusp condition (26). We will consider the horizontal distance between these two solutions. We denote by $r(u)$ the inverse of $u(r)$ (which is well defined because of Lemma 8) and by $s(u)$ the inverse of $v(r)$. We assume that $u(r) \not\equiv v(r)$ and our goal is to show that this is impossible, i.e. that there is at most one solution. We start with

Lemma 14. *Let $\lambda < Z^2/4$ and suppose $a(\cdot)$ satisfies A4. Assume that $r(u) - s(u) > 0$ on some interval I. Then, $r(u) - s(u)$ can have at most one critical point on I. Moreover this critical point is necessarily a strict minimum.*

Proof. It is convenient to make the following change of variable: let $u = e^{-q}$. Here q is an increasing function of r which goes to $+\infty$ as $r \to \infty$. Let us consider the distance $r(q) - s(q)$, where $r(q) = r(u(q))$, etc. We have

$$u_r = -e^{-q}/r_q\,, \tag{47}$$

$$u_{rr} = e^{-q}(r_q + r_{qq})/r_q^3\,, \tag{48}$$

where $r_q \equiv \dfrac{dr}{dq} = \dfrac{1}{q_r}$, etc. Introducing (47) and (48) into Eq. (29) we find that $r(q)$ satisfies

$$r_{qq} + r_q - \frac{2}{r} r_q^2 - e^q a(e^{-q}) r_q^3 + \frac{Z}{r} r_q^3 = 0\,. \tag{49}$$

The function $s(q)$ satisfies (49) too. Hence,

$$(r-s)_{qq} + (r-s)_q - \frac{2}{r} r_q^2 + \frac{2}{s} s_q^2 + \frac{Z}{r} r_q^3 - \frac{Z}{s} s_q^3 - e^q a(e^{-q})\,(r_q^3 - s_q^3) = 0\,.$$

At a critical point of $(r-s)(q)$ we have $r_q = s_q$ and therefore

$$(r-s)_{qq} = r_q^2(s^{-1} - r^{-1})(-2 + Z r_q) > 0\,. \tag{50}$$

The last inequality in (50) follows since $r > s$ on I and because $Zr_q - 2 = h/(-u')$ is positive, by Lemmas 8 and 12. Therefore any critical point of $(r-s)(q)$ in I is a minimum. It follows immediately from here that there is at most one such a point on I. □

Remark. The proof of Lemma 14 is based on the proof of Lemma 7 in [7].

An immediate consequence of the previous lemma is the following

Corollary 15. *Let $\lambda < Z^2/4$ and assume $a(\cdot)$ satisfies* A4. *Then, two different solutions of* (29) *u and v cannot intersect.*

Proof. Suppose, for contradiction, that u and v do intersect; that is, there exist r_1 and r_2, $r_2 > r_1$, such that $u(r_1) = v(r_1) = \alpha$ and $u(r_2) = v(r_2) = \beta$, $\beta < \alpha$. This means that $r(\alpha) - s(\alpha) = r(\beta) - s(\beta) = 0$. Without loss of generality we may assume $u > v$ on (r_1, r_2). Thus $r(u) - s(u) > 0$ on (β, α). Since $r(u) - s(u)$ is continuous, it must then have a maximum on (β, α), which contradicts the previous lemma. □

Lemma 16. *Let $\lambda < Z^2/4$ and suppose $a(\cdot)$ satisfies* A4. *Let u and v be two solutions of* (29), (26), *vanishing at ∞ with $u(0) > v(0)$. Then,*

$$\frac{d}{du}(r(u) - s(u))|_{v(0)} < 0 . \tag{51}$$

Proof. Since $u(0) > v(0)$, there exists $\xi > 0$ such that $u(\xi) = v(0)$. Then,

$$(r_q - s_q)(-\ln v(0)) = -\frac{u(\xi)}{u'(\xi)} + \frac{v(0)}{v'(0)} > 0 , \tag{52}$$

where the last inequality follows from (26) and Lemma 12. From (52) the lemma follows. □

After all these preliminary lemmas we conclude this section with our main result, namely the uniqueness of the positive radial solutions of Eq. (1) under the hypothesis A4.

Theorem 17. *Let $\lambda < Z^2/4$ and suppose $a(\cdot)$ satisfies* A4. *Then, there is at most one positive solution of* (29), (26) *which goes to zero at infinity.*

Proof. Assume, for contradiction, that there are two different solutions. Because of Lemma A.1 in the Appendix, $u(0) \neq v(0)$, and we may assume without loss of generality that $u(0) > v(0)$. Because of Corollary 15, there are only two possibilities to consider namely: i) u and v intersect only once; ii) u and v do not intersect in $[0, \infty)$. Let us first consider case i) and denote by ξ the intersection point. We then have,

$$(r - s)(-\ln u(\xi)) = 0 , \tag{53}$$

and

$$(r_q - s_q)(-\ln v(0)) > 0 , \tag{54}$$

where this last inequality follows from the previous lemma [Eq. (52)]. Since $(r - s)$ is a continuous function of q, Eqs. (53) and (54) imply that there is a maximum of $(r - s)(q)$ in the interval $(-\ln v(0), -\ln u(\xi))$, where $(r - s)(q)$ is positive. This contradicts Lemma 14. Thus, we need only discard the second possibility. Since $u(0) > v(0)$ and u and v do not intersect, $u > v$, all r. Here $(r - s)(-\ln v(0)) > 0$, and therefore, using Lemmas 16 and 14 we conclude

$$(r - s)_q(q) > 0 , \tag{55}$$

and

$$(r - s)(q) > 0 , \tag{56}$$

for all $q \geqq -\ln v(0)$. From the equation below (49) we have,

$$(r-s)_{qq} = M \cdot (r_q - s_q) + s_q^2 (s^{-1} - r^{-1}) \cdot (Z s_q - 2), \tag{57}$$

with

$$M = -1 + e^q a(e^{-q})(r_q^2 + s_q^2 + r_q s_q) + \frac{2}{r}(r_q + s_q) - \frac{Z}{r}(r_q^2 + r_q s_q + s_q^2).$$

When $u \to 0$, $q \to \infty$, and therefore A1 implies $e^q a(e^{-q}) \to \lambda$. Also, from Lemma 10, $r_q \to \lambda^{-1/2}$ and $s_q \to \lambda^{-1/2}$. Moreover, r and s go to infinity. Hence, $M \to 2$ as $q \to \infty$. On the other hand,

$$(Z s_q - 2) = h/(-u') > 0,$$

by Lemmas 8 and 12. Thus, for q large enough, $(r-s)_{qq} > 0$ [see Eq. (57)]. Let $g(q) \equiv (r-s)_q(q)$. We have shown that $g(q) > 0$, all $q \geqq -\ln v(0)$ [Eq. (55)]. Moreover, Lemma 10 implies $g(q) \to 0$ as $q \to \infty$, and we have just proved that $\frac{dg}{dq} > 0$ for q large enough. These three conditions on g cannot hold at the same time, so from here the lemma follows. □

4. Further Results and Applications to the TFDW Equation

To conclude this article we show some additional properties of the positive radial solutions of Eq. (1) and we apply the results we have obtained here to the TFDW equation, i.e. to Eq. (1) with $a(\cdot)$ given by (7). In Lemmas 18 and 19 below we prove some "negative" results, in fact we prove that under certain hypothesis on $F(u)$ there are no nontrivial positive solutions to Eq. (29), satisfying (26) and going to zero at infinity. On the other hand, in Lemma 20 we give some conditions on $F(u)$ which assure the existence of a nontrivial positive solution to (29), (26). In Lemma 21 we give a lower bound on the value of $u(0)$, which is better than the one obtained in Lemma 11. Finally in Lemma 22, we summarize the consequences of all these previous lemmas on the TFDW equation.

In the Lemmas 18 and 19 below, we relax the hypothesis A3 on $F(u)$ and we obtain:

Lemma 18. *Let $\lambda < Z^2/4$ and assume that $G(u)$ has $2k+1$ simple zeros, say $0 < w_1 < w_2 < \ldots < w_{2k+1} < \infty$. Define*

$$\alpha_1 \equiv \inf\{F(s)/s^2 | 0 \leqq s \leqq w_1\}. \tag{58}$$

Then, if $\alpha_1 < 0$, there is no nontrivial positive solution to Eq. (29), (26) *which goes to zero at infinity.*

Proof. By Lemma 13, iii), $h(r) \geqq 0$, all $r \geqq R$, with $u(R) = w_1$. Multiplying (29) by $u'(r)$, which is negative (Lemma 8), and integrating on r from $\xi > R$ up to infinity we get,

$$u'(\xi)^2 < F(u(\xi)), \tag{59}$$

all $\xi > R$. Hence, $F(s) > 0$ all $0 \leqq s \leqq w_1$, which contradicts $\alpha_1 < 0$. □

Lemma 19. *Let $\lambda > Z^2/4$ and suppose that $G(u)$ has $2k(k \geqq 1)$ simple zeros, say $0 < w_1 < w_2 < \ldots < w_{2k} < \infty$. Let*

$$\alpha_2 \equiv \inf\{F(s)/s^2 | w_1 < s\}. \tag{60}$$

Then, if $\alpha_2 > Z^2/4$, there is no nontrivial positive solution of Eq. (29), (26) *which goes to zero at infinity.*

Proof. Proceeding as in the proof of Lemma 12 one can show that the corresponding h has at most $2k$ simple zeros (including $r=0$) in $[0, \infty)$. Let ξ denote the largest zero of h. Because of Lemma 10, $h(r) \leqq 0$ for all $r \geqq \xi$. It is easy to see that $u(\xi) \geqq w_1$ (this is the analog of Lemma 13, iii). Multiplying (29) by $u'(r)$, which is negative (Lemma 8) and integrating on r from ξ up to infinity, we obtain

$$-(u'(\xi))^2 + F(u(\xi)) < 0. \tag{61}$$

Since $h(\xi) = 0$, (61) implies

$$Z^2/4 > F(u(\xi))/\xi^2,$$

which proves the lemma. □

After these two negative results we now give a criterion for having nontrivial positive solutions of (29).

Lemma 20. *Let $a(\cdot)$ satisfy* A1 *through* A3 *and define*

$$\beta \equiv \inf\left\{\frac{4}{A^2} \int_0^\infty F(Ae^{-y}) y^2 dy | A > 0\right\}. \tag{62}$$

Then, if $\beta < Z^2/4$, there is at least one nontrivial positive solution of (29), (26) *which goes to zero at infinity. In particular, if λ is such that $\alpha(\lambda) > 0$ and $\lambda < Z^2/4$, then there is at least one nontrivial positive solution of* (29). *(Here $\alpha(\lambda)$ is given by* (5)*).*

Proof. We need only show that if $\beta < Z^2/4$, there is a nontrivial ($u \neq 0$) minimum of $S[u]$ in E. It is easy to find a $\psi \in E$ such that $S[\psi] < 0$. (Note here that $S[0] \equiv 0$). Consider $\psi(r) = A\exp(-tr)$ with A and t positive. Obviously $\psi \in E$ and

$$S[\psi] = (\pi A^2/t)(1 - Zt^{-1} + f(A)t^{-2}), \tag{63}$$

with

$$f(A) \equiv \frac{4}{A^2} \int_0^\infty F(Ae^{-y}) y^2 dy. \tag{64}$$

Choosing $t = 2f(A)/Z$, which minimizes the parenthesis ( ) in (63) we get,

$$S[\psi] = (\pi A^2 Z/2f(A))(1 - (Z^2/4f(A)). \tag{65}$$

The first part of this lemma follows by choosing in (65) the A which minimizes $f(A)$. Now, (2) and (4) imply that for every $\varepsilon > 0$ there exists $\delta(\varepsilon)$ so that

$$F(t) \leqq (\lambda + \varepsilon)t^2, \tag{66}$$

all $t < \delta(\varepsilon)$. Clearly,

$$\beta \leqq \inf\left\{\frac{4}{A^2} \int_0^\infty F(Ae^{-y}) y^2 dy | 0 < A < \delta(\varepsilon)\right\} \leqq (\lambda + \varepsilon),$$

all $\varepsilon < 0$. Hence $\beta \leqq \lambda$, which proves the second part of this lemma. □

Lemma 21. *Let $\lambda < Z^2/4$ and assume $a(\cdot)$ satisfies* A1 *through* A4. *Let u be as in Lemma 9. Then, $u(0)$ is such that*

$$Z^2/4 < F(u(0))/u(0)^2 . \tag{67}$$

Remark. This result is stronger than the one obtained in Lemma 11.

Proof. Multiply (29) by u' and integrate over $(0, \infty)$. Then use Lemma 12 and Eq. (26) to get (67). □

Putting all these results thogether in the case where $a(\cdot)$ is given by (7) we obtain the following result:

Lemma 22. *Consider the* TFDW *Eq.* (1), *with $a(u)$ given by* (7). *Then,*

i) *For all $\lambda \in \left(\frac{15}{64}\frac{b^2}{a}, \frac{Z^2}{4}\right]$ there is a unique positive radial solution of* (1) *satisfying the cusp condition* (26) *and which goes to zero at infinity.*

ii) *For $\lambda < \frac{15}{64}\frac{b^2}{a}$ and $\lambda > \frac{Z^2}{4} + \frac{15}{64}\frac{b^2}{a}$ there is no nontrivial positive solution of* (1) *which goes to zero at infinity.*

iii) *For $Z^2/4 \leqq \lambda \leqq (Z^2/4) + (15/16)^4(b^2/4a)$, there is at least one nontrivial solution of* (1) *satisfying* (26) *and going to zero at infinity.*

Proof. It is a direct application of our previous results. □

Appendix

Lemma A.1. *Let u and v be two solutions of Eq.* (29), *with $u, v \in C^2(0, \infty)$, satisfying the cusp condition* (26). *Then, if $u(0) = v(0)$, $u \equiv v$.*

Proof. We need only show that $u \equiv v$ on a neighborhood of $r = 0$, since Eq. (29) is nonsingular away from $r = 0$. Let $\psi(r) = u(r) - v(r)$. Then $\psi(0) = 0$ and, because of (26), $\psi'(0) = 0$. Here, $\psi(r)$ satisfies the equation

$$-\psi''(r) - \frac{2}{r}\psi'(r) - \frac{Z}{r}\psi(r) + H(r) = 0 , \qquad r > 0 , \tag{68}$$

where $H(r) = a(u(r)) - a(v(r))$. Using Green's function for the operator $L\psi \equiv \psi'' + (2/r)\psi'$ with the initial conditions $\psi(0) = \psi'(0) = 0$, we find

$$\psi(r) = -\int_0^r (1 - (s/r))s\left(\frac{Z}{s}\psi(s) - H(s)\right)ds . \tag{69}$$

Because of A1, $|H(s)| \leqq M|\psi(s)|$, for $0 \leqq s < s_0$, where $s_0 > 0$ and $M = M(s_0)$. Therefore,

$$|\psi(r)| \leqq Z\int_0^r |\psi(s)|ds + \tfrac{1}{4}Mr\int_0^r |\psi(s)|ds . \tag{70}$$

From here it follows that $|\psi(r)| \equiv 0$ for $0 \leqq r < s_0$ (by Gronwall's inequality). □

Lemma A.2. *Let u be as in Lemma 9. Then $u'(r)<0$, all $r \geqq 0$.*

Proof. We have already proved that $u'(r) \leqq 0$ (see the proof of Lemma 8). Thus, we need only show that $u'(r) \neq 0$, all $r \geqq 0$. Assume, for contradiction, that $u'(\xi)=0$ for some $0<\xi<\infty$. Since $u \in C^2(0,\infty)$ and $u'(r) \leqq 0$, all r, we must have $u''(\xi)=0$. From Eq. (29) we then get,

$$u''(\xi+\delta)=M_1+M_2+M_3, \tag{71}$$

with

$$\begin{aligned} M_1(\delta) &\equiv -2u'(\xi+\delta)/(\xi+\delta), \\ M_2(\delta) &\equiv Z[u(\xi)/\xi-u(\xi+\delta)/(\xi+\delta)], \\ M_3(\delta) &\equiv a(u(\xi+\delta))-a(u(\xi)). \end{aligned} \tag{72}$$

Because of A1,

$$|M_3(\delta)|<N|u(\xi+\delta)-u(\xi)|, \tag{73}$$

all $0<\delta<\delta_0$, with $N=N(\xi,\delta_0)$. From the mean value theorem, we get

$$|u(\xi+\delta)-u(\xi)|=\left|\int_{\xi}^{\xi+\delta} u'(s)ds\right|<|u'(\xi+\theta\delta)|\cdot\delta<|u''(\xi+\theta\mu\delta)|\theta\delta^2, \tag{74}$$

for some $0<\theta,\mu<1$. Hence, $\lim_{\delta\to 0} M_3(\delta)/\delta=0$. In a similar way one can prove that $\lim_{\delta\to 0} M_1(\delta)/\delta=0$, whereas

$$\lim_{\delta\to 0} M_2(\delta)/\delta=Zu(\xi)/\xi^2>0, \tag{75}$$

where the last inequality follows from Lemma 6, iv). Going back to (71), we conclude that $u''(\xi+\delta)>0$ for $\delta>0$ small enough. Since $u'(\xi)=0$, this contradicts the fact that $u'(r)\leqq 0$, all r. □

Remark. Lemma A.2 still holds if we replace Z/r by any other strictly decreasing potential.

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