Commun. Math. Phys. 103, 637–668 (1986)

Communications in
Mathematical
Physics

Absence of Charged States in the U(1) Higgs Lattice Gauge Theory

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Abstract. We show that a sequence of dipole states of finite energy introduced by Fredenhagen and Marcu is chargeless upon removal of one of the charges to spatial infinity in certain subsets of the phase diagram of the U(1)-Higgs lattice gauge theory. It is also explicitly seen how this phenomenon is related to the existence of exponential clustering (i.e., of a mass gap). Related properties of dipole states are briefly discussed.

I. Introduction and Summary

In a beautiful paper, Fredenhagen and Marcu [1] showed that a certain sequence of dipole states of finite energy acquires a charge when one of the charges is removed to infinity in a subset of the phase diagram of a $Z(2)$-Higgs lattice gauge theory. They also proved that the same states are chargeless in certain subsets of the confinement/screening regions of the model.

In this paper we prove the absence of charge for the same sequence of dipole states introduced in [1] in a subregion of the screening/confinement diagram of the U(1)-Higgs lattice gauge theory. Our main motivation in so doing is that this theory involves additive (in contrast to multiplicative in the $Z(2)$ case, treated in [1]) charges, implying a different structure, more akin to the more interesting nonabelian gauge theories. In particular, the connection between the absence of charges and exponential clustering (i.e., a mass gap) seems to appear more directly in the present model (see Sect. III). This statement is related to a remarkable theorem of Swieca [2] (a rigorous version of which was formulated and proved in [3]), a general result in the framework of (continuum) relativistic quantum field theory which roughly states that in abelian gauge theories with a mass gap there

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are no charged states. Swieca's theorem requires locality and is therefore not applicable to nonabelian theories. It should therefore be especially interesting to extend the results of [1] and the present paper to the nonabelian case. In view of this aim, we present in Sect. II the transfer matrix formalism and the relevant results (used in Sect. III) in the general (nonabelian) setting, specializing to the U(1) case later. In particular, we study in the same section the dipole states analogous to those introduced for the $Z(2)$ model in [1], deriving their "finite energy property" in Proposition II.1. The latter property is *universal*, depending only on the perimeter law of the Wilson Loop[1], an important fact which, although not explicitly stated in [1], is implicit in the proof given there[2]. Finally, we make some remarks on the expected properties of other dipole states (finite energy property, disjointness from vacuum representation). They are illustrated by a short and self-contained discussion in the framework of the massless Schwinger model in Appendix A, which presents a somewhat different, mathematically precise version of arguments of Swieca [4] and Lowenstein and Swieca [5]. It should be remarked that we are concerned here with the energy of *dynamical* charged states, not the (potential) energy between static, external charges. The latter has a special, atypical behaviour in the massless Schwinger model (due to the zero electron mass), but not the former (see remarks in Appendix A).

Section III is the core of the present paper, based on [6]. There we present the proof of absence of charge for the sequence of dipole states defined in Sect. II, specialized to the U(1) model. It extends some results concerning polymer expansions in [1]. The main results of [1, 8] which have been used are either briefly stated in Sect. III or collected in Appendix B, which also presents a complete derivation of the crucial formula (B.12)[3]. Section IV is a conclusion, with a brief additional discussion of open problems.

II. The Model, Dipole States and the Finite Energy Property

Consider a spatial box $\Lambda \subset \mathbb{Z}^d$, where $d \geqq 1$ is the number of space dimensions, which we take as the time-zero slice of a space-time lattice $\varLambda = \Lambda \times \mathbb{Z}$. A point in $\varLambda$ will be denoted by n. Positively (respectively negatively) oriented links will be written (n, μ), where μ runs over the positive (negative) space directions $\hat{\mu} = \hat{e}_i$ $(\hat{\mu} = \hat{e}_i)$, $i = 1, \ldots, d$, or over the positive (negative) time direction $\hat{t} = \hat{e}_{d+1}$ $(-\hat{t} = -\hat{e}_{d+1})$. We write $B(\Lambda)$ for the set of positively oriented bonds (links) in Λ, and $P(\varLambda)$ for the set of positively oriented plaquettes in $\varLambda$. A special plaquette is denoted P_s, a temporal plaquette P_t. To each bond $(n, \mu) \in B(\Lambda)$ (respectively to each $n \in \Lambda$) we associate a faithful representation $\mathrm{U}(n, \mu)$ [respectively $\phi(n)$] of a compact Lie group G. The $\mathrm{U}(n, \mu)$ are interpreted as gluon fields, the $\phi(n)$ as Higgs fields. Our Hilbert space is

$$\mathscr{H}_\Lambda = L^2(d\mu_\Lambda),$$

1 Which is expected whenever matter fields are present, due to pair creation [8, 13, 14]

2 See also [16] for similar remarks and a more extensive discussion of this point

3 We should like to thank Dr. K. Fredenhagen for showing us this derivation, and suggesting its inclusion in the appendix

where $d\mu_\Lambda = d\mu_u \otimes d\mu_\phi$, with

$$d\mu_u = \bigotimes_{(n,\mu)\in B(\Lambda)} d\mu_{u(n,\mu)}, \qquad d\mu_\phi = \bigtimes_{n\in\Lambda} d\mu_{\phi(n)}$$

is the (product) Haar measure. We define the transfer matrix in the temporal gauge as the integral operator on $\mathscr{H}_\Lambda$ defined by

$$T_\Lambda = e^{A_\Lambda/2} F_\Lambda e^{A_\Lambda/2}, \tag{II.1a}$$

where A_Λ is the multiplication operator given by

$$\begin{aligned} A_\Lambda = 2\beta_g & \sum_{P_s\in P(\Lambda)} \mathrm{Re}(\mathrm{Tr}\{U_{P_s} - \mathbb{1}\}) \\ & + 2\beta_h \sum_{(n,\mu)\in B(\Lambda)} \mathrm{Re}(\mathrm{Tr}\{\phi^{-1}(n+\mu)\, U(n,\mu)\, \phi(n) - \mathbb{1}\}), \end{aligned} \tag{II.1b}$$

and F_Λ is an integral operator with kernel given by

$$\begin{aligned} F_\Lambda(\{U,\phi\},\{U',\phi'\}) = \exp\Big[& 2\beta_g \sum_{(n,\mu)\in B(\Lambda)} \mathrm{Re}\{\mathrm{Tr}\, U^{-1}(n,\mu)\, U'(n,\mu) - \mathbb{1}\} \\ & + 2\beta_h \sum_{n\in\Lambda} \mathrm{Re}(\mathrm{Tr}\{\phi'^{-1}(n)\,\phi(n) - \mathbb{1}\})\Big]. \end{aligned} \tag{II.1c}$$

Above, β_g and β_h are the gluon and Higgs coupling constants and

$$U_{P_s} = \bigotimes_{(n,\mu)\in P_s} \gamma_{(n,\mu)} U,$$

where

$$\gamma_{(n,\mu)} U = \begin{cases} U(n,\mu) & \text{if the orientations of } (n,\mu) \text{ and } P_s \text{ are the same} \\ U(-(n,\mu)) \equiv U^{-1}(n,\mu) & \text{otherwise}. \end{cases}$$

The above construction, with the Higgs fields as matrices, is not quite standard, although the corresponding transfer matrix has a more symmetric form, and an additional invariance (coming from the trace). In the more usual one, the Higgs fields are elements of a vector space, and conditions ensuring "complete symmetry breakdown [8, p. 57] must be imposed. We shall refer to the above cases as "matrix" or "vector space" formulations, for brevity. The following properties of T_Λ are known [7, 17]:

a) T_Λ is a self-adjoint, trace class operator;

b) $\|T_\Lambda\| \leqq 1$ is a simple eigenvalue. The corresponding normalized eigenstate Ω_Λ is interpreted as the ground state (vacuum);

c) T_Λ is invertible. The inverse T_Λ^{-1} is an unbounded self-adjoint operator. In the matrix formulation this follows immediately from [7]. See [17] for a general proof of invertibility of the transfer matrix in the thermodynamic limit.

If A is a local observable, i.e., a bounded operator on $\mathscr{H}_\Lambda$, it follows that [1]

$$\omega_\Lambda(A) \equiv (\Omega_\Lambda, A\Omega_\Lambda) = \lim_{q\to\infty} \mathrm{Tr}\{A(T_\Lambda)^q E_\Lambda^{(0)} (T_\Lambda)^q\}/Z_{\Lambda,q}, \tag{II.2a}$$

where

$$Z_{\Lambda,q} \equiv \mathrm{Tr}\{(T_\Lambda)^q E_\Lambda^{(0)} (T_\Lambda)^q\} \tag{II.2b}$$

is the "partition function" and $E_{\Lambda}^{(0)}$ is any bounded positive operator. Choose $E_{\Lambda}^{(0)}$ as an integral operator with kernel

$$\begin{aligned} E_{\Lambda}^{(0)}(\{U,\phi\},\{U',\phi'\}) = \exp\Big\{ & -\tfrac{1}{2}(2\beta_g) \sum_{P_s \in P(\Lambda)} \mathrm{Re}(\mathrm{Tr}\{U_{P_s} - \mathbb{1}\}) \\ & + \tfrac{1}{2} \sum_{(n,\mu)\in B(\Lambda)} (2\beta_h)\,\mathrm{Re}(\mathrm{Tr}\{\phi^{-1}(n+\mu)\,U(n,\mu)\,\phi(n) - \mathbb{1}\})\Big\} \\ \cdot \exp\Big\{ & \tfrac{1}{2} \sum_{P_s \in P(\Lambda)} (2\beta_g)\,\mathrm{Re}(\mathrm{Tr}\{U'_{P_s} - \mathbb{1}\}) \\ & + \tfrac{1}{2} \sum_{(n,\mu)\in B(\Lambda)} (2\beta_h)\,\mathrm{Re}(\mathrm{Tr}\{\phi'^{-1}(n+\mu)\,U'(n,\mu)\,\phi'(n) - \mathbb{1}\})\Big\}. \end{aligned} \tag{II.3}$$

This choice corresponds to imposing free boundary conditions. We suppose that AT_{Λ} may be written as an integral operator with kernel

$$(AT_{\Lambda})(\{U_0,\phi_0\};\{U_1,\phi_1\}), \tag{II.4a}$$

and define then the "second kernel" of A, denoted by $[A]$, through the formula

$$[A]\,(\{U_0,\phi_0\},\{U_1,\phi_1\}) \equiv (AT_{\Lambda})(\{U_0,\phi_0\};\{U_1,\phi_1\})/T_{\Lambda}(\{U_0,\phi_0\};\{U_1,\phi_1\}). \tag{II.4b}$$

With the above choice of $E_{\Lambda}^{(0)}$, we find then from (II.2):

$$\begin{aligned} \omega_{\Lambda}(A) = \lim_{q\to\infty} \int \prod_{m=-q}^{q} d\mu_{U_{(m)}}\, d\mu_{\phi_{(m)}} [A]\,(\{U_{(0)},\phi_{(0)}\};\{U_{(1)},\phi_{(1)}\}) \\ \exp\left\{\sum_{m=-q}^{q-1} S_{\Lambda}(m)\right\} \Big/ Z_{\Lambda,q}, \end{aligned} \tag{II.5a}$$

where

$$Z_{\Lambda,q} = \int \prod_{m=-q}^{q} d\mu_{U_{(m)}}\, d\mu_{\phi_{(m)}} \exp\left[\sum_{m=-q}^{q-1} S_{\Lambda}(m)\right]. \tag{II.5b}$$

Above, $U_{(m)}$ and $\phi_{(m)}$ are the fields, and, correspondingly, $d\mu_{U_{(m)}}$ and $d\mu_{\phi_{(m)}}$ the Haar measures restricted to the m'th time slice, and

$$\begin{aligned} S_{\Lambda}(m) & \sum_{P_s \in P(\Lambda)} (2\beta_g)\,\mathrm{Re}(\mathrm{Tr}\{U_{P_{s(m)}} - \mathbb{1}\}) \\ & + \sum_{(n,\mu)\in B(\Lambda)} (2\beta_h)\,\mathrm{Re}(\mathrm{Tr}\{\phi_{(m)}^{-1}(n+\mu)\,U_{(m)}(n,\mu)\,\phi_{(m)}(n) - \mathbb{1}\}) \\ & + \sum_{(n,\mu)\in B(\Lambda)} (2\beta_g)\,\mathrm{Re}(\mathrm{Tr}\{U_{(m)}^{-1}(n,\mu)\,U_{(m+1)}(n,\mu) - \mathbb{1}\}) \\ & + \sum_{n\in\Lambda} (2\beta_h)\,\mathrm{Re}(\mathrm{Tr}\{\phi_{(m+1)}^{-1}(n)\,\phi_{(m)}(n) - \mathbb{1}\}). \end{aligned} \tag{II.6}$$

Let $\{\lambda_b\}$ be a basis of self-adjoint elements for the Lie algebra of G in the chosen representation. In a way similar to [7], the generators of the (time-independent) gauge transformations (in the vector space formulation)

$$(G_{\{\tilde{T}\}}f)(\{U,\phi\}) = f(\{\tilde{T}_{n+\mu}U(n,\mu)\,\tilde{T}_n^{-1},\, \tilde{T}_n\phi(n)\})$$

with $\tilde{T}_n = \exp[-i\mathscr{E}^b(n)\lambda_b]$, $\mathscr{E}^b(n) \in \mathbb{R}$, $f \in \mathscr{H}_\Lambda$, are given by

$$G_b(n) = \sum_{\hat{\mu}=e_i, i=1,\dots,d} [E_b^*(n,\hat{\mu}) - E_b(n-\hat{\mu},\hat{\mu})] - P_b(n), \tag{II.7}$$

where

$$(E_b^*(n_0,\hat{\mu}_0)f)(\{U,\phi\}) = \lim_{\varepsilon\to 0}\frac{1}{i\varepsilon}[f(\{U_\varepsilon,\phi\}) - f(\{U,\phi\})],$$

and

$$U_\varepsilon(n,\hat{\mu}) = \begin{cases} U(n_0, +\hat{\mu}_0)e^{i\varepsilon\lambda_b} & \text{if } (n,\hat{\mu}) = (n_0,\hat{\mu}_0) \\ U(n,\hat{\mu}) & \text{if } (n,\hat{\mu}) \neq (n_0,\hat{\mu}_0) \end{cases}.$$

Similarly

$$(E_b(n_0,\hat{\mu}_0)f)(\{U,\phi\}) = \lim_{\varepsilon\to 0}\frac{1}{i\varepsilon}[f(\{U'_\varepsilon,\phi\}) - f(\{U,\phi\})],$$

where

$$U'(n,\ ) = \begin{cases} e^{i\varepsilon\lambda_b}U(n_0,\hat{\mu}_0) & \text{if } (n,\hat{\mu}) = (n_0,\hat{\mu}_0) \\ U(n,\hat{\mu}) & \text{if } (n,\hat{\mu}) \neq (n_0,\hat{\mu}_0) \end{cases},$$

and

$$(P_b(n_0)f)(\{U,\phi\}) = \lim_{\varepsilon\to 0}\frac{1}{i\varepsilon}[f(\{U,\phi_\varepsilon\}) - f(\{U,\phi\})],$$

with

$$\phi_\varepsilon(n) = \begin{cases} e^{i\varepsilon\lambda_b}\phi(n) & \text{if } n = n_0 \\ \phi(n) & \text{if } n \neq n_0 \end{cases}.$$

We may decompose $\mathscr{H}_\Lambda$ into irreducible components relative to the product group $\left(\bigtimes_{n\in\Lambda} G_n\right)$:

$$\mathscr{H} = \bigoplus \mathscr{H}_{\{\tau_n\}}.$$

$G_b(n)$ acts on the subspace $\mathscr{H}_{\{\tau_n\}}$ according to the irreducible representation τ_n, which is identified with the "external charge" at n. Let $P_{\{\tau_n\}}$ denote the orthogonal projector onto $\mathscr{H}_{\{\tau_n\}}$, and P_0 the projection operator onto the subspace of states with zero external charge at each point.

Lemma II.1. $P_0\Omega_\Lambda = \Omega_\Lambda$.

Proof. The proof follows from the Peter-Weyl theorem, which provides an explicit formula for P_0. It is simplified by the choice of $E_\Lambda^{(0)}$ in (II.2). □

It is easy to verify that

$$[P_{\{\tau_n\}}, T_\Lambda] = 0, \tag{II.8}$$

which means that the external charges are "static."

We now consider dipole states. In the vector space formulation a suitable definition of such states along the lines of [1] would be

$$\Psi_{r,\Lambda} = \sum_{a=1}^{V} \sum_{b=1}^{V} \Psi_{r,\Lambda}^{a,b}, \tag{II.9a}$$

$$\Psi_{r,\Lambda}^{a,b} \equiv \Phi_a^\dagger(\mathbf{O})\, \Phi_b(\mathbf{x}_r)\, T_\Lambda^r(U(2r))_{ab}\, T_\Lambda^{-r}\Omega_\Lambda . \tag{II.9b}$$

We assume that $\Psi_{r,\Lambda} \neq 0$. This is expected, and verified in the cases $G = Z(2)$ and $G = U(1)$. Above, $\mathbf{O}$ is the origin of the coordinate system, $\mathbf{x}_r \equiv (2r, 0, \ldots, 0)$ and

$$U(2r) \equiv U(0, \hat{e}_1) \times \ldots \times U(\mathbf{O} + (2r-1)\hat{e}_1, \hat{e}_1) . \tag{II.10}$$

Notice that Ω_Λ belongs to the domain of T_Λ^{-r}, since $\|T_\Lambda\| \neq 0$. To each site $\mathbf{n}$ we attach a copy $V_\mathbf{n}$ of a (finite-, say V-) dimensional vector space V carrying a faithful unitary representation of G. The $\{\Phi_a(\mathbf{n})\}$ above are a basis of $V_\mathbf{n}$, $\Phi^\dagger$ denotes the adjoint of a vector $\Phi \in V$, and we assume that the Higgs measure ensures that $\Phi^\dagger \Phi = 1$, for simplicity, as in [8, p. 57]. Finally, a condition ensuring "complete symmetry breakdown" [8, p. 57] is assumed.

We now wish to define (and prove) a property of the above sequence of dipole states ensuring that the limiting charged state has, if it exists, "finite energy." Unfortunately, for continuous groups, the (uniform in Λ) norm-boundedness of the "automorphism" $\alpha_i^\Lambda(A) = T_\Lambda^{-1} A T_\Lambda$, for A in the field algebra $\mathscr{F}$ [1], does not follow as in the discrete case, because T_Λ^{-1} is unbounded (see also [17] for further discussion). We therefore modify the definition of the "finite-energy property" [1, Proposition 6.1] as follows.

By a) to c), there exists a positive self-adjoint operator S_Λ such that

$$T_\Lambda = e^{-S_\Lambda} .$$

For $\Lambda' \subseteqq \Lambda$, let $A \in \mathscr{F}^{\Lambda'}$, the local subalgebra of $\mathscr{F}$ corresponding to region Λ'. Its representative on $\mathscr{H}_\Lambda$ will also be denoted by the same symbol. We define $\hat{\mathscr{D}}$ to be the set of functions f of a real variable such that their Fourier transform $\hat{f}$ is in the Schwartz space $\mathscr{D}$ of infinitely differentiable functions of compact support, and, correspondingly, a "time-smeared" operator $A_\Lambda(f)$ associated to A by

$$A_\Lambda(f) \equiv \int_{-\infty}^{\infty} dt\, f(t)\, \alpha_t^\Lambda(A), \qquad f \in \hat{\mathscr{D}}, \tag{II.11a}$$

where

$$\alpha_t^\Lambda(A) \equiv e^{itS_\Lambda} A e^{-itS_\Lambda} . \tag{II.11b}$$

Since f is rapidly decreasing, the integral in (II.11a) exists in the strong sense. In addition, given $\varepsilon > 0$ and $\psi \in \mathscr{H}_\Lambda$, one may show that there exists a positive $f \in \hat{\mathscr{D}}$ such that

$$\int_{-\infty}^{\infty} dt\, f(t) = 1 \tag{II.12a}$$

and

$$\|(A_\Lambda(f) - A)\Psi\| \leqq \varepsilon . \tag{II.12b}$$

Lemma II.2. *For any $n \in \mathbb{Z}_+$, ψ in the domain $D(T_\Lambda^{-n})$ of T_Λ^{-n} and $f \in \hat{\mathscr{D}}$, $A_\Lambda(f)\psi \in D(T_\Lambda^{-n})$, and there exists a constant C depending only on f and n such that*

$$\|T_\Lambda^{-n} A_\Lambda(f)\Psi\| \leqq C\|A\| \, \|T_\Lambda^{-n}\Psi\| . \tag{II.13}$$

Proof. By the Paley-Wiener theorem, f is an entire analytic function and the spectral theorem implies in a straightforward way that $A_\Lambda(f) \in D(T_\Lambda^{-n})$, and

$$T_\Lambda^{-n} A_\Lambda(f) T_\Lambda^n = \int_{-\infty}^{\infty} dt \alpha_t^\Lambda(A) f(t+in) . \tag{II.14}$$

Again by the Paley-Wiener theorem, $\forall m \in \mathbb{Z}_+$, $\exists C_m < \infty$, $\alpha > 0$ such that

$$|f(z)| \leqq C_m(1+|z|)^{-m} \exp(\alpha|\mathrm{Im}\, z|) \qquad z \in \mathbb{C} .$$

Hence

$$\int_{-\infty}^{\infty} dt |f(t+ni)| \leqq C_2 e^{\alpha n} \int_{-\infty}^{\infty} dt (1+t^2)^{-1} \equiv C(n,f) \equiv C ,$$

and by (II.11b) and (II.14):

$$\|T_\Lambda^{-n} A_\Lambda(f) T_\Lambda^n\| \leqq c\|A\| ,$$

which implies (II.13) for $\psi \in D(T_\Lambda^{-n})$. □

Choose, now, the operator A above as $A \equiv A^{a,b} \equiv \Phi_a^\dagger(\mathbf{O})\, \Phi_b(\mathbf{x}_r)$ ($\|A^{a,b}\| \leqq 1$) and define

$$\varrho_{ab} \equiv T_\Lambda^r (U(2r))_{ab} T_\Lambda^{-r} \Omega_\Lambda , \tag{II.15a}$$

$$\Psi_{r,\Lambda}^{a,b}(f) \equiv A_\Lambda^{a,b}(f) \varrho_{ab} , \tag{II.15b}$$

$$\Psi_{r,\Lambda}(f) \equiv \sum_{a=1}^{V} \sum_{b=1}^{V} \Psi_{r,\Lambda}^{a,b}(f) . \tag{II.15c}$$

By (II.15a), $\varrho_{ab} \in D(T_\Lambda^{-n})$ if $r \geqq n$ (which we assume henceforth), and by (II.15) and Lemma II.2,

$$\|T_\Lambda^{-n} \Psi_{r,\Lambda}^{a,b}(f)\| \leqq \|T_\Lambda^{-n} \varrho_{ab}\| . \tag{II.16}$$

By (II.12), $\Psi_{r,\Lambda}(f) \neq 0$ for suitable f if $\Psi_{r,\Lambda} \neq 0$. We may therefore define the quantity:

$$E_{\Lambda,n,r}(f) \equiv \frac{(\Psi_{r,\Lambda}(f), T_\Lambda^{-n} \Psi_{r,\Lambda}(f))}{\|\Psi_{r,\Lambda}(f)\|^2} . \tag{II.17}$$

We have that $T_\Lambda^{-n} = e^{nS_\Lambda}$, where S_Λ is the Hamiltonian in the limit of continuous time (studied in [7] for the pure case). By (II.12), for $\Psi \in \mathscr{H}_\Lambda$ and a suitable delta sequence $\{f_m\}_{m \in \mathbb{Z}_+} \subset \hat{\mathscr{D}}$, $\lim_{\Omega \to \infty} A_\Lambda^{a,b}(f_m)\Psi = A^{a,b}\Psi$. We expect that $\Psi_{r,\Lambda}^{a,b} \in D(T_\Lambda^{-n})$, $T_\Lambda^{-n} \Psi_{r,\Lambda}^{a,b}(f_m)$ converges, and then necessarily to $T_\Lambda^{-n} \Psi_{r,\Lambda}^{a,b}$, because (T_Λ^{-1}) is a closed operator. These facts would imply that

$$E_{\Lambda,n,r}(f_m) \xrightarrow[m \to \infty]{} (\Psi_{r,\Lambda}, T_\Lambda^{-n} \Psi_{r,\Lambda}) / \|\Psi_{r,\Lambda}\|^2 .$$

We are presently, however, unable to prove even that $\Psi_{r,\Lambda} \in D(T_\Lambda^{-n})$. Nevertheless, the energy of the string ϱ_{ab}, as well as that of the fields at the endpoints, is independent of r (see the forthcoming proof). These remarks may render the following definition plausible:

Definition II.1. The sequence of states $\Psi_{r,\Lambda}$ has the finite-energy property (f.e.p.) if, for each positive integer n and $f \in \hat{\mathscr{D}}$ satisfying (II.12a) there exists a constant $C_n(f)$ independent of Λ and r such that

$$E_{\Lambda,n,r}(f) \leqq C_n(f) .$$

The basic element of the following proof is the assumption that the Wilson loop has perimeter decay. This is expected whenever $\beta_h > 0$, but the proof sketched in [8, Theorem 3.14 (4)] is inconclusive: while a proof along the lines of [20] might be feasible, it is, as yet, open (we thank the referee for these remarks). We therefore state a

Perimeter Assumption. There exists a subset A of the screening/confinement region of the (β_g, β_h) phase diagram, such that if $(\beta_g, \beta_n) \in A$,

$$W_\Lambda(M_r) \geqq \exp(-\alpha r) ,$$

where $\alpha > 0$ is independent of Λ, r, but depends on A. Above, $W_\Lambda(M_r)$ denotes the expectation value of the Wilson loop observable, which will be defined shortly. We also make the simplifying assumption (as in [8, p. 57]) that G acts transitively on the unit sphere in $\mathbb{C}^v$.

Proposition II.1. *Under the above Perimeter Assumption, $\Psi_{r,\Lambda}$ has the f.e.p. if $(\beta_g, \beta_h) \in A$.*

Proof. By the Schwartz inequality for both the scalar products and the sums over the indices, together with (II.16), we find

$$E_{\Lambda,n,r}(f) \leqq v \,.\, g_{r,\Lambda} \cdot h_{r,\Lambda} ,$$

where

$$g_{r,\Lambda} \equiv \frac{\left(\sum_{a,b=1}^{V} \| T_\Lambda^{r-n}(U(2r))_{ab} \Omega_\Lambda \|^2 \right)^{1/2}}{\left(\sum_{a,b=1}^{V} \| T_\Lambda^{r}(U(2r))_{ab} \Omega_\Lambda \|^2 \right)^{1/2}} ,$$

$$f_{r,\Lambda} \equiv \frac{\left(\sum_{a,b=1}^{V} \| T_\Lambda^{r}(U(2r))_{ab} \Omega_\Lambda \|^2 \right)^{1/2}}{\left\| \sum_{a,b=1}^{V} A_\Lambda^{a,b}(f)\, T_\Lambda^{r}(U(2r))_{ab} \Omega_\Lambda \right\|} .$$

The numerator in $f_{r,\Lambda}$ is bounded by

$$v .\, \| T_\Lambda^{r}(U(2r))_{a_0,b_0} \Omega_\Lambda \| \equiv v .\, \sup_{a,b} \| T_\Lambda^{r}(U(2r))_{ab} \Omega_\Lambda \| .$$

By assumption, G acts transitively on the unit sphere in $\mathbb{C}^v$, and hence there exists a (r-dependent) gauge transformation yielding:

$$\phi_a(\mathbf{O})=1 \quad \text{if} \quad a=a_0, \qquad \phi_a(\mathbf{O})=0 \quad \text{if} \quad a \neq a_0,$$

$$\phi_b(\mathbf{x}_r)=1 \quad \text{if} \quad b=b_0, \qquad \phi_b(\mathbf{x}_r)=0 \quad \text{if} \quad b \neq b_0.$$

In this gauge the denominator of $f_{r,\Lambda}$ equals $\|T^r_\Lambda(U(2r))_{a_0b_0}\Omega_\Lambda\|$, by (II.12a). By gauge invariance of both numerator and denominator of $f_{r,\Lambda}$, we thus get

$$f_{r,\Lambda} \leqq V.$$

We now apply the Schwartz inequality for both sums and scalar products to $g_{r,\Lambda}$ successively in the manner of [1, Proposition 6.1] obtaining, together with the above estimate for $f_{r,\Lambda}$:

$$E_{\Lambda,n,r}(f) \leqq \text{const}[W_\Lambda(M_r)]^{-n/r}, \tag{II.18a}$$

where

$$W_\Lambda(M_r)=\sum_{a,b}\|T^r_\Lambda(U(2r))_{ab}\Omega_\Lambda\|^2$$

is the expectation value (in spatial volume Λ) of the Wilson-loop observable corresponding to the oriented loop M_r in Fig. 1, which will occur many times in the sequel:

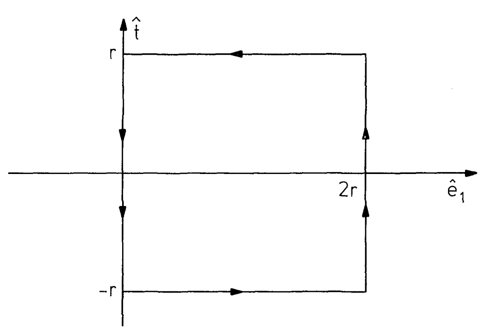

Fig. 1. The oriented loop M_r

By the "Perimeter Assumption" there exists a subset of the screening/confinement region of the (β_g, β_h) phase diagram where the Wilson loop has perimeter decay:

$$W_\Lambda(M_r) \geqq \exp(-\alpha r), \tag{II.18b}$$

where $\alpha > 0$ is independent of Λ, r (but depending on A). Inequalities (II.18a) and (II.18b) together yield the f.e.p. □

We may interpret the above result physically: placing the string as far away in time as the spatial separation between the charges involves distributing the flux lines of the electric field spatially in such a way that the energy is uniformly bounded, avoiding the formation of a permanent "linear flux tube." The important point is that this phenomenon is *universal*, depending only on the perimeter law.

This was not explicitly stated in [1], but is implicit in their proof.[4] What happens if the string is chosen to lie at $t=0$? It is hard to prove that Proposition II.1 does *not* hold. Since the state is now charged, it may be expected that the corresponding representation be disjoint from the vacuum representation, and that the f.e.p. does not hold, with the formation of a permanent linear flux tube. These properties are illustrated in the Schwinger model in Appendix A, which is a mathematically more precise version of arguments of [4, 5].

We shall now specialize some results to $G=\mathrm{U}(1)$, writing

$$u(n,\mu)=\exp[i\theta(n,\mu)]\,, \qquad \phi(n)=\exp[i\tau(n)]\,,$$

with $-\pi<\theta(n,\mu)\leqq\pi\ \forall(n,\mu)\in B(\mathbf{\Lambda})$, and $-\pi<\tau(n)\leqq\pi\ \forall n\in\mathbf{\Lambda}$. There are essentially no modifications, except for the normalization in the nonabelian case, $\mathrm{tr}(\lambda_a\lambda_b)=\frac{1}{2}\delta_{ab}$, which is no longer valid here. Hence all expressions involving a trace in the group representation should be multiplied by a factor 1/2. It is also convenient to evaluate the Euclidean integrals in the unitary gauge, defined by the change of variable

$$U(x_m,\hat{\mu})\equiv\theta(n,\hat{\mu})_{(m)}+\tau(n)_{(m)}-\tau(n+\hat{\mu})_{(m)}\,, \quad U(x_m,\hat{t})\equiv\tau(n)_{(m+1)}-\tau(n)_{(m)}\,,$$

where, now, $x_m\equiv(n,m)\in Z^{d+1}$, with $n\in\mathbf{\Lambda}$, and m denotes the time-slice. The vacuum expectation value of gauge-invariant operators A is, in the unitary gauge, given by

$$\omega_{\mathbf{\Lambda}}(A)=\lim_{q\to\infty}\frac{\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\prod_{(x,\mu)\in B(\Lambda_q)}dU(x,\mu)[A](\{U\})\exp(S_{\Lambda_q})}{\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\prod_{(x,\mu)\in B(\Lambda_q)}dU(x,\mu)\exp(S_{\Lambda_q})}\,, \tag{II.19}$$

with

$$S_{\Lambda_q}=\beta_g\sum_{P\in P(\Lambda_q)}(\cos U_P-1)+\beta_h\sum_{(x,\mu)\in B(\Lambda_q)}\{\cos[U(x,\mu)]-1\}\,. \tag{II.20}$$

Above, (x,μ) is an oriented link in the region

$$\Lambda_q\equiv\mathbf{\Lambda}x[-q,q]\subset\mathbb{Z}^{d+1}\,,$$

$B(\Lambda_q)$ denotes the set of positively oriented links in Λ_q, $P(\Lambda_q)$ the set of positively oriented plaquettes in Λ_q, and

$$U_P\equiv\sum_{(x,\mu)\in P}U(x,\mu)\gamma_{(x,\mu)}\,, \qquad P\in P(\Lambda_q)\,,$$

where

$$\gamma^P_{(x,\mu)}=\begin{cases}+1 & \text{if the orientations of } (x,\mu) \text{ and } P \text{ agree}\,,\\ -1 & \text{otherwise}\,.\end{cases}$$

We shall also frequently write $U_b=U(x,\mu)$, where b denotes a bond (x,μ), and

$$\prod_{(x,\mu)\in B(\Lambda_q)}dU(x,\mu)[dU]_{B(\Lambda_q)}\,.$$

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4 See also [16]

The generator $G(n)$ corresponding to (II.9) is written

$$G(n) = -i \sum_{\hat{\mu}=\hat{e}_i,\, i=1,\ldots,d} \left[\frac{\partial}{\partial\theta(n,\hat{\mu}} - \frac{\partial}{\partial\theta(n-\hat{\mu},\hat{\mu})}\right] + i\frac{\partial}{\partial\tau(n)}. \tag{II.21}$$

From (II.21) we see that the operator

$$Q_n \equiv -i\partial/\partial\tau(n) \tag{II.22a}$$

should be interpreted as the charge density operator of the matter field at site n. The self-adjoint operator

$$Q_{\mathbf{V}} \equiv \sum_{n\in\mathbf{V}} Q_n \tag{II.22b}$$

measures therefore the charge in a volume $\mathbf{V}$ of space. It is easy to verify that $Q_{\mathbf{V}}T_{\mathbf{\Lambda}}$ $(\mathbf{\Lambda}\supset\mathbf{V})$ is a bounded operator. In fact, its "second kernel" is given by

$$\begin{aligned}&[Q_{\mathbf{V}}](\{U,\phi\},\{U',\phi'\})\\ &= i\beta_h\Big\{\tfrac{1}{2}\sum_{(n,\hat{\mu})\in B(\mathbf{\Lambda})}\gamma_{(n,\hat{\mu})}\sin[\tau(n)-\tau(n+\hat{\mu})+\theta(n,\hat{\mu})]\\ &\quad+\sum_{n\in\mathbf{V}}\sin[\tau(n)-\tau'(n)]\Big\},\end{aligned}$$

where

$$\gamma_{(n,\hat{\mu})} = \begin{cases}+1 & \text{if} \quad n\in\mathbf{V},\\ -1 & \text{if} \quad (n+\hat{\mu})\in\mathbf{V}.\end{cases}$$

Lemma II.3. *$\Omega_{\mathbf{\Lambda}}$ lies in the domain of $Q_{\mathbf{V}}$ $(\mathbf{\Lambda}\supset\mathbf{V})$ and*

$$\omega_{\mathbf{\Lambda}}(Q_{\mathbf{V}}) = (\Omega_{\mathbf{\Lambda}}, Q_{\mathbf{V}}\Omega_{\mathbf{\Lambda}}) = 0. \tag{II.23}$$

Proof. Since $\Omega_{\mathbf{\Lambda}}$ is the eigenvector of $T_{\mathbf{\Lambda}}$ corresponding to eigenvalue $\|T_{\mathbf{\Lambda}}\|$, and $Q_{\mathbf{V}}T_{\mathbf{\Lambda}}$ is a bounded operator, $\Omega_{\mathbf{\Lambda}}$ belongs to the domain of $Q_{\mathbf{V}}$. The same usual proof then yields

$$\omega_{\mathbf{\Lambda}}(Q_{\mathbf{V}}) = \lim_{q\to\infty}[(T_{\mathbf{\Lambda}})^q Q_{\mathbf{V}}(T_{\mathbf{\Lambda}})^q E_{\mathbf{\Lambda}}^{(0)}]/Z_{\mathbf{\Lambda},q}.$$

Hence, passing to the unitary gauge

$$\omega_{\mathbf{\Lambda}}(Q_{\mathbf{V}}) = \lim_{q\to\infty}\alpha_q,$$

where

$$\begin{aligned}\alpha_q \equiv Z_{\mathbf{\Lambda},q}^{-1}\int_{-\pi}^{\pi}\ldots\int_{-\pi}^{\pi}[dU]_{B(\Lambda_q)}(i\beta_h)\\ \cdot\Big\{\sum_{(n,\hat{\mu})\in\delta B(\mathbf{V})}\gamma(n,\hat{\mu})/2\sin[U(n,\hat{\mu})]+\sum_{n\in\mathbf{V}}\sin[U(n,\hat{t})]\Big\}e^{S_M},\end{aligned}$$

where $\delta B(\mathbf{V})$ denotes the boundary of $\mathbf{V}$ (i.e., the set of all bonds with only one vertex in $\mathbf{V}$). But $\alpha_q = 0$ because S_{Λ_q} is even under the change of variable $U(x,\mu) \to -U(x,\mu)$, while the rest of the integrand is odd.

A version of (II.9) in the U(1) theory is

$$\Psi_{r,\mathbf{\Lambda}} \equiv F_r\Omega_{\mathbf{\Lambda}}, \tag{II.24}$$

where

$$F_r = \phi^*(\mathbf{0})\phi(\mathbf{x}_r) T_\Lambda^r U_{2r} T_\Lambda^{-r}. \tag{II.25}$$

Above, the star denotes complex conjugate, and U_{2r} is defined as in (II.10). Note that in U(1) case T_Λ^{-1} is well-defined. Define the operator $\phi_{\mathbf{V}}$ which measures the flux of the electric field on the boundary of $\mathbf{V}$:

$$\Phi_V \equiv \sum_{n \in \mathbf{V}} \Phi_n, \tag{II.26a}$$

$$\phi_n \equiv (-i) \sum_{\hat{\mu} = \hat{e}_i,\, i=1,\dots,d} \left[\frac{\partial}{\partial \theta(n, \hat{\mu})} - \frac{\partial}{\partial \theta(n-\hat{\mu}, \hat{\mu})} \right]. \tag{II.26b}$$

Lemma II.4. *$\Psi_{r,\Lambda}$ is gauge-invariant and lies in the domain of both $\phi_{\mathbf{V}}$ and $Q_{\mathbf{V}}$. Further,*

$$(\Psi_{r,\Lambda}, Q_{\mathbf{V}} \Psi_{r,\Lambda}) = (\Psi_{r,\Lambda}, \phi_{\mathbf{V}} \Psi_{r,\Lambda}). \tag{II.27}$$

Proof. It follows from definitions (II.22), (II.24), and (II.25) and the fact that $Q_n T_\Lambda$ $(n \in \Lambda)$ is a bounded operator that $\Psi_{r,\Lambda}$ belongs to the domain of Q_n. Similarly $\Phi_n T_\Lambda$ is also a bounded operator and $\Psi_{r,\Lambda}$ is in the domain of ϕ_n. Hence, by (II.21), $\Psi_{r,\Lambda}$ belongs to the domain of $G(n)$ $(n \in \Lambda)$. By Lemma II.1 and (II.10),

$$e^{\alpha G_n} \Psi_{r,\Lambda} = \Psi_{r,\Lambda}, \qquad n \in \Lambda, \qquad \alpha \in \mathbb{R}. \tag{II.28}$$

By the above remarks, (II.28) may be differentiated at $\alpha = 0$ yielding

$$\phi_n \Psi_{r,\Lambda} = Q_n \Psi_{r,\Lambda}, \qquad n \in \Lambda$$

from which (II.27) follows.

III. Absence of Charge and Exponential Clustering in the U(1) Theory

Now we consider the flux operator defined in (II.26) and the expected value of $\phi_{\mathbf{V}}$ in the states $\Psi_{r,\Lambda}/\|\Psi_{r,\Lambda}\|$ defined in (II.24), (II.25),

$$\begin{aligned} \omega_r(\phi_{\mathbf{V}}) &\equiv (F_r \Omega_\Lambda, \phi_{\mathbf{V}} F_r \Omega_\Lambda)/(F_r \Omega_\Lambda, F_r \Omega_\Lambda) \\ &= \sum_{(n,\hat{v}) \in \partial \mathbf{V}} (-i) s(n, \hat{v}) \\ &\quad \cdot \left(\Omega_\Lambda, (U_{2r})^\dagger (T_\Lambda)^r \frac{\partial}{\partial \theta(n, \hat{v})} (T_\Lambda)^r (U_{2r}) \Omega_\Lambda \right) \Big/ (\Omega_\Lambda, (U_{2r})^\dagger (\cdot\, T_\Lambda)^{2r} (U_{2r}) \Omega_\Lambda), \end{aligned} \tag{III.1}$$

where

$$s(n, \hat{v}) = \begin{cases} +1 & \text{if } n \in \mathbf{V}, \\ -1 & \text{if } n + \hat{v} \in \mathbf{V}. \end{cases}$$

In the unitary gauge we have:

$$\omega_r(\phi_V) = \lim_{q\to\infty} \sum_{(n,\hat{v})\in\partial\mathbf{V}} s(n,\hat{v}) \int_{-\pi}^{\pi} [dU]_{B(\Lambda_{q+r})}$$
$$\cdot \exp\{S_{\Lambda_{q+r}}\}\left\{\beta_g\left[-\sum_{P\supset(n,v)} \sin(U_p)\bar{\gamma}^P_{(n,v)}\right] - \frac{\beta_h}{2}\sin(U(n,\hat{v}))\right\}$$
$$\cdot \sin(U_{M_r}) \Big/ \int_{-\pi}^{\pi} [dU]_{B(\Lambda_{q+r})} \exp\{S_{\Lambda_{q+r}}\}\cos(U_{M_r}), \qquad \text{(III.2)}$$

where

$$\bar{\gamma}^P_{(n,\hat{v})} = \begin{cases} \frac{1}{2}\gamma^P_{(n,\hat{v})} & \text{if } P \text{ is a spatial plaquette,} \\ 1 & \text{if } P \text{ is temporal in the direction of increasing time,} \\ 0 & \text{if } P \text{ is temporal in the direction of decreasing time} \end{cases}$$

and

$$U_{M_r} = \sum_{(n,\hat{v})\in M_r} \gamma^{M_r}_{(n,\hat{v})} U_{(n,\hat{v})}, \qquad \text{(III.3)}$$

where M_r is the oriented loop shown in Fig. 1, and

$$\gamma^{M_r}_{(n,\hat{v})} = \begin{cases} +1 & \text{if the orientations of } (n,\hat{v}) \text{ and } M_r \text{ agree,} \\ -1 & \text{otherwise.} \end{cases} \qquad \text{(III.4)}$$

In expression (III.2) we arrive at $\cos U_{M_r}$ in the denominator and at the $\sin U_{M_r}$ in the numerator using the fact that the remaining functions under the integrations are even (respectively odd) under the change of variables $U_b \to -U_b$ for all $b \in B(\Lambda_{q+r})$.

In the numerator of expression (III.2) we find the sum

$$\sum_{(n,\hat{v})\in\partial\mathbf{V}} \sum_{P_S\supset(n,\hat{v})} \sin(U_{P_S}) s(n,\hat{v})\gamma^{P_S}(n,\hat{v}), \qquad \text{(III.5)}$$

which may be written as

$$\sum_{P_S\in P^*} \sin(U_{P_S}) \left\{ \sum_{\substack{(n,\hat{v})\in P_s \\ (n,\hat{v})\in\partial\mathbf{V}}} s(n,\hat{v})\gamma^{P_s}(n,\hat{v}) \right\}, \qquad \text{(III.6)}$$

where P^* is the set of all the spatial plaquettes which have at least one bond in $\partial\mathbf{V}$.

Lemma III.1. *If P_S is a spatial plaquette, then*

$$\sum_{\substack{(n,\hat{v})\in P_s \\ (n,\hat{v})\in\partial\mathbf{V}}} s(n,\hat{v})\gamma^{P_s}(n,\hat{v}) = 0. \qquad \text{(III.7)}$$

Proof. The proof is a simple verification of each possible case.

This lemma says that in the sums in expression (III.2) only temporal plaquettes in the direction of increasing time contribute.

Hence, expression (III.2) may be simplified to

$$\begin{aligned}\omega_r(\phi_{\mathrm{V}})_\Lambda &= \sum_L g_L \frac{\int [dU]_{B(\Lambda)} \exp\{S_\Lambda\} \sin U_{M_r} \sin U_L}{\int [dU]_{B(\Lambda)} \exp\{S_\Lambda\} \cos U_{M_r}} \\ &= \sum_L g_L \langle \sin U_{M_r} \sin U_L \rangle_\Lambda / \langle \cos U_{M_r} \rangle_\Lambda ,\end{aligned} \tag{III.8}$$

where U_L reads to $U_{\bar{P}(n,\hat{v})}$ or $U_{(n,\hat{v},\bar{P}(n,\hat{v})}$ being a temporal increasing time plaquette which contains the bond $(n,\hat{v}) \in \partial \mathrm{V}$ and g_L is the corresponding factor, which may be $\pm\beta_g$ or $\beta_h/2$.

It is obvious that we may write (III.8) as

$$\omega_r(\phi_V) = \sum_L (g_L/2) \frac{\langle \cos(U_{M_r} - U_L) \rangle_\Lambda}{\langle \cos U_{M_r} \rangle_\Lambda} - \sum_L (g_L/2) \frac{\langle \cos(U_{M_r} + U_L) \rangle_\Lambda}{\langle \cos U_{M_r} \rangle_\Lambda} . \tag{III.9}$$

From this point, our aim is to consider the limit $r \to \infty$ of the expression above and, therefore, attempt to provide a polymer expansion of expression (III.9). The action S_Λ may be written (by a trivial addition of a constant) as

$$S_\Lambda = \beta_g \sum_{P \in P(\Lambda)} (\cos U_P + 1) + \beta_h \sum_{b \in B(\Lambda)} \cos U_b . \tag{III.10}$$

We define for each $(n,\hat{v}) \in B(\Lambda)$ the measure

$$dU^*(n,\hat{v}) = dU(n,\hat{v}) e^{\beta_h \cos U(n,v)} \Big/ \int_{-\pi}^{\pi} dU \exp(\beta_h \cos U) , \tag{III.11}$$

so that $\int_{-\pi}^{\pi} dU^* = 1$. We define also for each $P \in P(\Lambda)$

$$\varrho_P(U) = \exp[\beta_g(\cos U_P + 1)] - 1 , \tag{III.12}$$

so that $\varrho_P(U) \geqq 0$.

With the use of these definitions we write

$$\langle \cos(U_{M_r} + U_L) \rangle_\Lambda = Z_\Lambda^{-1} \sum_{C \subset P(\Lambda)} \int [dU^*]_{B(\Lambda)} \left[\prod_{P \in C} \varrho_P(U) \right] \cos(U_{M_r} + U_L) . \tag{III.13}$$

Since the measure dU^* is normalized to one, we may write simply

$$\begin{aligned}\langle \cos(U_{M_r} + U_L) \rangle_\Lambda = Z_\Lambda^{-1} \sum_{C \subset P(\Lambda)} \int [dU^*]_{(\partial C \cup M_r \cup L)} \left[\prod_{P \in C} \varrho_P(U) \right] \\ \cdot \cos(U_{M_r} + U_L) ,\end{aligned} \tag{III.14}$$

where ∂C is the set of all the bonds of $C \subset P(\Lambda)$ and $[dU^*]_{(A)}$ means $\prod_{b \in A} dU_b^*$, A being a set of bonds.

Let H be a set of bonds. $|H|$ denotes the number of elements of H. For $\beta_h > 0$ we define

$$N(|H|) = \left[\int_{-\pi}^{\pi} dU^* \cos U \right]^{|H|} \tag{III.15}$$

which is a positive, monotonically increasing function of β_h, satisfying

$$N(|H|)<1\,;\qquad \lim_{\beta_h\to\infty} N(|H|)=1\,;\qquad N(|H|)|_{\beta_h=0}=0\,.$$

We now consider the ratio

Let $b\in M_r/\partial C$ be a bond. Then, from the definition of U_{M_r},

$$\begin{aligned}
&\frac{\langle\cos(U_{M_r}+U_L)\rangle_\Lambda}{N(|M_r|)}\\
&=Z_\Lambda^{-1}\sum_{C\subset P(\Lambda)}\int[dU^*]_{(\partial C\cup M_r\cup L)}\left[\prod_{P\in C}\varrho_P(U)\right]\frac{\cos(U_{M_r}+U_L)}{N(|M_r|)}\\
&=Z_\Lambda^{-1}\sum_{C\subset P(\Lambda)}\int[dU^*]_{(\partial C\cup L)}\left[\prod_{P\in C}\varrho_P(U)\right]\left\{\frac{\int[dU^*]_{(M_r/\partial C)}\cos(U_{M_r}+U_L)}{N(|M_r|)}\right\}.
\end{aligned}\tag{III.16}$$

$$\begin{aligned}
\cos(U_L+U_{M_r})&=\cos(U_L+U_{M_r/\{b\}}\pm U_b)\\
&=\cos(U_L+U_{M_r/\{b\}})\cos U_b\pm\sin(U_L+U_{M_r/\{b\}})\sin U_b\,.
\end{aligned}\tag{III.17}$$

(The sign depends on the orientation of b alone M_r.) Since

$$[dU^*]_{(M_r/\partial C)}=[dU^*]_{(M_r/\partial C)/\{b\}}\times dU_b^*\quad\text{and}\quad\int_{-\pi}^{\pi}dU_b^*\sin U_b=0\,,$$

it follows that

$$\frac{\int[dU^*]_{(M_r/\partial C)}\cos(U_{M_r}+U_L)}{N(|M_r|)}=\frac{\int[dU^*]_{((M_r/\partial C)/\{b\})}\cos(U_{M_r/\{b\}}+U_L)}{N(|M_r|-1)}.\tag{III.18}$$

Repeating the process for all b's $\in M_r/\partial C$, we conclude that

$$\frac{\int[dU^*]_{(M_r/\partial C)}\cos(U_{M_r}+U_L)}{N(|M_r|)}=\frac{\cos(U_{M_r\cap\partial C}+U_L)}{N(|M_r\cap\partial C|)}.\tag{III.19}$$

So, we have

$$\begin{aligned}
\frac{\langle\cos(U_{M_r}+U_L)\rangle_\Lambda}{N(|M_r|)}&=Z_\Lambda^{-1}\sum_{C\subset P(\Lambda)}\int[dU^*]\sum_{(\partial C\cup L)}\left[\prod_{P\in C}\varrho_P(U)\right]\\
&\quad\cdot\frac{\cos(U_{M_r\cap\partial C}+U_L)}{N(|M_r\cap\partial C|)}.
\end{aligned}\tag{III.20}$$

In analogous fashion,

$$\begin{aligned}
\frac{\langle\cos(U_{M_r})\rangle_\Lambda}{N(|M_r|)}&=Z_\Lambda^{-1}\sum_{C\subset P(\Lambda)}\int[dU^*]_{(\partial C\cup L)}\left[\prod_{P\in C}\varrho_P(U)\right]\\
&\quad\cdot\frac{\cos(U_{M_r\cap\partial C})}{N(|M_r\cap\partial C|)}.
\end{aligned}\tag{III.21}$$

Definition (III.1). $\mathrm{CONN}(L)=\{B\in P(\Lambda)$ all the connected parts of ∂B have a non-empty intersection with L (L understood as a set of bonds)$\}$.

Through Definition (III.1), (III.21) may be written as

$$\frac{\langle\cos(U_{M_r}+U_L)\rangle_\Lambda}{N(|M_r|)} = Z_\Lambda^{-1}\sum_{B\in\mathrm{CONN}(L)}\int[dU^*]_{(\partial B\cup L)}\sum_{\substack{C\in P(\Lambda)\\ \partial C\cap\partial B=\phi\\ \partial C\cap L=\phi}}\int[dU^*]_{(\partial C)}\left\{\prod_{P\in C}\varrho_P(U)\right\} \cdot\frac{\cos(U_{M_r\cap(\partial C\cup\partial B)}+U_L)}{N(|M_{r\cap(\partial C\cup\partial B)}|)}. \tag{III.22}$$

From the fact that C and B are disjoint, it follows that $M_r\cap(\partial C\cup\partial B)=(M_r\cap\partial C)\cup(M_r\cap\partial B)$, a disjoint union. So we find that

$$\cos(U_{M_r\cap(\partial C\cup\partial B)}+U_L)=\cos(U_{M_r\cap\partial C})\cos(U_{M_r\cap\partial B}+U_L) -\sin(U_{M_r\cap\partial C})\sin(U_{M_r\cap\partial B}+U_L). \tag{III.23}$$

But we have

$$\int[dU^*]_{(\partial C)}\left\{\prod_{P\in C\cup B}\varrho_P(U)\right\}\sin(U_{M_r\cap\partial C})\sin(U_{M_r\cap\partial B}+U_L) = \left\{\prod_{P\in B}\varrho_P(U)\right\}\sin(U_{M_r\cap\partial B}+U_L)\int[dU^*]_{(\partial C)}\left\{\prod_{P\in C}\varrho_P(U)\right\} \cdot\sin(U_{M_r\cap\partial C})=0. \tag{III.24}$$

In addition, $N(|M_r\cap(\partial C\cup\partial B)|)=N(|M_r\cap\partial C|)N(|M_r\cap\partial B|)$, and we arrive at

$$\langle\cos(U_{M_r}+U_L)\rangle_\Lambda/N(|M_r|) = Z_\Lambda^{-1}\sum_{B\in\mathrm{CONN}(L)}\int[dU^*]_{(\partial B\cup L)}\left\{\prod_{P\in B}\varrho_P(U)\right\}\frac{\cos(U_{M_r\cap\partial B}+U_L)}{N(|M_r\cap\partial B|)} \left\{\sum_{\substack{C\in P(\Lambda)\\ \partial C\cap\partial B=\phi\\ \partial C\cap L=\phi}}\int[dU^*]_{(\partial C)}\left\{\prod_{P\in C}\varrho_P(U)\right\}\cos(U_{M_r\cap\partial C})/N(|M_r\cap\partial C|)\right\}. \tag{III.25}$$

Each set C may be decomposed in the set of its connected parts (two plaquettes are said to be connected if they have a common bond), which we denote by $\{C_\gamma\}$. We therefore have

$$\int[dU^*]_{(\partial C)}\left\{\prod_{P\in C}\varrho_P(U)\right\}\cos(U_{M_r\cap\partial C}) = \int\prod_\gamma\{[dU^*]_{(\partial C_\gamma)}\}\left\{\prod_{P\in C}\varrho_P(U)\right\}\cos\left(\sum_\gamma U_{M_r\cap\partial C_\gamma}\right). \tag{III.26}$$

Again we decompose

$$\cos\sum_\gamma U_{M_r\cap\partial C_\gamma}=\prod_\gamma\cos(U_{M_r\cap\partial C_\gamma}) +(\text{terms with at least one factor }\sin(U_{M_r\cap\partial C_\gamma})),$$

and we conclude that

$$\int[dU^*]_C\left\{\prod_{P\in C}\varrho_P(U)\right\}\cos(U_{M_r\cap\partial C})/N(|M_r\cap\partial C|)=\prod_{\{C_\gamma\}}\mu_r(C_\gamma), \tag{III.27}$$

where

$$\mu_r(C_\gamma)=\int [dU^*]_{(\partial C_\gamma)}\left\{\prod_{P\in C}\varrho_P(U)\right\}\frac{\cos(U_{M_r\cap\partial C_\gamma})}{N(|M_r\cap\partial C_\gamma|)}. \tag{III.28}$$

Now we are able to express (III.25) in terms of a polymer expansion. Two plaquettes are said to be connected if they have a common bond. The polymers of our model are connected sets of plaquettes. Two polymers are said to be "incompatible" if they have, at least, one bond in common and "compatible" otherwise. To each polymer γ we associate an activity given by $\mu_r(C_\gamma)$ defined in (III.28), C_γ being the set of plaquettes which forms γ. (In Appendix B we present some general results on polymer expansions which will be used here.) Hence, we conclude from (III.28), (III.27), and (III.25) that

$$\begin{aligned}\frac{\langle\cos(U_{M_r}+U_L)\rangle_\Lambda}{N(|M_r|)}&=Z_\Lambda^{-1}\sum_{B\in\mathrm{CONN}(L)}\int [dU^*]_{(\partial B\cup L)}\left\{\prod_{P\in B}\varrho_P(U)\right\}\\&\quad\cdot\frac{\cos(U_{M_r\cap\partial B}+U_L)}{N(|M_r\cap\partial B|)}\sum_{\substack{\Gamma\in\mathscr{G}_\Lambda\\ \Gamma\sim B\\ \Gamma\cap L=\phi}}(\mu_r)^\Gamma,\end{aligned} \tag{III.29}$$

where $\Gamma\cap L=\phi$ means $\gamma\cap L=\phi$ for all $\gamma\in\Gamma$. For all notation used above see Appendix B.

According to (III.29) we also have

$$\langle\cos(U_{M_r})\rangle_\Lambda/N(|M_r|)=Z_\Lambda^{-1}\sum_{\Gamma\in\mathscr{G}_\Lambda}(\mu_r)^\Gamma. \tag{III.30}$$

Using (III.30) and (III.29) we finally arrive at

$$\begin{aligned}\frac{\langle\cos(U_{M_r}+U_L)\rangle_\Lambda}{\langle\cos U_{M_r}\rangle_\Lambda}&=\sum_{B\in\mathrm{CONN}(L)}\int [dU^*]_{(\partial B\cup L)}\left\{\prod_{P\in B}\varrho_P(U)\right\}\\&\quad\cdot\frac{\cos(U_{M_r\cap\partial B}+U_L)}{N(|M_r\cap\partial B|)}\left[\sum_{\Gamma\in\mathscr{G}_\Lambda}(\mu_r)^\Gamma b_{B,L}^\Gamma\Big/\sum_{\Gamma\in\mathscr{G}_\Lambda}(\mu_r)^\Gamma\right],\end{aligned} \tag{III.31}$$

where

$$b_{B,L}(\gamma)=\begin{cases}0 & \text{if}\quad \gamma\nsim B\quad\text{or if}\quad \gamma\cap L\neq 0,\\ 1 & \text{otherwise}.\end{cases}$$

According to Definition (B.2) we may write

$$\left[\sum_{\Gamma\in\mathscr{G}_\Lambda}(\mu_r)^\Gamma b_{B,L}^\Gamma\Big/\sum_{\Gamma\in\mathscr{G}_\Lambda}(\mu_r)^\Gamma\right]=\varrho(B\cup L).$$

Using Lemma B.5 we write (III.31) as

$$\begin{aligned}\frac{\langle\cos(U_{M_r}+U_L)\rangle_\Lambda}{\langle\cos(UM_r)\rangle_\Lambda}&=\sum_{B\in\mathrm{CONN}(L)}\int [dU^*]_{(\partial B\cup L)}\left\{\prod_{P\in B}\varrho_P(U)\right\}\\&\quad\cdot\frac{\cos(U_{M_r\cap\partial B}+U_L)}{N(|M_r\cap\partial B|)}\exp\left[\sum_\Gamma C_\Gamma(\mu_r)^\Gamma(b_{B,L}^\Gamma-1)\right],\end{aligned} \tag{III.32}$$

provided the series $\sum_{\Gamma} C_{\Gamma}(\mu_r)^{\Gamma}(b^{\Gamma}_{B,L}-1)$ is absolutely convergent. This point may be seen as follows.

We have

$$\left|\sum_{\Gamma} C_{\Gamma}(\mu_r)^{\Gamma}(b^{\Gamma}_{B,L}-1)\right| \leqq \sum_{\Gamma} |C_{\Gamma}|\,|\mu_r^{\Gamma}|(1-b^{\Gamma}_{B,L}). \tag{III.33}$$

The sets Γ which contribute in the right side of (III.33) are those for which $\exists \gamma \in \Gamma$, such that either $\gamma \nsim B$ or $\gamma \cap L = \phi$. The set of γ's which satisfy these conditions is contained in the set of γ's which satisfy either $\gamma \nsim B$ or $\gamma \nsim P_L$, where P_L is the smaller set of plaquettes such that $\partial P_L \supset L$. So, using $\Gamma_0 = B \cup P_L$, we conclude that

$$\left|\sum_{\Gamma} C_{\Gamma}(\mu_r)^{\Gamma}(b^{\Gamma}_{B,L}-1)\right| \leqq \sum_{\Gamma \nsim \Gamma_0} |C_{\Gamma}|\,|\mu_r^{\Gamma}| \leqq C'' \|\Gamma_0\|, \tag{III.34}$$

where C'' is a constant independent of Γ_0.

According to Appendix B (Lemma B.6) the last inequality in (III.34) holds if $\|\mu_r\| \leqq \|\mu_c\|$.

We have

$$\|\mu_r\| = \sup_{\gamma}\left(\int [dU^*]_{(\partial C_\gamma)} \left\{\prod_{P \in C_\gamma} \varrho_P(U)\right\} \cos(U_{M_r \cap \partial C_\gamma})/N(|M_r \cap \partial C_\gamma|)\right)^{1/|C_\gamma|}$$

$$= (e^{2\beta_g}-1)\sup_{\gamma}\left\{\left[\int_{-\pi}^{\pi} dU^* \cos U\right]^{-(|M_r \cap \partial C_\gamma|/|C_\gamma|)}\right\}. \tag{III.35}$$

Using $|M_r \cap \partial C_\gamma|/|\partial C_\gamma| \leqq |\partial C_\gamma|/|C_\gamma| \leqq 4$, we arrive at

$$\|\mu_r\| \leqq \left(\int_{-\pi}^{\pi} dU^* \cos U\right)^{-4} (e^{2\beta_g}-1), \tag{III.36}$$

and the condition $\|\mu_r\| \leqq \|\mu_c\|$ is obtained if

$$\left(\int_{-\pi}^{\pi} dU^* \cos U\right) \geqq [(e^{2\beta_g}-1)/\|\mu_c\|]^{1/4}. \tag{III.37}$$

In Fig. 2 we show a region of the phase diagram of the model where we have, according to our naive estimate, the condition $\|\mu_r\| \leqq \|\mu_c\|$ satisfied uniformly in r.

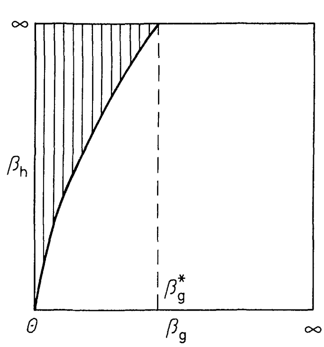

Fig. 2

In Fig. 2, $\beta_g^* = 1/2\ln(1+\|\mu_c\|)$.

Now our interest is to consider the limit $r \to \infty$ of expression (III.32). The first step is to search for an upper bound, independent of r for the exponential in (III.32). It follows from Lemma B.3 (Appendix B):

$$\begin{aligned}|\varrho(B \cup L)| &\leqq \exp[F_1(-\ln\|\mu\|)(\|B \cup L\|)] \\ &\leqq \exp[F_1(-\ln\omega)(|B|+|P_L|)],\end{aligned} \tag{III.38}$$

where

$$\omega \equiv (e^{2\beta_g}-1)\Big/\left[\int_{-\pi}^{\pi} dU^* \cos U\right]^4 .$$

We therefore obtain from (III.32) the estimate

$$\begin{aligned}\left|\frac{\langle\cos(U_{M_r}+U_L)\rangle}{\langle\cos(U_{M_r})\rangle}\right| &\leqq \sum_{B\in \mathrm{CONN}(L)} (e^{2\beta_g}-1)\left[\int_{-\pi}^{\pi} dU^* \cos U\right]^{-|M_r\cap\partial B|} \\ &\quad \cdot \exp\{F_1[-\ln\omega](|B|+|P_L|)\}.\end{aligned} \tag{III.39}$$

Again, as $|M_r \cap \partial B| \leqq 4|B|$, it follows that

$$\begin{aligned}\left|\frac{\langle\cos(U_{M_r}+U_L)\rangle}{\langle\cos U_{M_r}\rangle}\right| &\leqq \exp[F_1(-\ln\omega)|P_L|] \sum_{B\in \mathrm{CONN}(L)} \omega^{|B|} \exp[F_1(-\ln\omega)|B|] \\ &\leqq \exp[F_1(-\ln\omega)|P_L|]\,|L| \sum_{n=0}^{\infty} \omega^n e^{F_1(-\ln\omega)n+kn},\end{aligned} \tag{III.40}$$

where K is a constant which does not depend on r, but depends on the dimension.

The last series (III.39) converges if

$$\ln\omega + F_1(-\ln\omega) + K < 0, \tag{III.41}$$

or

$$a > F_1(a) + K, \tag{III.42}$$

where $a = -\ln\omega$. From the fact that $F_1(a)$ is decreasing there exists a^*, solution of $a^* = F_1(a^*) + K$, such that inequality (III.42) is always satisfied for $a > a^*$. The condition $a > a^*$ means

$$(e^{2\beta_g}-1)\left(\int_{-\pi}^{\pi} dU^* \cos U\right)^{-4} < e^{-a^*} \equiv \omega^*(<1), \tag{III.43}$$

or

$$\left(\int_{-\pi}^{\pi} dU^* \cos U\right) < [(e^{2\beta_g}-1)/\omega^*]^{1/4}. \tag{III.44}$$

Collecting (III.43) and (III.37) we arrive at the following theorem.

Theorem III.1. *There is a sub-region of the phase diagram of the model, namely, the region given by*

$$\left(\int_{-\pi}^{\pi} dU^* \cos U\right)^4 > [\max(e^{a^*}, \|\mu_c\|^{-1})](e^{2\beta_g}-1), \tag{III.45}$$

where the following estimate holds uniformly in r:

$$\left|\sum_{B\in \mathrm{CONN}(L)} \int [dU^*]_{(\partial B\cup L)} \left\{\prod_{P\in B} \varrho_P(U)\right\} \frac{\cos(U_{M_r\cap\partial B}+U_L)}{N(|M_r\cap\partial B|)} \right.$$
$$\left. \cdot \exp\left\{\sum_{\Gamma} C_\Gamma(\mu_r)^\Gamma (b^\Gamma_{B,L}-1)\right\}\right|$$
$$\leqq 4\exp[F_1(-\ln\omega)] \sum_{n=0}^{\infty} \omega^n \exp[(F_1(-\ln\omega)+K)n]. \tag{III.46}$$

Remarks. 1) Region (III.45) has the same form of the one shown in Fig. 2, replacing β_g^* for $y\equiv 1/2\ln[\min(e^{-a^*}, \|\mu_c\|)+1]$.

2) Above we used $|P_L|=1$ and $|L|=\leqq 4$.

Corollary III.1. *For the region given in Theorem III.1 the limit* $\lim_{r\to\infty} \langle\cos(U_{M_r}+U_L)\rangle/\langle\cos(U_{M_r})\rangle$ *exists and is given by*

$$\lim_{r\to\infty} \frac{\langle\cos(U_{M_r}+U_L)\rangle}{\langle\cos(U_{M_r})\rangle}$$
$$= \sum_{B\in \mathrm{CONN}(L)} \int [dU^*]_{(\partial B\cup L)} \left\{\prod_{P\in B} \varrho_P(U)\right\} \frac{\cos(U_{M_\infty\cap\partial B}+U_L)}{N(|M_\infty\cap\partial B|)}$$
$$\cdot \left\{\sum_{\Gamma\in\mathscr{G}} (\mu_\infty)^\Gamma b^\Gamma_{B,L} \Big/ \sum_{\Gamma\in\mathscr{G}} (\mu_\infty)^\Gamma\right\}, \tag{III.47}$$

where

$$\mu_\infty(\gamma) = \int [dU^*]_{(\partial C_\gamma)} \left\{\prod_{P\in C_\gamma} \varrho_P(U)\right\} \frac{\cos(U_{M_\infty\cap\partial C_\gamma})}{N(M_\infty\cap\partial C_\gamma)} \tag{III.48}$$

and M_∞ is the time axis with origin at **0**.

Proof. We begin defining

$$i_L(B\cap M_r) \equiv \int [dU^*]_{(\partial B\cup L)} \left\{\prod_{P\in B} \varrho_P(U)\right\} \frac{\cos(U_{M_r\cap\partial B}+U_L)}{N(|M_r\cap\partial B|)},$$
$$j_L(B, M_r) \equiv \exp\left[\sum_{\Gamma} C_\Gamma(\mu_r)^\Gamma (b^\Gamma_{B,L})\right], \tag{III.49}$$

so that

$$\frac{\langle\cos(U_{M_r}+U_L)\rangle}{\langle\cos U_{M_r}\rangle} = \sum_{B\in \mathrm{CONN}(L)} i_L(B\cap M_r) j_L(B, M_r). \tag{III.50}$$

Now, we rewrite (III.50) in the form:

$$\frac{\langle\cos(U_{M_r}+U_L)\rangle}{\langle\cos U_{M_r}\rangle} = \sum_{\substack{B\in \mathrm{CONN}(L)\\ |B|<l}} i_L(B\cap M_r) j_L(B, M_r)$$
$$+ \sum_{\substack{B\in \mathrm{CONN}(L)\\ |B|\geqq l}} i_L(B\cap M_r) j_L(B, M_r). \tag{III.51}$$

According to Theorem III.1,

$$\left|\sum_{\substack{B\in\mathrm{CONN}(L)\\|B|\geqq l}} i_L(B\cap M_r)j_L(B,M_r)\right| \leqq 4\exp[F_1(-\ln\omega)]\sum_{n=l}^{\infty}\omega^n e^{(F_1(-\ln\omega)+K)n}$$
$$=\exp[(\ln\omega+F_1(-\ln\omega)+K)l]4\exp[F_1(-\ln\omega)]\sum_{n=0}^{\infty}\omega^n e^{(F_1(-\ln\omega)+K)n}$$
$$=e^{-m(\omega)l}f(\omega). \tag{III.52}$$

The bound given by (III.52), again, does not depend on r.

Since the sum $\sum_{\substack{B\in\mathrm{CONN}(L)\\|B|<l}}$ is finite we have

$$\left|\lim_{r\to\infty}\frac{\langle\cos U_{M_r}+U_L\rangle}{\langle\cos U_{M_r}\rangle}-\sum_{\substack{B\in\mathrm{CONN}(L)\\|B|<l}}\lim_{r\to\infty} i(B\cap M_r)j(B,M_r)\right|\leqq e^{-m(\omega)l}f(\omega). \tag{III.53}$$

The limit $\lim_{r\to\infty} i(B\cap M_r)$ is clearly given by $i(B\cap M_\infty)$.

For limit $\lim_{r\to\infty} j(B,M_r)$, we need a more careful analysis. The square M_r may be written as the union $M_r=L_r\cup\bar{M}_r$, where $L_r=M_r\cap M_\infty$. The series $\sum_\Gamma C_\Gamma(\mu_r)^\Gamma(b^\Gamma_{B,L}-1)$ was proven to be absolutely convergent in the region given in Theorem III.1, and we may write

$$\sum_\Gamma C_\Gamma(\mu_r)^\Gamma(b^\Gamma_{B,L}-1)=\sum_{\substack{\Gamma\\ \Gamma\cap\bar{M}_r\neq\emptyset}} C_\Gamma(\mu_r)^\Gamma(b^\Gamma_{B,L}-1)$$
$$+\sum_{\substack{\Gamma\\ \Gamma\cap\bar{M}_r=\emptyset}} C_\Gamma(\mu_r)^\Gamma(b^\Gamma_{B,L}-1). \tag{III.54}$$

But

$$\left|\sum_{\substack{\Gamma\\ \Gamma\cap\bar{M}_r\neq\emptyset}} C_\Gamma(\mu_r)^\Gamma(b^\Gamma_{B,L}-1)\right|\leqq\sum_{\substack{\Gamma\\ \Gamma\not\subset P_{\bar{M}_r}\cup B\cup P_L}}|C_\Gamma|\,|(\mu_r)^\Gamma|, \tag{III.55}$$

where $P_{\bar{M}_r}$ is the smallest set of plaquettes such that $P_{\bar{M}_r}\supset\bar{M}_r$. From Theorem B.1, given in Appendix B, we conclude that the sets Γ which contribute in the right side of (III.55) satisfy $\|\Gamma\|>d_r$, d_r being the smallest distance from $B\cup L$ to $\bar{M}_r$. Hence, using Corollary B.2 (see Appendix B) we arrive at

$$\left|\sum_{\substack{\Gamma\\ \Gamma\cap\bar{M}_r\neq\emptyset}} C_\Gamma(\mu_r)^\Gamma(b^\Gamma_{B,L}-1)\right|\leqq(|P_{\bar{M}_r}|+|B|+|L|)\left(\frac{\|\mu_r\|}{\|\mu_c\|}\right)^{d_r}C''. \tag{III.56}$$

Since in the region given in Theorem III.1 we have $\|\mu_r\|/\|\mu_c\|<1$, uniformly in r, we conclude that

$$\lim_{r\to\infty}\left|\sum_{\substack{\Gamma\\ \Gamma\cap\bar{M}_r\neq\emptyset}} C_\Gamma(\mu_r)^\Gamma(b^\Gamma_{B,L}-1)\right|=0 \tag{III.57}$$

and

$$\lim_{r\to\infty} j(B,M_r)=\exp\left\{\sum_\Gamma C_\Gamma(\mu_\infty)^\Gamma(b^\Gamma_{B,L}-1)\right\}. \tag{III.58}$$

Therefore, Corollary III.1 arises from (III.53) upon taking the limit $l \to \infty$. □

Returning from (III.47) to (III.9) we conclude for the region given in Theorem III.1,

$$\lim_{r\to\infty} \omega_r(\phi_{\mathbf{V}}) = \sum_{L} \sum_{B\in\mathrm{CONN}(L)} g_L \int [dU^*]_{(\partial B\cup L)} \left\{\prod_{P\in B} \varrho_P(U)\right\} \frac{\sin(U_{M_\infty\cap\partial B})\sin U_L}{N(|M_\infty \cap \partial B|)} \cdot \left\{\sum_{\Gamma\in\mathscr{G}} b^{\Gamma}_{B,L}(\mu_\infty)^{\Gamma} \Big/ \sum_{\Gamma\in\mathscr{G}} (\mu_\infty)^{\Gamma}\right\}. \tag{III.59}$$

Theorem III.2. *Under the conditions in which* (III.59) *holds,*

$$\lim_{\mathbf{V}\uparrow\mathbb{Z}^d} \left[\lim_{r\to\infty} \omega_r(\phi_{\mathbf{V}})\right] = 0 .$$

Proof. Due to the factors $\sin(U_{M_\infty\cap\partial B})\sin U_L$ which appear in the integrals in (III.59) the only sets B which contribute are those for which we have simultaneously $\partial B \cap L \neq \emptyset$ (which already occurs) and $\partial B \cap M_\infty \neq \emptyset$. Since all the connected parts of ∂B, which we denote by ∂B_a, satisfy $\partial B_a \cap L \neq \emptyset$, we conclude that the sets B which contribute in (III.59) satisfy $|\partial B| \geqq \mathrm{DIST}(L, M_\infty)$; $\mathrm{DIST}(L, M_\infty)$ being the smallest distance from L to M_∞. Hence, proceeding as before, we have

$$\begin{aligned} &\left|\lim_{r\to\infty} \omega_r(\phi_{\mathbf{V}})\right| \\ &\leqq \exp[F_1(-\ln\omega)] \sum_L g_L |L| \sum_{n=\mathrm{DIST}(L,M_\infty)}^{\infty} \exp[(\ln\omega + F_1(-\ln\omega) + K)n] \\ &\leqq 4\exp[F_1(-\ln\omega)] \left\{\sum_L g_L \exp[(\ln\omega + F_1(\ln\omega) + K)\,\mathrm{DIST}(L, M_\infty)] f(\omega)\right\} \end{aligned} \tag{III.60}$$

with $f(\omega)$ defined as in (III.52).

Since the number of terms being summed in $\sum_L$ increases only with the area of $\mathbf{V}$, we conclude that the last expression in (III.60) converges to zero when $\mathbf{V} \uparrow \mathbb{Z}^d$ and $\mathrm{DIST}(L, M_\infty) \to \infty$, with decay governed by an exponential clustering with mass gap given by $m > -(\ln\omega + F_1(-\ln\omega) + K)$. □

IV. Conclusion and Open Problems

There is a central point in the proof presented in Sect. III. In the limit $r \to \infty$, the expression $\omega_r(\phi_v)$, given by (III.59), is such that the support of the integrand/summand occurring there has the following geometrical structure:

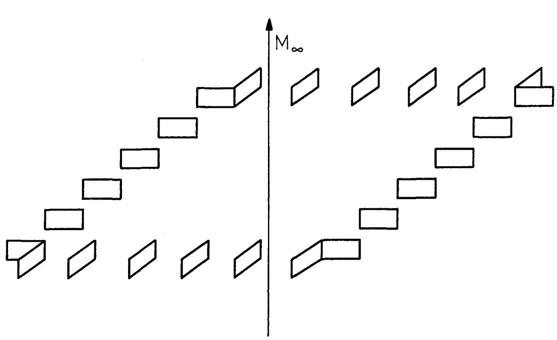

Fig. 3

The number of sets L which are summed grows, as $\mathbf{V} \nearrow \mathbb{Z}^d$, as the area of $\mathbf{V}$, and hence the existence of a mass gap in the model provides sufficient fall-off to compensate for this growth, implying at the same time the absence of charges. A very similar effect happens in connection with the Goldstone phenomenon in nonrelativistic theories (where locality, as here, plays no role). Let $j_\mu(t, \vec{x})$ be a conserved current, and $Q_R(t) = \int d\vec{x} f_R(\vec{x}) j_0(t, \vec{x})$, the associated charge, where $f_R \in \mathcal{D}$ is a regulator function such that ($\Delta > 0$, independent of R)

$$f_R(\vec{x}) = \begin{cases} 1, & |\vec{x}| \leqq R, \\ 0 & \text{if } \; |\vec{x}| \geqq R + \Delta . \end{cases}$$

Define

$$q_R(t) \equiv (\Omega, [Q_R(t), A]\Omega), \tag{IV.1}$$

where A is an observable and Ω the ground state. Then, formally (see also [15] for a more careful discussion)

$$\begin{aligned} &\frac{d}{dt} \int d\vec{x} f_R(\vec{x}) (\Omega, [j_0(t, \vec{x}), A]\Omega) \\ &= \int\limits_{R \leqq |\vec{x}| \leqq R+\Delta} d\vec{x} (\Omega, [\vec{j}(t, \vec{x}), A]\Omega) \cdot \nabla f_R(\vec{x}) . \end{aligned} \tag{IV.2}$$

It may be expected from (IV.2) that, if the truncated vacuum expectation value $(\Omega, \vec{j}(t, \vec{x}), A\Omega) - (\Omega, \vec{j}(t, \vec{x})\Omega)(\Omega, A\Omega)$, and, hence, the expectation value of the commutator $(\Omega, [\vec{j}(t, \vec{x}), A]\Omega)$ falls off faster than $|\vec{x}|^{-2}$ (Coulomb case), compensating in this way for the area growth which results from integration over the region $R \leqq |\vec{x}| \leqq R + \Delta$, the limit be time-independent (if it exists). In addition, if, for some observable, $q \neq 0$ (spontaneous symmetry breakdown), the forces must be long-range, i.e., the commutator cannot decay faster than Coulomb (Goldstone theorem). Rigorous proofs of the Goldstone theorem for quantum spin systems which follow this intuition have been given ([19], see also [20]). Our proof may also be viewed as a version of Swieca's theorem mentioned in the introduction to continuous gauge theories on a lattice, in the following restricted sense: due to the existence of a mass gap, the states associated to a *special* sequence of dipole states of finite energy are chargeless. Although other, possibly charged, states cannot be ruled out, we believe that the "universal" finite-energy property of these states conveys them a fundamental role, as discussed in the introduction, Sect. II and reference [16]. An open technical problem of some interest is to eliminate the "time-smearing" in Definition II.1 and Proposition II.1.

What is the main difference in behaviour of the dipole states in the screening-confinement and charged regions? In the charged sector of the $Z(2)$ model [1] the same setting depicted in Fig. 2 occurs, but there the "memory" that $\{M_r\}$ (see Fig. 1) is a sequence of squares is not lost. In the screening/confinement regions of both the U(1) and $Z(2)$ models, only M_∞ "remains." These differences are intimately connected with the order of the limits involved: firstly $\mathbf{\Lambda} \nearrow \mathbb{Z}^d$, then $r \to \infty$, and finally $\mathbf{V} \nearrow \mathbb{Z}^d$. A discussion of the difference in behaviour of the dipole states in the screening and confinement regimes of the $Z(2)$ model, which also strongly depends on the above sequential order of the limits, may be found in [1].

Finally, it remains an interesting problem to formulate similar criteria to characterize charged states in nonabelian gauge theories. In any case, we expect from Proposition II.1 and further refinements that the sequence of dipole states introduced in Sect. II might be suitable for this formulation.

Appendix A

In this appendix we study dipole states in the massless Schwinger model [5, 9, 10], with the string placed at the line $t=0$, illustrating the points at the end of Sect. II. It may be viewed as a mathematically precise version, somewhat different in form, of the arguments in [4, 5]. In a special (noncovariant) gauge ([5]; eventual mathematical gaps may be filled in as in [9]), the theory is isomorphic to a massive Boson field theory in Fock space $\mathscr{F}$, with the formal Hamiltonian

$$H=\frac{1}{2}\int dx :\left[\pi^2(x,0)+\left(\frac{d\varphi(x,0)}{dx}\right)^2+\frac{e^2}{\pi}\varphi(x,0)\right]:, \tag{A.1}$$

where the dots denote Wick ordering, e is the electric charge in the original model, and φ is a free Boson field (of mass $e/\sqrt{\pi}$), with π its conjugate momentum. The formal charge-density operator at zero time is [5]

$$j^0(x)=\frac{1}{\sqrt{\pi}}\frac{d\varphi(x,0)}{dx}. \tag{A.2}$$

If $f\in\mathscr{S}_{\mathbb{R}}(\mathbb{R})$, the Schwartz space of real-valued infinitely differentiable functions of fast decrease, then $\varphi(f)$ and $\pi(f)$ are essentially self-adjoint on the dense subset $\mathscr{D}$ of $\mathscr{F}$ consisting of finite particle vectors, and their self-adjoint closures (on $\mathscr{D}$) will be denoted by the same symbols. Let $g_{R,\varepsilon}$ and f_S denote infinitely differentiable functions of compact support, such that

$$g_{R,\varepsilon}(x)=\begin{cases}1 & \text{if } 0\leqq x\leqq R,\\ 0 & \text{if } x\leqq -\varepsilon \text{ or } x\geqq R+\varepsilon,\end{cases} \tag{A.3}$$

$$f_S(x)=\begin{cases}1 & \text{if } -S\leqq x\leqq S,\\ 0 & \text{if } x\geqq S+\delta \text{ or } x\leqq -S-\delta.\end{cases} \tag{A.4}$$

If Ω is the Fock vacuum, we define a dipole state (corresponding to the string at $t=0$ [4, 5]) by

$$\Psi_{R,\varepsilon}=T_{R,\varepsilon}\Omega, \tag{A.5}$$

where

$$T_{R,\varepsilon}=\exp[i\sqrt{\pi}\,\pi(g_{R,\varepsilon})]. \tag{A.6}$$

(In contrast to [5] we keep ε fixed as $R\to\infty$.) Define a (self-adjoint) charge operator by

$$Q_S=j^0(f_S)=-\frac{1}{\sqrt{\pi}}\varphi(f_S'), \tag{A.7}$$

where the prime denotes first derivative.

Consider, now, the family of states [11]

$$\omega_{R,\varepsilon}(\,\cdot\,)=(\Psi_{R,\varepsilon},\,\cdot\,\Psi_{R,\varepsilon})$$

on the Weyl (CCR) algebra $\mathfrak{A}$ [11], generated by the operators $U(f)\equiv\exp[i\varphi(f)]$, and $V(g)\equiv\exp[i\pi(g)]$ $(f, g\in\mathscr{S}_{\mathbb{R}}(\mathbb{R}))$, which satisfy the CCR in Weyl form:

$$\begin{aligned} &U(f_1+f_2)=U(f_1)U(f_2),\\ &V(g_1+g_2)=V(g_1)V(g_2),\\ &U(f)V(g)=\exp[-i(f,g)]V(g)U(f), \end{aligned} \tag{A.8}$$

where $(f,g)\equiv\int\limits_{-\infty}^{\infty}dx\bar{f}(x)g(x)$.

By usual compactness arguments there exists at least one limit (possibly along a net [11])

$$\omega_\varepsilon=\text{“}\lim_{R\to\infty}\text{”}\,\omega_{R,\varepsilon}\,. \tag{A.9}$$

We assume that $\exp(i\alpha Q_S)\in\mathfrak{A}$ for $\alpha\in\mathbb{R}$ [which is verified in the case at hand by (A.7)], and say that a state ω on $\mathfrak{A}$ is *charged* [with respect to the vacuum, which we assume to have zero charge, a fact which holds in the present case, by (A.7) and the following definition] if

$$\frac{1}{i}\frac{\partial}{\partial\alpha}\omega(e^{i\alpha Q_S})|_{\alpha=0}\equiv\omega(Q_S) \tag{A.10}$$

exists for each $S<\infty$ and

$$\exists\lim_{S\to\infty}\omega(Q_S)\neq0\,. \tag{A.11}$$

Proposition A.1. *Any state* $\omega_\varepsilon(\,\cdot\,)$ *of the form* (A.8) *has unit charge. Further,*

$$(\Psi_{R,\varepsilon},H\Psi_{R,\varepsilon})=\tfrac{\pi}{2}\int\limits_{-\infty}^{\infty}dx\left[g'_{R,\varepsilon}(x)^2+\frac{e^2}{\pi}g_{R,\varepsilon}(x)^2\right]. \tag{A.12}$$

Proof. By (A.8), if $0<\varepsilon<S$ and $S+\delta<R$,

$$\omega_{R,\varepsilon}(e^{i\alpha Q_S})=e^{i\alpha(f'_S,g_{R,\varepsilon})}(\Omega,e^{-i\alpha/\sqrt{\pi}\varphi(f'_S)}\Omega)=e^{i\alpha(f'_S,g_{R,\varepsilon})}\cdot e^{-\alpha^2/2\pi\|f'_S\|_2^2},$$

where $\|f\|_2^2=(f,f)$. Hence, for any state ω_ε given by (A.9)

$$\omega_\varepsilon(e^{i\alpha Q_S})=e^{i\alpha(f'_S,g_\varepsilon)}e^{-\alpha^2/2\pi\|f'_S\|_2^2}, \tag{A.13}$$

where g_ε is an infinitely differentiable function such that

$$g_\varepsilon(x)=\begin{cases}1 & \text{if}\quad x\geqq0,\\ 0 & \text{if}\quad x\leqq-\varepsilon.\end{cases} \tag{A.14}$$

By (A.13) and (A.14)

$$\frac{1}{i}\frac{\partial}{\partial\alpha}\omega_\varepsilon(e^{i\alpha Q_S})\bigg|_{\alpha=0}=(f'_S,g_\varepsilon)=1\,,$$

hence, by definitions (A.10) and (A.11), ω_ε has unit charge. Equation (A.13) is a standard Fock space computation. □

This proposition shows that ω_ε is charged and has infinite energy, because the first term in (A.12) is bounded as $R\to\infty$ (for $\varepsilon>0$ fixed) and the second one diverges linearly with R. Both properties lead us to expect that the corresponding representation be disjoint from the vacuum representation (a fact which was not completely established for the finite-energy charged sector in the $Z(2)$ theory for technical reasons [1]). In fact, the representation for R, ε fixed is defined by the map

$$U(f)\to U_{R,\varepsilon}(f)\equiv T_{R,\varepsilon}U(f)T_{R,\varepsilon}^{-1}=\exp[i\sqrt{\pi}(f,g_{R,\varepsilon})],$$
$$V(f)\to V_{R,\varepsilon}(f)\equiv T_{R,\varepsilon}U(f)T_{R,\varepsilon}^{-1}=V(f).$$

Hence $\omega_\varepsilon(\cdot)$ corresponds to the representation (U',V'), defined by

$$\begin{aligned}U'(f)&\equiv\exp[i\sqrt{\pi}(f,g_\varepsilon)],\\ V'(f)&\equiv V(f),\end{aligned}\tag{A.15}$$

where g_ε is given by (A.14).

Proposition A.2. *The representation* (U',V') *defined by* (A.15) *is disjoint from the Fock representation* (U,V).

Proof. Suppose U,U' unitarily equivalent. Then there exists a unitary operator $T:\mathscr{F}\to\mathscr{F}$ such that

$$U'(f)=TU(f)T^*.\tag{A.16}$$

Choose a sequence

$$f_n(x)\equiv\frac{1}{n}e^{-x^2/4n^2}g_{1/n}(x),\qquad n\geqq1,\tag{A.17}$$

with $g_{\varepsilon=1/n}$ as in (A.14). Then $f_n\in\mathscr{S}_{\mathbb{R}}(\mathbb{R})$, and $\|f_n\|_2\xrightarrow[n\to\infty]{}0$. Hence $\varphi(f_n)\xrightarrow[n\to\infty]{}0$ on $\mathscr{D}$ and therefore

$$U(f_n)\to\mathbb{1}\tag{A.18}$$

on $\mathscr{F}$. It follows that

$$TU(f_n)T^*\xrightarrow[n\to\infty]{}\mathbb{1}\tag{A.19}$$

on $\mathscr{F}$. By (A.15)

$$U'(f_n)=\exp\left[i\sqrt{\pi}\int_{-\pi}^{\pi}f_n(x)dx\right]U(f_n)\xrightarrow[n\to\infty]{}\exp(i\pi)\mathbb{1}=-\mathbb{1}\tag{A.20}$$

by (A.17) and (A.18). Hence, (A.18) and (A.20) contradict (A.16). The same proof shows disjointness. □

The infinite energy of charged *dynamical* states is expected in both massless (such as the model treated in this appendix) and massive models, in the screening/confinement region (which covers the whole phase diagram in the case of two-dimensional systems). The situation is different if one is concerned about the (potential) energy V of static, external charges, as a function of their separation l.

There, due to polarization effects (pair creation), V rises linearly until l is of order the inverse mass of the particles (matter fields), and saturates for larger l. A semiclassical calculation in the massive Schwinger model yields this result ([13]; see also [14] for a similar calculation in lattice gauge theories). In this latter respect, the massless Schwinger model is, of course, pathological, because there the potential rises linearly with distance, without saturation.

Appendix B. Polymer Expansions

This appendix is devoted to present some of the basic results concerning the method of polymer expansions. We follow in part the notation used in [1]. The proofs which are not presented here were onlined in [1, 8].

We consider a set $\mathcal{G}^c$ of geometric elements called polymers. Among the polymers we establish a symmetric relation called "compatibility" (or "incompatibility") denoted by $\gamma \sim \gamma'$ (respectively $\gamma \nsim \gamma'$), where $\gamma, \gamma' \in \mathcal{G}^c$. The relation holds for all pairs of polymers and we also have $\gamma \nsim \gamma$: each polymer is incompatible with itself. In all the proofs we will suppose $\mathcal{G}^c$ is a finite set.

A set of polymers Γ is said to be "admissible" if $\gamma \sim \gamma'$ for all $\gamma, \gamma' \in \Gamma$ with $\gamma \neq \gamma'$. We denote by $\mathcal{G}$ the set formed by all the admissible sets of polymers.

To each polymer we associate an "activity" $\mu(\gamma)$, in principle a complex number, and we denote

$$\mu^\Gamma = \prod_{\gamma \in \Gamma} (\mu(\gamma))^{\Gamma(\gamma)},$$

where $\Gamma(\gamma)$ is the multiplicity of γ in the set Γ (not necessarily an admissible one). Notation: if Γ is a multi-index [8], $\Gamma! = \prod_{\gamma \in \mathcal{G}^c} \Gamma(\gamma)!$ and $n(\Gamma) = \sum_{\gamma \in \mathcal{G}^c} \Gamma(\gamma)$.

If Γ and Γ' are two sets of polymers, we denote $\Gamma \sim \Gamma'$ if $\gamma \sim \gamma'\ \forall \gamma \in \Gamma, \gamma' \in \Gamma'$ and $\Gamma \nsim \Gamma'$ otherwise. For $\Gamma \in \mathcal{G}$ we define $\mathrm{CONN}(\Gamma) \equiv \{\Gamma' \in \mathcal{G} | \gamma' \nsim \Gamma\ \forall \gamma' \in \Gamma'\}$.

The partition function of a polymer model is given by

$$Z = \sum_{\Gamma \in \mathcal{G}} \mu^\Gamma. \tag{B.1}$$

and we define

$$\varrho(\Gamma) = Z^{-1} \sum_{\substack{\Gamma' \in \mathcal{G} \\ \Gamma' \sim \Gamma}} \mu^{\Gamma'}. \tag{B.2}$$

To each polymer γ we associate a conveniently defined size denoted by $|\gamma|$, $|\gamma| \in \mathbb{N}$. We define $\|\Gamma\| = \sum_\gamma \Gamma(\gamma)|\gamma|$. Following [1] we assume there exists a convex differentiable monotonically decreasing function $F_0 : (b_0, \infty) \to \mathbb{R}_+$, $b_0 \in \mathbb{R}$, such that for each $\Gamma \in \mathcal{G}$ and $b > b_0$,

$$\sum_{\gamma \nsim \Gamma} e^{-b|\gamma|} \leqq F_0(b)\, \|\Gamma\|. \tag{B.3}$$

We will consider here polymer models for which the following property holds: if $\Gamma \in \mathcal{G}$ and $N(\Gamma, s)$ denotes the number of polymers of size s incompatible with Γ, then there exists C such that

$$N(\Gamma, s) \leqq \|\Gamma\| C^s. \tag{B.3a}$$

It's straightforward to prove that in this case we may choose $F_0(b) = Ce^{-b}(1-Ce^{-b})^{-1}$, $b_0 = \ln C$.

It is a well known fact that (B.3a) holds for the polymer model which occurs in this paper.

We define $\|\mu\| = \sup_{\gamma} |\mu(\gamma)|^{1/|\gamma|}$, so that $|\mu(\gamma)| \leqq \|\mu\|^{|\gamma|}$.

Lemma B.1. *Let a_c be the smallest solution of $F_0'(a_c) = -1$ provided it exists, and $a_c = b_0$ otherwise, and suppose*

$$\|\mu\| \leqq \|\mu_c\| \equiv e^{-(a_c + F_0(a_c))}.$$

Then

$$\varrho(\Gamma_0) = \lim_{n\to\infty} \sum_{\substack{\Gamma_1,\ldots,\Gamma_n \in \mathscr{G} \\ \Gamma_i \in \mathrm{CONN}(\Gamma_{i-1}),\, i=1,\ldots,n}} (-\mu)^{\Gamma_1 + \ldots + \Gamma_n} \tag{B.4}$$

as a convergent series.

Proof. May be found in [1]. □

Note. For the models in which (B.3a) holds we have

$$\|\mu_c\| = C^{-1} \tfrac{1}{2}(3-\sqrt{5})e^{-1/2(\sqrt{5}-1)} < 1. \tag{B.4a}$$

Definition. $F_1 : (a_c + F_0(a_c), \infty) \to \mathbb{R}_+$ is such that

$$F_1(a + F_0(a)) = F_0(a). \tag{B.5}$$

Lemma B.2. *F_1 is monotonically decreasing.*

Proof. Follows easily from the definition of F_1 and the assumed properties of F_0. □

Lemma B.3. *If $\|\mu\| \leqq \|\mu_c\|$, then the following bound for $\varrho(\Gamma_0)$ holds:*

$$|\varrho(\Gamma_0)| \leqq \exp[F_1(-\ln\|\mu\|)\,\|\Gamma_0\|]. \tag{B.6}$$

Proof. Follows from Lemma B.1, Definition (B.3) and Definition (B.5). □

Let us define the function $g : \mathscr{G}_c \times \mathscr{G}_c \to \{0, -1\}$ by

$$g(\gamma, \gamma') = \begin{cases} 0 & \text{if } \gamma \sim \gamma', \\ -1 & \text{if } \gamma \nsim \gamma'. \end{cases} \tag{B.7}$$

It follows easily that the partition function (B.1) may be written as

$$Z = \sum_{\Gamma \subset \mathscr{G}_c} \mu^{\Gamma} \phi(\Gamma), \tag{B.8}$$

where

$$\phi(\Gamma) \equiv \prod_{\substack{i<j \\ \gamma_i, \gamma_j \in \Gamma}} [1 + g(\gamma_i, \gamma_j)]. \tag{B.9}$$

Let $f_1, f_2 : \{\Gamma \subset \mathscr{G}_c\} \to \mathbb{C}$ be functions defined on multi-indices. We define a $*$ product between f_1 and f_2 by

$$(f_1 * f_2)(X) = \sum_{X_1 + X_2 = X} f_1(X_1) f_2(X_2). \tag{B.10}$$

Definition.

$$\mathbb{1}(X)=\begin{cases}1 & \text{if} \quad X=\emptyset, \\ 0 & \text{otherwise}.\end{cases} \tag{B.10a}$$

We denote by $\mathscr{F}_0$ the set of functions on multi-indices satisfying $f_0(\phi)=0$ and by $\mathscr{F}_1$ the set of functions on multi-indices satisfying $f_1(\Gamma)=\mathbb{1}(\Gamma)+f_0(\Gamma)$ for some $f_0 \in \mathscr{F}_0$.

Definitions.

a) $$(\operatorname{Exp} f)(X)=\sum_{n \geqq 0} \frac{f^{*n}(X)}{n!} \quad \text{for} \quad f \in \mathscr{F}_0,$$

b) $$(\operatorname{Log} f)(X)=\sum_{n \geqq 1} \frac{(-1)^{n+1}}{n} f^{*n}(X) \quad \text{for} \quad f \in \mathscr{F}_1,$$

where $f^{*0}(X) \equiv \mathbb{1}(X)$.

Properties. $\operatorname{Exp}(\operatorname{Log} f_1)=f_1$; $\operatorname{Log}(\operatorname{Exp} f_0)=f_0$ for $f_1 \in \mathscr{F}_1$ and $f_0 \in \mathscr{F}_0$. We note that in definitions a) and b) above the sums are finite.

Let $\phi(\Gamma)$ be as defined in (B.9). We define the "Ursell function" $\phi^T(\Gamma)$ by

$$\phi^T(\Gamma)=(\operatorname{Log}\phi)(\Gamma).$$

Following [1] we will prefer sometimes to use the symbol C_Γ for $\phi^T(\Gamma)$.

Given a set Γ of polymers we may define a graph which we call "graph of incompatibilities," in the following way: we attribute to each polymer in Γ a vertex and join two vertices by a line provided the corresponding polymers are incompatible.

Theorem B.1. *$C_\Gamma=0$ if the "graph of incompatibilities" which corresponds to Γ is not connected. (The proof is outlined in* [8, *Lemma 3.5*].*)*

Lemma B.4. *Let F be a function on multi-indices and ω a function on polymers. Then*

$$\exp\left(\sum_X F(X)\omega^X\right)=\sum_X (\operatorname{Exp} F)(X)\omega^X, \tag{B.11}$$

provided $\sum_X F(X)\omega^X$ is an absolutely convergent series, i.e., $\sum_X |F(X)||\omega^X|<\infty$.

Proof. $\exp\left(\sum_X F(X)\omega^X\right)=\sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_X F(X)\omega^X\right)^n$. Since the series in parenthesis is absolutely convergent, we may rewrite the right-hand side above as

$$\sum_X \omega^X\left(\sum_{n=0}^{\infty}(1/n!)F^{*n}(X)\right)=\sum_X \omega^X(\operatorname{Exp} F)(X). \quad \square$$

Corollary B.1.

$$Z=\exp\left(\sum_\Gamma C_\Gamma \mu^\Gamma\right),$$

provided $\sum_\Gamma C_\Gamma \mu^\Gamma$ is absolutely convergent.

Proof. Follows from Lemma B.4, putting $F(X)=\phi^T(X)=C_X$. $\square$

Lemma B.5. *Let $\mathscr{G}_c$ be a finite set and b a function on polymers. Then*

$$\left(\sum_{X\in\mathscr{G}}\phi(X)b^X\mu^X\right)\Big/\left(\sum_{X\in\mathscr{G}}\phi(X)\mu^X\right)=\exp\left\{\sum_X C_X(b^X-1)\mu^X\right\},$$

provided $\sum_X C_X(b^X-1)\mu^X$ *is absolutely convergent.*

Proof. Since $\sum_X \phi^T(X)(b^X-1)\mu^X$ is absolutely convergent, we may write

$$\sum_X \phi^T(X)(b^X-1)\mu^X=\lim_{\{Y\}}\sum_X \phi^T(X)I_{\{Y\}}(X)(b^X-1)\mu^X,$$

where I is the characteristic function of the set $\{Y\}$, a subset of the set of all multi-indices. The limit $\lim_{\{Y\}}$ means the limit whereby $\{Y\}$ converges to the set of all multi-indices. Since the series is absolutely convergent, this limit may be taken in an arbitrary way.

Now, using Lemma B.4

$$\begin{aligned}&\exp\left(\sum_X \phi^T(X)(b^X-1)\mu^X\right)\\&=\lim_{\{Y\}}\left\{\left[\sum_X \mathrm{Exp}(\phi^T I_{\{Y\}})(X)\mu^X b^X\right]\Big/\left[\sum_X \mathrm{Exp}(\phi^T I_{\{Y\}})(X)\mu^X\right]\right\}\\&=\sum_{X\in\mathscr{G}}\mu^X b^X\Big/\sum_{X\in\mathscr{G}}\mu^X. \quad\square\end{aligned}$$

Lemma B.6. *There exists a constant c independent of Γ_0 such that*

$$\sum_{\Gamma\nsim\Gamma_0}|C_\Gamma|\,|\mu^\Gamma|\leqq c\|\Gamma_0\|. \tag{B.12}$$

Proof. (The details given below, which were omitted in [1], were communicated to us by Dr. K. Fredenhagen.) From (B.2) we obtain as in [1, p. 110]

$$\varrho(\Gamma)=\sum_{\Gamma'\in\mathrm{CONN}(\Gamma)}(-\mu)^{\Gamma'}\varrho(\Gamma') \tag{B.13}$$

(formula (A.4) of [1]). We now use identity (A.13) of [1] (the sign is incorrect there):

$$\begin{aligned}\sum_{\substack{\Gamma':\mathscr{G}^c\to\mathbb{N}\\ \Gamma'\sim\Gamma}}C_{\Gamma'}\mu^{\Gamma'}&=\ln\varrho(\Gamma)=\ln Z(0)-\ln Z(1)\\&=\int_0^1 d\lambda\sum_{\gamma\in\mathscr{G}^c,\,\gamma\nsim\Gamma}\mu(\gamma)\varrho_\lambda(\{\gamma\}),\end{aligned} \tag{B.14}$$

where $Z(\lambda)$ and ϱ_λ are partition function and correlation functions, respectively, corresponding to the activities $\mu_\lambda(\gamma)=\lambda\mu(\gamma)$ for $\gamma\nsim\Gamma$, and $\mu_\lambda(\gamma)=\mu(\gamma)$ otherwise, $\gamma\in\mathscr{G}_c$. Since $\|\mu_\lambda\|\leqq\|\mu\|$ and F_1 is decreasing, it follows that $F_1(-\ln\|\mu_\lambda\|)\leqq F_1(-\ln\|\mu\|)$. From the inequality

$$|\varrho_\lambda(\{\gamma\})|\leqq\exp[F_1(-\ln\|\mu_\lambda\|)|\gamma|],$$

which follows from (B.6), we obtain

$$|\varrho_\lambda(\{\gamma\})|\leqq\exp[F_1(-\ln\|\mu\|)|\gamma|]. \tag{B.15}$$

Let the activities be negative: $\mu(\gamma)=-|\mu(\gamma)|$. Then by (B.4) we obtain $\varrho(\Gamma)\geqq 0$.

Inserting (B.13) into the right-hand side of (B.14) there results a power series in $|\mu(\gamma)|$ with just negative coefficients. By comparison with the left-hand side of (B.14), it then follows that

$$-c_\Gamma=(-1)^\Gamma|c_\Gamma|. \tag{B.16}$$

Inserting now (B.16) into (B.14), and using (B.3), (B.15) and the definition of F_1, we arrive at (B.12), with $c=F_1(-\ln\|\mu\|)$. □

Corollary B.2. *If* $\|\mu\|\leqq\|\mu_c\|$ *there exists a constant c such that*

$$\sum_{\substack{\Gamma\sim\Gamma_0\\ \|\Gamma\|\geqq n}}|c_\Gamma|\,|\mu|^\Gamma\leqq c\left[\frac{\|\mu\|}{\|\mu_c\|}\right]^n\|\Gamma_0\|. \tag{B.17}$$

Proof.

$$\sum_{\substack{\Gamma\nsim\Gamma_0\\ \|\Gamma\|\geqq n}}|c_\Gamma|\,|\mu^\Gamma|\leqq\left[\frac{\|\mu\|}{\|\mu_c\|}\right]^n\sum_{\substack{\Gamma\nsim\Gamma_0\\ \|\Gamma\|\geqq n}}|c_\Gamma|\,\|\mu_c\|^{\|\Gamma\|}$$

$$\leqq[\|\mu\|/\|\mu_c\|]^n\sum_{\Gamma\nsim\Gamma_0}|c_\Gamma|\,\|\mu_c\|^{\|\Gamma\|}$$

$$\leqq[\|\mu\|/\|\mu_c\|]^nF_1(-\ln\|\mu_c\|)=c\left[\frac{\|\mu\|}{\|\mu_c\|}\right]^n\|\Gamma_0\|,$$

$$c=F_1(-\ln\|\mu_c\|)=F_0(a_c).\quad\square$$

Acknowledgement. We are very much indebted to Dr. K. Fredenhagen for his interest and, in particular, for several very useful suggestions, as well as for pointing out an error in a previous version of the manuscript.

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Communicated by K. Osterwalder

Received July 1, 1985; in revised form September 10, 1985