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Random Fields on Riemannian Manifolds: A Constructive Approach

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Abstract. We extend to "Euclidean" fields on a wide class of Riemannian manifolds two results which have proven to be crucial in the construction of interacting quantum fields in the flat case, namely local regularity properties of the free covariance in two dimensions and Osterwalder-Schrader positivity.

1. Introduction and Outline

The peculiarity and the depth of the physical problems posed by Quantum Field Theory on a gravitational background [1] and the need of providing models of existing axiomatic proposals [2, 3], motivate the opportunity of a constructive approach.

In this paper we extend some techniques of constructive quantum field theory to the case of two dimensional properly Riemannian ("Euclidean") manifolds.

The specific case of the two dimensional sphere (vs. two dimensional de Sitter universe) was considered in [4]. Here we rather rely on general intrinsic properties of the heat kernel on the manifold as the main tool to overcome the difficulties due to the non-availability of explicit expressions for the free Green's functions.

Our arguments which, because of the lack of translational invariance, must do without the tool of Fourier transformation, shed, we hope, some light also on the classical arguments for the conventional flat case.

The "Euclidean" approach of [1] is exceedingly convenient in the case (which we will always be considering in the following) of a paracompact, complete, C^∞ Riemannian manifold M.

As pointed out in [5], the uniqueness of the free covariance $C=(-\Delta_M+m^2)^{-1}$ (Δ_M being the Laplace-Beltrami operator on M), which follows from the essential self-adjointness of Δ_M in $C_0^\infty(M)$ [6] gives an unambiguous starting point for the "Euclidean" construction, as opposite to the ambiguities arising in the choice of a free vacuum on a manifold of Lorentzian signature [7].

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In particular, the Gaussian random field φ of covariance C on M is uniquely defined by the characteristic functional $S: f \in C_0^\infty(M) \to S(f) = \exp - \frac{1}{2} C(f,f)$ via the Bochner-Minlos theorem.

The following representation of the free covariance C [1, 5]:

$$C(x,y) = \int_0^{+\infty} e^{-tm^2} p(t,x,y) dt, \tag{1}$$

$p(t,x,y)$ being the heat kernel on the manifold M, provides a convenient starting point for the ultraviolet regularization in terms of the multiscale decomposition of [8]. The covariance $C(x,y)$ is given by

$$C(x,y) = \sum_{\ell=0}^{\infty} C_\ell(x,y), \tag{2}$$

where, for some fixed constant γ larger than 1,

$$C_\ell(x,y) = \int_0^{+\infty} (\exp(-m^2\gamma^{2\ell}t) - \exp(-m^2\gamma^{2\ell+2}t)) p(t,x,y) dt \tag{3}$$

is the kernel of the operator

$$C_\ell = (-\Delta_M + m^2\gamma^{2\ell})^{-1} - (-\Delta_M + m^2\gamma^{2\ell+2})^{-1}. \tag{4}$$

We introduce the regularized covariance

$$C^{(k)}(x,y) = \sum_{\ell=0}^{(\log_\gamma k)-1} C_\ell(x,y), \tag{5}$$

where, of course, k is such that $\log_\gamma k$ ranges on the positive integers. $C^{(k)}$ represents the covariance of the field with length cutoff $\gamma(mk)^{-1}$, the analog, in the flat case, of a momentum cutoff of order mk. In this sense $C^{(k)}$ compares with the $\delta_k C \delta_k$ of [9, p. 124].

The following estimates extend to Riemannian manifolds in two dimensions the local regularity (LR) properties of the free Euclidean covariance which in [9] are at the basis of the control of the ultraviolet divergences in the two dimensional flat case:

Theorem 1. *If* $\dim M = 2$, *for every* $1 \leqq q < +\infty$ *and for every compact set K in M:*

$$\sup_{x\in M} \| C(x,\cdot)\|_{L^q(K,dV)} < +\infty, \tag{LR1}$$

$$\|C^{(k)}(\cdot,\cdot) - C(\cdot,\cdot)\|_{L^q(K\times K, dV\otimes dV)} \leqq O(k^{-2/q}), \tag{LR2}$$

$$\sup_{x\in K} C^{(k)}(x,x) \leqq O(\log_\gamma k), \tag{LR3}$$

where dV is the Riemannian volume on M.

The second result which we prove is Osterwalder-Schrader (O-S) positivity of the free covariance C for manifolds with suitable symmetry. As it is well known this result implies O-S positivity for theories with $P(\varphi)_2$ interaction.

Theorem 2. *Suppose that* i) $M = V_+ \cup V_- \cup \partial V$, *where* $V_\pm$ *are smooth submanifolds of M with common boundary* ∂V; ii) *there exists an isometric map* $\theta: M \to M$ *such that*

$\theta V_{\pm} = V_{\mp}$ and $\theta\partial V = \partial V$ pointwise; iii) *having set $(\theta f)(x) = f(\theta x)$, θ commutes with Δ_M and $\mathbf{n}\cdot\operatorname{grad}\theta f = -\mathbf{n}\cdot\operatorname{grad} f$ on ∂V, where $\mathbf{n}$ is the outer normal to ∂V_-, then $C(\theta f, f) \geqq 0$ for every $f \in C_0^\infty(M)$ with $\operatorname{supp} f \subset V_+$.*

Theorem 1 is proven in Sect. 2, Sect. 3 will deal with the proof of O-S positivity, while Sect. 4 will address itself to open problems.

2. Local Regularity Properties

LR1 and LR3 reflect the logarithmic nature of the singularity of the covariance $C(x, y)$ at coinciding points $\left(C(x,y) \sim -\frac{1}{2\pi}\log_e md(x,y)\right)$, which, in turn, follows from the asymptotic behavior of the heat kernel as $t \downarrow 0$ [10, 11]:

$$p(t,x,y) \sim (4\pi t)^{-\nu/2} H(x,y) \exp\left(-\frac{d^2(x,y)}{4t}\right), \tag{6}$$

uniformly on all compact sets in $M \times M$ which do not intersect the cut locus of M. Here $\nu = \dim M$, $d(\cdot\,,\cdot)$ is the geodesic distance and $H(\cdot\,,\cdot)$ the Ruse invariant.

The more technical property LR2, essential in the removal of the ultraviolet cutoff in a $P(\varphi)_2$ interaction, follows from the strong Markov property of the Brownian motion on M having Δ_M as generator and $p(t,x,y)$ as density of the transition semigroup with respect to the Riemannian volume dV.

Having fixed the compact set $K \subset M$, we define:

$$A_K(t) = \sup_{x \in M,\, y \in K} t^{\nu/2} p(t,x,y). \tag{7}$$

We observe, first of all, that there exists $\bar{t} = \bar{t}(K) > 0$ such that:

$$B_K(\bar{t}) = \sup_{0 < t \leqq \bar{t}} A_K(t) < +\infty \tag{8}$$

$\Big($in the flat case, by direct inspection of the heat kernel, one has the stronger estimate $\sup\limits_{t,x,y} t^{\nu/2} p(t,x,y) < +\infty\Big)$.

Notice that the local equivalence (6) only ensures the existence of positive constants $c(K)$, $t(K)$, $r(K)$ such that for $0 < t \leqq t(K)$, $(x,y) \in K \times K$ *and* $d(x,y) \leqq r(K)$:

$$p(t,x,y) \leqq c(K) t^{-\nu/2} \exp\left(-\frac{d^2(x,y)}{4t}\right). \tag{9}$$

Removing the last condition [x and y close enough, so that (x,y) is not in the cut locus of M] requires Azencott's argument [12] to propagate local estimates:

Lemma. *If $(x,y) \in M \times K$, then there exists $\bar{t}(K) > 0$ such that:*

$$p(t,x,y) \leqq c(K) t^{-\nu/2} \exp -\frac{r^2(K)}{4t} \tag{10}$$

for $0 < t \leqq \bar{t}(K)$ and $d(x,y) > r(K)$.

The proof, given in [12], is based on the strong Markov property of the Brownian motion on M starting at x with respect to the first hitting time of a sphere centered at y and with radius smaller than $r(K)$.

The inequalities (9) and (10) then imply (8).

Next we prove that, for every $t>0$,

$$B_K(t) \leqq B_K(\bar{t})2^{\nu/2\,[\log_2 \max(1,t/\bar{t})]} = B_K(t)f(\bar{t}). \tag{11}$$

Indeed, by the Chapman-Kolmogorov equation, for $(x,y)\in M\times K$:

$$p(t,x,y)=\int_M p\left(\frac{t}{2},x,z\right)p\left(\frac{t}{2},z,y\right)dV(z) \leqq \left(\frac{t}{2}\right)^{-\nu/2} A_K\left(\frac{t}{2}\right). \tag{12}$$

Therefore:

$$A_K(t) \leqq 2^{\nu/2} A_K\left(\frac{t}{2}\right), \tag{13}$$

and, by iteration on n:

$$A_K(t) \leqq 2^{n\nu/2} A_K\left(\frac{t}{2^n}\right), \tag{14}$$

so that (11) follows, for $t>\bar{t}$, from the choice $n=\left[\log_2\left(\frac{t}{\bar{t}}\right)\right]$, the smallest integer larger than $\log_2\left(\frac{t}{\bar{t}}\right)$.

Set, now

$$F(t,x,y)=t^{\nu/2}p(t,x,y) \tag{15}$$

and

$$d\mu(t)=N^{-1}(\exp-m^2t-\exp-m^2\gamma^2t)t^{-\nu/2}dt, \tag{16}$$

where N normalizes $d\mu$ to a probability measure on $(0,+\infty)$ (for $\nu<4$, of course). Equation (11) implies then that:

$$C_\ell(x,y) \leqq \gamma^{\ell(\nu-2)}NB_K(\bar{t}) \int_0^{+\infty} f(t)d\mu(t) = \text{const}\,\gamma^{\ell(\nu-2)} \tag{17}$$

because $d\mu$ falls off exponentially and $f(t)$ is polynomially bounded.

In particular, for $\nu=2$,

$$\sup_{x\in K} C^{(k)}(x,x) \leqq O(\log_\gamma k). \tag{18}$$

For $1\leqq q<+\infty$, $y\in K$, Jensen's inequality implies:

$$\begin{aligned}(C_\ell(x,y))^q \leqq{}& \gamma^{\ell q(\nu-2)}N^qB_K(\bar{t})^{q-1}\\ &\cdot\int_0^{+\infty} F(\gamma^{-2\ell}t,x,y)f(\gamma^{-2\ell}t)^{q-1}d\mu.\end{aligned} \tag{19}$$

As $f(\gamma^{-2\ell}t)\leqq f(t)$ and

$$\int_K F(\gamma^{-2\ell}t,x,y)dV(y) \leqq (t\gamma^{-2\ell})^{\nu/2}, \tag{20}$$

it follows that:

$$\|C_\ell(x,\cdot)\|_{L^q(K,dV)} \leqq \text{const}\,\gamma^{\ell\left(v-2-\frac{v}{q}\right)}. \tag{21}$$

As

$$C(x,y) = \sum_{\ell=0}^{\infty} C_\ell(x,y), \tag{22}$$

Minkowski's inequality implies then, for $v=2$:

$$\sup_{x\in M} \|C(x,\cdot)\|_{L^q(K,dV)} < +\infty, \tag{23}$$

namely LR1 holds.

As to LR2, if K_1 and K_2 are compact sets in M, the previous estimates show that:

$$\|C_\ell(\cdot,\cdot)\|_{L^q(K_1\times K_2, dV\otimes dV)} \leqq V(K_1)^{1/q}\,\text{const}\,\gamma^{\ell\left(v-2-\frac{v}{q}\right)}. \tag{24}$$

Therefore, for $v=2$:

$$\begin{aligned}&\|C(\cdot,\cdot)-C^{(k)}(\cdot,\cdot)\|_{L^q(K_1\times K_2, dV\otimes dV)}\\ &\leqq \text{const}\sum_{\ell=\log_\gamma k}^{\infty} \gamma^{-2\ell/q} = O(k^{-2/q}),\end{aligned} \tag{25}$$

which proves LR2.

As a side remark to the previous considerations we wish to observe that in the multiscale decomposition of the free field, the conventional scaling law:

$$C_\ell(x,y) = C_0(\gamma^\ell x, \gamma^\ell y) \tag{26}$$

is to be supplemented by the observation that any length R appearing in the theory through an expression of the metric tensor as

$$g^R_{\alpha\beta}(x) = g_{\alpha\beta}\left(\frac{x}{R}\right) \tag{27}$$

gets itself involved in (26), which is therefore better rewritten as:

$$C_\ell(x,y,R) = C_0(\gamma^\ell x, \gamma^\ell y, \gamma^\ell R). \tag{28}$$

So, at least in this sense, fields on different scales are carried by similar but different Riemannian manifolds.

One interesting question in this respect is whether the comparison theorems in stochastic differential geometry [13], giving local growth estimates of the heat kernel and, therefore, of the free covariance with the curvature, extend to some class of interacting models and whether they admit a consistent reinterpretation in terms of correlation inequalities.

3. Osterwalder-Schrader Positivity

Much in the same spirit as the electrostatic example of [14] and the considerations of [15], the proof of Theorem 2 shows that O-S positivity is a simple potential theoretic fact relative to the operator $-\Delta_M + m^2$.

Let f be a real smooth function with compact support contained in V_+ and $U=(-\Delta_M+m^2)^{-1}f$ the potential of f.

From the hypothesis $[\theta,\Delta_M]=0$ and Green's formula, it follows that:

$$\begin{aligned}C(\theta f,f)&=\int_M \theta f(-\Delta_M+m^2)^{-1}f dV=\int_{V_-}\theta f U dV\\&=\int_{V_-}U(-\Delta_M+m^2)\theta U dV=\int_{V_-}[U(-\Delta_M+m^2)\theta U-\theta U(-\Delta_M+m^2)U]dV\\&=\int_{\partial V_-}(\theta U\operatorname{grad}U-U\operatorname{grad}\theta U)\cdot\mathbf{n}dS.\end{aligned}\tag{29}$$

Furthermore, as $\theta U=U$ and $\mathbf{n}\cdot\operatorname{grad}\theta U=-\mathbf{n}\cdot\operatorname{grad}U$ on ∂V_-:

$$C(\theta f,f)=2\int_{\partial V_-}U\operatorname{grad}U\cdot\mathbf{n}dS.\tag{30}$$

By Gauss' theorem:

$$\int_{\partial V_-}U\operatorname{grad}U\cdot\mathbf{n}dS=\int_{V_-}\operatorname{div}(U\operatorname{grad}U)dV=\int_{V_-}\left(g^{\alpha\beta}\frac{\partial U}{\partial x^\alpha}\frac{\partial U}{\partial x^\beta}+m^2U^2\right)dV.\tag{31}$$

In the last step we have used the fact that $\Delta_M U=m^2U$ in V_-.

From (30) and (31) the requested result follows:

$$C(\theta f,f)=2\int_{V_-}\left(g^{\alpha\beta}\frac{\partial U}{\partial x^\alpha}\frac{\partial U}{\partial x^\beta}+m^2U^2\right)dV\geqq 0.\tag{32}$$

As a final remark, we observe that θ's satisfying hypotheses of Theorem 2 do exist for two dimensional Riemannian manifolds which are surfaces of revolution and are realized as reflections with respect to planes through any axis of symmetry. Such manifolds possess, at least, an SO(2) symmetry and are the natural "Euclidean" analogues of the manifolds with Lorentz symmetry considered in [2].

4. Conclusions and Outlook

What Theorems 1 and 2 really prove is the feasibility of the construction of two dimensional interacting "Euclidean" fields also on curved properly Riemannian manifolds following the strategy familiar in the conventional flat case.

One major problem is open, namely the absence, as far as we know, of a reconstruction theorem for the quantum theory in the physical curved space-time.

Difficulties of such a reconstruction program are very well explained, for instance, in [16].

In some sense the theory is at the same stage as it was, in the flat case, at the beginning of the Euclidean program in the early proposal of Symanzik [17]. The "Euclidean" approach, once the reconstruction program is accomplished, should select one distinguished state of the quantum theory (a "vacuum") which, as tested on a submanifold with static metric of the space-time (for instance the exterior Schwarzschild region in the Kruskal space-time), should exibit the thermal properties analyzed by Hawking.

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Note added in proof. As to the reconstruction problem, we observe that the properly Riemannian construction on the manifolds considered in Sect. 3 provides a *periodic* (in the imaginary time coordinate on a static wedge) *O-S positive* stochastic process, one for which the general analysis of A. Klein and L. Landau in J. Funct. Anal. **42**, 368 (1981) ensures the possibility of uniquely reconstructing a stochastically positive KMS system. Otherwise stated, Theorem 2 substantiates assumption C.2 of G. Sewell, Phys. Lett. **79** A, 23 (1980), providing a whole class of models for which suitably adapted "Euclidean" axioms, including O-S positivity, hold.