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The Reciprocal of a Borel Summable Function is Borel Summable

G. Auberson and G. Mennessier

Departement de Physique Mathématique*, U.S.T.L., F-34060 Montpellier Cedex, France

Abstract. It is proven that if a function f is Borel summable in some angular region and has a non-vanishing derivative at the origin, then its reciprocal f^{-1} is also Borel summable in a region which has essentially the same angular extent.

Formal manipulations of divergent (or presumably divergent) power series are frequently done when non-exactly solvable problems are treated through the perturbation method, especially in quantum mechanics and in quantum field theory. In some favourable circumstances, i.e. when the particular quantity to be expanded perturbatively turns out to be a Borel summable function, these manipulations can be justified by appealing to general properties of such functions. Admittedly, for the most interesting cases, e.g. for the non-abelian gauge theories, it is very unlikely that the Green functions (say) enjoy the required Borel summability property [1]. In this context, it has been argued however [2], that a proper use of renormalization group methods (especially through the freedom in the choice of the renormalization scheme) might improve the situation, at least for the quantities of physical interest, so that the issue seems (at least to us!) still inconclusive[1].

In any case, we believe that it is interesting to gather as many results as possible about the general properties of Borel summable functions, which potentially may give a firm basis to the above mentioned formal manipulations. This was precisely the purpose of our work in [3]. In that paper however, an aspect of Borel summability was not touched upon, namely the problem of inverting a Borel summable function, which may be of some relevance in the renormalization

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* Physique Mathématique et Théorique, Unité associée au CNRS No. 768

1 To avoid a possible confusion, let us notice that we are using the expression "Borel summable" in the full mathematical sense (and not simply to mean that the perturbation *series* is formally Borel summable). This convention is not shared by all authors (for instance an explicit distinction is made by Stevenson [2] who calls "Borel recoverable" a "Borel summable" function in our acceptation)

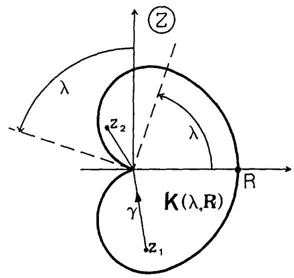

Fig. 1. The analyticity domain $K(\lambda, R)$ of $f(z)$. Also shown is an example of the integration path γ used in the proof of Eq. (10)

process and when one is dealing with power expansions in terms of a running coupling constant [4]. The gap will be filled in the present paper, leading to a very simple answer. Actually, it will be shown that the Borel summability of a function f is preserved by inversion, with the *only* proviso that the first derivative of f does not vanish at the origin (such a condition obviously must hold, in order to insure the local invertibility of f at the origin). In particular, this trivially implies that the coefficients of the asymptotic power expansion of f^{-1} are obtained by inverting formally the corresponding expansion of f. The precise theorem is stated and proved below, within the notation of [3]. However, in order to make the present account essentially self-contained, we first recall the definition of the relevant classes of Borel summable functions.

Let $K(\lambda, R)$ be the kidney-shaped region (Fig. 1),

$$K(\lambda, R) = \bigcup_{|\theta| \leqq \lambda} e^{i\theta} K(0, R) \quad (0 \leqq \lambda \leqq \tfrac{\pi}{2}), \tag{1}$$

where $K(0, R)$ is the open disk $\mathrm{Re}(1/z) > 1/R$.

Definition. Given $R, \sigma > 0$ and λ, f is said to belong to $N-(\lambda, R, \sigma)$ if

i) $f(z)$ is analytic in $K(\lambda, R)$,

ii) $f(z)$ admits there the asymptotic expansion

$$f(z) = \sum_{n=0}^{M-1} f_n z^n + R_M(z), \qquad M = 0, 1, \ldots, \tag{2}$$

where the remainders $R_M(z)$ are subjected to

$$|R_M(z)| \leqq A(\lambda, R, \sigma) M! |\sigma z|^M \quad \forall z \in K(\lambda, R), \tag{3}$$

$[R_0(z)$ is identified with $f(z)]$.

The class $\mathcal{N}$ is then defined by

$$\mathcal{N} - (\lambda, R, \sigma) = \bigcap_{\substack{0 < R' < R \\ \sigma' > \sigma}} N - (\lambda, R', \sigma'). \tag{4}$$

The functions in $\mathcal{N}$ are known to be Borel summable, as a consequence of a theorem of Nevanlinna [5]. Restricting now to (properly normalized) functions f such that $f'(0) \neq 0$, we have the

Theorem. *Let f be a function in $\mathcal{N} - (\lambda, R, \sigma)$ with $f_0 = 0, f_1 = 1$. Then there exists a R_1 such that $g \equiv f^{-1}$ is well defined in $K(\lambda, R_1)$, and a σ_1 such that*

$$g \in \mathcal{N} - (\lambda, R_1, \sigma_1) \quad (\textit{with } g_0 = 0,\ g_1 = 1).$$

Proof. First of all, Eqs. (2)–(4) imply that if $f \in \mathcal{N}-(\lambda, R, \sigma)$, then for any σ' and R' such that $0 < R' < R$, $\sigma' > \sigma$,

$$|R_M(z)| \leqq AM!|\sigma' z|^M \quad \forall z \in K(\lambda, R'), \qquad M=0,1,\ldots, \tag{5}$$

$$|f_n| \leqq An!\sigma'^n, \qquad n=1,2,\ldots, \tag{6}$$

where we have written A for $A(\lambda, R', \sigma')$.

i) Univalence of f

In order to define f^{-1}, one has first to show that f is univalent in $K(\lambda, r)$ for sufficiently small r.

Consider two distinct points z_1 and z_2 in $K(\lambda, r)$, and their images $u_1 = f(z_1)$, $u_2 = f(z_2)$. Then, from Eq. (2) (with $f_0 = 0$, $f_1 = 1$),

$$u_2 - u_1 = z_2 - z_1 + z_2^2 \varrho(z_2) - z_1^2 \varrho(z_1), \tag{7}$$

where $\varrho(z) \equiv R_2(z)/z^2$. It is an immediate consequence of the definitions that the function ϱ also belongs to $\mathcal{N}-(\lambda, R, \sigma)$. Now, Eq. (7) yields

$$\begin{aligned} |u_2 - u_1| &\geqq |z_2 - z_1| - (|z_2^2 - z_1^2||\varrho(z_2)| + |z_1|^2 |\varrho(z_2) - \varrho(z_1)|) \\ &\geqq |z_2 - z_1| - 4A\sigma'^2 r |z_2 - z_1| - r^2 |\varrho(z_2) - \varrho(z_1)|, \end{aligned} \tag{8}$$

where we have used $|z_{1,2}| < r$ and $|\varrho(z_2)| \leqq 2A\sigma'^2$ [Eq. (5)]. $|\varrho(z_2) - \varrho(z_1)|$ can also be bounded from above in terms of $|z_2 - z_1|$ by rewriting it as

$$|\varrho(z_2) - \varrho(z_1)| = \left| \int_\gamma dz\, \varrho'(z) \right| \leqq \ell \sup_{z \in K(\lambda, r)} |\varrho'(z)|, \tag{9}$$

where γ is some rectifiable path of length ℓ joining z_1 to z_2 in $K(\lambda, r)$. According to Theorem 4 (or 5) of [3], $\varrho \in \mathcal{N}-(\lambda, R, \sigma)$ implies the absolute boundedness of ϱ' in $K(\lambda, r)$. Moreover, obvious geometrical reasons (Fig. 1) allow us to choose the path γ in such a way that $\ell \leqq |z_2 - z_1|/\cos\lambda$. Thus, *if* $\lambda < \frac{\pi}{2}$,

$$|\varrho(z_2) - \varrho(z_1)| \leqq \frac{B}{\cos\lambda} |z_2 - z_1|, \tag{10}$$

and

$$|u_2 - u_1| \geqq |z_2 - z_1| \left(1 - 4A\sigma'^2 r - \frac{B}{\cos\lambda} r^2 \right) > 0 \text{ for } r \text{ small enough}. \tag{11}$$

Therefore $u_1 \neq u_2$. Notice that is required (but not necessarily sufficient) to have

$$r < \frac{1}{4A\sigma'^2}. \tag{12}$$

When $\lambda = \frac{\pi}{2}$, the previous argument fails, and needs to be somewhat refined, e.g. as follows:

– If $\operatorname{Im} z_1 \cdot \operatorname{Im} z_2 \geqq 0$, or if $\operatorname{Re} z_1$, $\operatorname{Re} z_2 \geqq 0$, the bound (10) remains valid with $1/\cos\lambda$ replaced by 1.

– If $\operatorname{Im} z_1 \cdot \operatorname{Im} z_2 < 0$ and $\operatorname{Re} z_1 \cdot \operatorname{Re} z_2 \leqq 0$, it is still valid with $1/\cos\lambda$ replaced by $\sqrt{2}$.

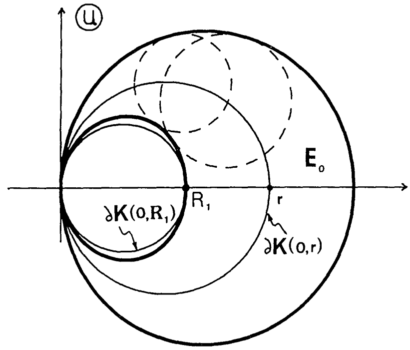

Fig. 2. The set E_0 defined in Eq. (13), when $r<\frac{1}{2}A\sigma'^2$

– If $\operatorname{Im}z_1 \cdot \operatorname{Im}z_2<0$ and $\operatorname{Re}z_1$, $\operatorname{Re}z_2 \leqq 0$, one necessarily has (say in the case $\operatorname{Im}z_1>0$)

$$\operatorname{Im}z_1>\frac{|z_1|^2}{r}, \quad \operatorname{Im}z_2<-\frac{|z_2|^2}{r}.$$

Hence

$$\begin{aligned}\operatorname{Im}(u_2-u_1)&=\operatorname{Im}(z_2-z_1)+\operatorname{Im}[z_2^2\varrho(z_2)-z_1^2\varrho(z_1)]\\&<-\frac{1}{r}(|z_2|^2+|z_1|^2)+2A\sigma'^2(|z_2|^2+|z_1|^2).\end{aligned}$$

Because of Eq. (12), this implies $\operatorname{Im}(u_2-u_1)<0$, and allows us again to infer that $u_1 \neq u_2$.

The univalence of the function f restricted to $K(\lambda, r)$ is thus established. As a result, f is a homeomorphic mapping of $K(\lambda, r)$ onto $\Omega_r \equiv f(K,(\lambda, r))$, and f^{-1} is holomorphic in Ω_r. Notice also that $0 \in \bar{\Omega}_r$.

ii) There Exists a Region $K(\lambda, R_1)$ Contained in Ω_r

Let $\partial\Omega_r$ be the boundary of Ω_r: $\partial\Omega_r=f(\partial K(\lambda, r))$.

Clearly, $\partial\Omega_r$ is contained in the set

$$E_\lambda \equiv \{u \,|\, |u-z| \leqq 2A\sigma'^2|z|^2, z \in \partial K(\lambda, r)\}. \tag{13}$$

Consider first E_0, which appears as a crescent the boundary of which may be obtained as the envelope of a family of circles (Fig. 2). It is easy to show (e.g. by working with the variable $1/u$) that E_0 does not intersect the disk $K(0, R_1)$, where $R_1=r-2A\sigma'^2r^2$. Next, by performing the rotations involved in Eq. (1), one finds that E_λ does not intersect $K(\lambda, R_1)$.

Thus $\partial\Omega_r$, which is a Jordan curve entirely contained in E_λ, does not intersect $K(\lambda, R_1)$. On the other hand, $\Omega_r \cap K(\lambda, R_1) \neq \emptyset$, since for z real and small enough, $|f(z)-z| \leqq 2A\sigma'^2z^2$ implies $|\arg f(z)|<\frac{\pi}{2}$. Then, as a consequence of the Jordan theorem, the (connected!) region $K(\lambda, R_1)$ is entirely contained in Ω_r.

iii) Borel Summability of $g \equiv f^{-1}|K(\lambda, R_1)$

Let us first define recursively the coefficients g_n by the relations resulting from the formal inversion of the series (2):

$$\left.\begin{aligned} g_1 &= 1 \\ g_n &= -\sum_{q=1}^{n-1} g_q \sum_{\substack{p_1+\ldots+p_q=n \\ p_i \geqq 1}} f_{p_1} f_{p_2} \ldots f_{p_q}, \qquad n=2,3,\ldots, \end{aligned}\right\} \tag{14}$$

and the remainders $\Gamma_M(u)$ by the formula

$$g(u) = \sum_{n=1}^{M-1} g_n u^n + \Gamma_M(u) \quad (u \in K(\lambda, R_1)). \tag{15}$$

One has to show that the $\Gamma_M(u)$'s are bounded by expressions analogous to those of Eq. (3). By inserting in Eq. (15) the g_n's of Eq. (14) and the expression (2) of $u = f(z)$, one finds

$$\left.\begin{aligned} \Gamma_0(u) &= \Gamma_1(u) = g(u) \\ \Gamma_M(u) &= -\sum_{m=1}^{M-1} g_m R_M^m(z), \qquad M=2,3,\ldots, \end{aligned}\right\} \tag{16}$$

where $R_M^m(z)$ is the M^{th} order remainder of the function $[f(z)]^m$. Now, according to the Eq. (A.1) of [3], these remainders are bounded by

$$|R_M^m(z)| \leqq \tfrac{1}{2}(3^m - 1)(M-m+1)!\, A^m |\sigma' z|^M \quad (z \in K(\lambda, r)). \tag{17}$$

In order to derive bounds on the coefficients g_n, it is convenient to set

$$\tilde{\sigma} = A\sigma'^2 \tag{18}$$

and to rewrite Eq. (6) in the weaker form:

$$|f_n| \leqq n!\, \tilde{\sigma}^{n-1}, \qquad n=1,2,\ldots. \tag{19}$$

These inequalities are true because Eq. (6) together with $f_1 = 1$ imply $A\sigma' \geqq 1$. Using them in Eq. (14) yields

$$\left.\begin{aligned} g_1 &= 1 \\ |g_n| &\leqq \tilde{\sigma}^n \sum_{q=1}^{n-1} \tilde{\sigma}^{-q} S_n^q |g_q|, \qquad n=2,3,\ldots, \end{aligned}\right\} \tag{20}$$

where the numbers S_n^q are defined by

$$S_n^q = \sum_{\substack{p_1+\ldots+p_q=n \\ p_i \geqq 1}} p_1!\, p_2! \ldots p_q!. \tag{21}$$

We then rely on the following inequalities, which are proven in the appendix:

$$S_n^q < 2\frac{n!}{q!} \quad \text{for all} \quad n=1,2,\ldots \quad \text{and} \quad 1 \leqq q \leqq n. \tag{22}$$

They lead to

$$|g_n| \leqq n!\, 2\tilde{\sigma}^n \sum_{q=1}^{n-1} \frac{\tilde{\sigma}^{-q}}{q!} |g_q|, \tag{23}$$

which allows us to deduce recursively from Eq. (20),

$$|g_n| \leqq n!2(3\tilde{\sigma})^{n-1}, \qquad n=1,2,\dots. \tag{24}$$

Indeed, Eq. (24) is trivially true for g_1, and, if it is true for

$$g_1,\dots,g_{n-1}: |g_n| \leqq \tfrac{4}{3} n!\,\tilde{\sigma}^{n-1} \sum_{q=1}^{n-1} 3^q < 2n!(3\tilde{\sigma})^{n-1}.$$

Using now Eqs. (17) and (24) in Eq. (16), we obtain for $M \geqq 2$,

$$\begin{aligned} |\Gamma_M(u)| &\leqq \frac{1}{3\tilde{\sigma}} |\sigma' z|^M \sum_{m=1}^{M-1} m!(M-m+1)!(9A\tilde{\sigma})^m \\ &\leqq \frac{1}{3\tilde{\sigma}} |\sigma' z|^M \sum_{m=1}^{M-1} m!(M-m+1)!(9A\tilde{\sigma})^{M-1} \end{aligned} \tag{25}$$

[since $9A\tilde{\sigma} = 9(A\sigma')^2 \geqq 9$].

Noticing that $\sum_{m=1}^{M-1} m!(M-m+1)! \leqq 3(M!)$ and recalling Eq. (18), we arrive at

$$|\Gamma_M(u)| \leqq \frac{A}{9} M!|\sigma_1' z|^M, \qquad M=2,3,\dots, \tag{26}$$

where

$$\sigma_1' = 9A^2\sigma'^3. \tag{27}$$

It remains to bound the right-hand side of Eq. (26) in terms of u rather than z. This is easily achieved by observing that $u \in K(\lambda, R_1)$ implies both

$$|z-u| \leqq 2A\sigma'^2|z|^2 \quad [\text{since } z=g(u) \in K(\lambda, r)], \tag{28}$$

and

$$|u| < R_1 = r - 2A\sigma'^2 r^2 < \frac{1}{8A\sigma'^2} \quad [\text{in view of Eq. (12)}]. \tag{29}$$

Hence

$$|z| \leqq \frac{1-\sqrt{1-8A\sigma'^2|u|}}{4A\sigma'^2} < 2|u|, \tag{30}$$

and finally,

$$|\Gamma_M(u)| \leqq A_1(\lambda, R_1, \sigma') M!|2\sigma_1' u|^M \quad \forall u \in K(\lambda, R_1), \qquad M=0,1,2,\dots. \tag{31}$$

This means that

$$g \in N-(\lambda, R_1, 2\sigma_1') \quad \forall \sigma' > \sigma. \tag{32}$$

Therefore

$$g \in \mathcal{N}-(\lambda, R_1, \sigma_1), \tag{33}$$

with

$$\sigma_1 = 18 \inf_{\sigma' > \sigma} [A^2(\lambda, r, \sigma')\sigma'^3]. \tag{34}$$

The proof of the theorem is completed.

Let us conclude by two remarks.

1. In the previous proof, no attempt has been made to optimize the values of the parameters σ_1 and R_1. For instance, there are numerical indications that the factor of 3 appearing in Eq. (24) could be somewhat lowered. Also, the coefficient 2 in front of $|u|$ in Eq. (30) clearly can be made arbitrarily close to 1 by choosing R_1 small enough.

2. In [3], besides $\mathcal{N}-(\lambda, R, \sigma)$, a larger class of Borel summable functions was introduced, called $\mathcal{W}-(\lambda, R, \sigma)$. It turns out that no essentially new result concerning f^{-1} is obtained if f is taken in $\mathcal{W}-(\lambda, R, \sigma)$ rather than in $\mathcal{N}-(\lambda, R, \sigma)$. This is because

$$\mathcal{W}-(\lambda, R, \sigma) \subset \mathcal{N}(\lambda', R, \sigma/\sin(\lambda'-\lambda)) \quad \text{for all} \quad \lambda' > \lambda .$$

Appendix

Proof of the Inequalities

$$S_n^q < 2(n!/q!), \qquad n=1,2,\ldots; \qquad 1 \leqq q \leqq n . \tag{A.1}$$

We shall use the following recursive formula, which immediately follows from the definition (21)

$$S_n^q = \sum_{p=1}^{n-q+1} p!\, S_{n-p}^{q-1} \quad (2 \leqq q \leqq n) . \tag{A.2}$$

a) Equation (A.1) is true for $q=n$, $n-1$, $n-2$.

Indeed, $S_n^n = 1$, and for $q=n-1$ Eq. (A.2) reads $S_n^{n-1} = S_{n-1}^{n-2} + 2$, a recurrence easily solved to give

$$S_n^{n-1} = 2(n-1) < 2n . \tag{A.3}$$

One derives similarly

$$S_n^{n-2} = 2n(n-2) < 2n(n-1) . \tag{A.4}$$

b) Equation (A.1) is true for $q=1,2,3$,

$$S_n^1 = n! < 2(n!) \tag{A.5}$$

$$\begin{aligned} S_n^2 &= \sum_{p=1}^{n-1} p!(n-p)! = \text{sum of } (n-1) \text{ terms} \leqq (n-1)! \\ &\leqq (n-1)(n-1)! < n! \end{aligned} \tag{A.6}$$

$$\begin{aligned} S_n^3 &= S_{n-1}^2 + \sum_{p=2}^{n-2} p!\, S_{n-p}^2 \leqq (n-2)(n-2)! + \sum_{p=2}^{n-2} p!\, S_{n-p}^2 \\ &= (n-2)(n-2)! + \text{sum of } (n-3) \text{ terms } \leqq 2!(n-2)! \\ &< (n-2)(n-2)! + 2(n-3)(n-2)! = (3n-8)(n-2)! \\ &< \frac{3(3n-8)}{n(n-1)} \cdot \frac{n!}{3} \end{aligned} \tag{A.7}$$

Thus

$$S_n^3 < \frac{n!}{3}, \tag{A.8}$$

provided that $n \geqq 6$. But Eq. (A.8) is also true for $n=3,4,5$ because of a).

c) In the remaining interval $4 \leqq q \leqq n-3$, the proof proceeds inductively over q. Assuming Eq. (A.1) to be true for $S_m^{q-1} \forall m \geqq q-1$ and applying Eq. (A.2), one obtains

$$S_n^q < \frac{2}{(q-1)!}\{[(n-1)!+2!(n-2)!]+[3!(n-3)!+ \ldots +(n-q+1)!(q-1)!]\}. \tag{A.9}$$

Now, amongst the $(n-q-1)$ terms of the second bracket, the largest one is $3!(n-3)!$, since $(q-1) \geqq 3$. Hence

$$S_n^q < \left[\frac{q}{n} + \frac{2q}{n(n-1)} + \frac{6q(n-q-1)}{n(n-1)(n-2)}\right] 2\frac{n!}{q!}, \tag{A.10}$$

so that we only need to show that the bracket of Eq. (A.10) is not larger than 1. By eliminating n in favor of $r = n-q$, the required inequality takes the form:

$$(r-2)q^2 + (2r^2 - 11r + 10)q + r(r-1)(r-2) \geqq 0 \quad \forall q \geqq 4, \quad r \geqq 3. \tag{A.11}$$

That Eq. (A.11) is true is obvious if one rearranges it as

$$(r-2)[q(q-5)+r(r-1)] + 2qr(r-3) \geqq 0. \tag{A.12}$$

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