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Existence of Stark Ladder Resonances

Jim Agler[1,2] and Richard Froese

Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA

Abstract. We show that resonances, in the translation analytic sense of Herbst and Howland, exist for the one dimensional Stark Hamiltonian, $-d^2/dx^2+q(x)+\varepsilon x$, with $q(x)$ a trigonometric polynomial, provided ε is sufficiently large.

1. Introduction

The problem of describing the motion of a particle in a one dimensional periodic solid pervaded by a uniform electric field has received considerable attention in the physics literature (see, e.g. [1,10,11]). Controversy has centered about the existence of so-called Wannier states, or Stark ladder resonances, which were described by Wannier in [9]. The purpose of this paper is to prove that for periodic potentials given by trigonometric polynomials resonances in the translation analytic sense of Herbst and Howland [5] exist for large values of the electric field.

The Hamiltonian for the system in question is

$$H(\varepsilon)=-\frac{d^2}{dx^2}+\varepsilon x+q(x),$$

acting in $L^2(\mathbb{R})$, where $q(x)=\sum_n c_n e^{inx}$ is a real valued trigonometric polynomial. Here ε is the strength of the electric field. It is known that for $\varepsilon\neq 0, \sigma(H(\varepsilon))=\mathbb{R}$ and is purely absolutely continuous [3,4]. To describe translation analyticity we begin with the unitary group of translations

$$(T(a)f)(x)=f(x+a) \tag{1.1}$$

for $a\in\mathbb{R}$ and note that

$$T(-a)H(\varepsilon)T(a)=-\frac{d^2}{dx^2}+\varepsilon x+q(x-a)-\varepsilon a.$$

Since $q(x)$ has an analytic extension to complex x we can define the complex

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[1] Permanent Address: UCSD—San Diego, CA 92093, USA

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translated Hamiltonian

$$H(\varepsilon,\alpha) = -\frac{d^2}{dx^2} + \varepsilon x + q(x - i\alpha) - i\varepsilon\alpha$$

for $\alpha \in \mathbb{R}$. In the case at hand, since $q(x - i\alpha)$ is bounded in x, $H(\varepsilon, \alpha)$ can be defined as a closed operator on $\mathscr{D}(p^2 + \varepsilon x)$. Herbst and Howland prove (for a much larger class of q) that $\sigma_{\text{ess}}(H(\varepsilon,\alpha)) \subseteq \mathbb{R} - i\varepsilon\alpha$ and $\sigma_{\text{disc}}(H(\varepsilon,\alpha)) \subseteq \{z: -i\varepsilon\alpha \leqq \operatorname{Im} z \leqq 0\}$. The eigenvalues in $\sigma_{\text{disc}}(H(\varepsilon, \alpha)) \backslash (\mathbb{R} - i\varepsilon\alpha)$ are the resonances. They don't move as α changes once they have been "uncovered" by the line containing essential spectrum. They can also be shown to be poles of suitable matrix elements of the resolvent of $H(\varepsilon)$ continued into the lower half plane. What we prove is that for large values of $\varepsilon, H(\varepsilon,\alpha)$ indeed has eigenvalues not on the line $\mathbb{R} - i\varepsilon\alpha$, for some value of α. Since there is the unitary equivalence

$$T(-2\pi n)H(\varepsilon, \alpha)T(2\pi n) = H(\alpha, \varepsilon) - 2\pi\varepsilon n$$

for $n \in \mathbb{Z}$, the existence of one resonance implies the existence of a whole sequence, the so-called Stark ladder.

We briefly mention some of the folklore surrounding this problem. The spectrum of $H(0)$ has the familiar band structure. Let P_n project onto the n^{th} band subspace of $H(0)$. Then for $\varepsilon \neq 0$ the operators $P_n H(\varepsilon) P_n$ and thus $\sum_n P_n H(\varepsilon) P_n$ have discrete spectrum. It is thought these bound states should persist as resonances when the off-diagonal parts $P_n H(\varepsilon) P_m, n \neq m$ are added to $\sum_n P_n H(\varepsilon) P_n$ to give $H(\varepsilon)$. Previous work has centered about proving that the off-diagonal parts are small as $\varepsilon \downarrow 0$ [2,7]. So far this approach has not led to a proof of the existence of Stark resonances.

We make no use of the above intuition. Instead we introduce an operator inspired by the Birman–Schwinger operators common in Schrödinger operator theory. The fact that this operator is compact follows from work in [4].
Our results are summarized in the following theorem.

Theorem 1. *Let $q(x)$ be the trigonometric polynomial given by*

$$q(x) = \sum_{n=-N}^{N} c_n e^{inx}, \quad c_n \in \mathbb{C},$$

where $c_0 = 0$, and $c_n \neq 0$ for at least one $n > 0$. Let $p = -id/dx$. For $\varepsilon > 0$ and $\alpha \in \mathbb{R}$ define the closed operator

$$H(\varepsilon, \alpha) = p^2 + \varepsilon x + q(x - i\alpha) - i\varepsilon\alpha \tag{1.2}$$

acting in $L^2(\mathbb{R})$ with domain $\mathscr{D}(H(\varepsilon,\alpha)) = \mathscr{D}(p^2 + \varepsilon x)$. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon > \varepsilon_0$ there exists $\alpha \in \mathbb{R}$ such that $H(\alpha,\varepsilon)$ has an eigenvalue not contained in the line $\mathbb{R} - i\varepsilon\alpha$.

In this theorem $q(x)$ need not be real on the real axis. However if it happens to be real on the real axis, then the results of Herbst and Howland apply and Theorem 1 asserts the existence of Stark resonances. We suspect, but cannot prove, that in fact ε_0 may be taken to be zero in Theorem 1.

2. *A Birman–Schwinger Type Operator*

Let the symbol $[\cdot]^{1/2}$ denote a branch of the square root, fixed throughout the rest of the paper, which is analytic in the lower half plane. Throughout this paper $\varepsilon > 0$ and $\operatorname{Im}\mu > 0$. Define

$$M(\varepsilon,\mu) = [p^2 + \varepsilon x - \mu]^{-1/2}, \tag{2.1}$$

where $p = -id/dx$. Let

$$S(\varepsilon,\mu,z) = M(\varepsilon,\mu) \sum_{n=-N}^{N} c_n z^n e^{inx} \quad M(\varepsilon,\mu), \tag{2.2}$$

where the c_n are the coefficients in $q(x)$ (see Theorem 1). We have the following analogue of the Birman–Schwinger principle.

Lemma 1. *Fix* $\alpha \in \mathbb{R}$ *and* $\varepsilon > 0$. *Let* $S(\varepsilon,\mu,z)$ *be as above and let* $H(\varepsilon,\alpha)$ *be given by (1.2). Then* $H(\varepsilon,\alpha)$ *has an eigenvalue at* λ iff $S(\varepsilon,\mu,z)$ *has an eigenvalue at* -1, *with*

$$z = e^{\alpha + ia}, \quad \mu = \lambda + i\varepsilon\alpha + \varepsilon a$$

for some choice of $a \in \mathbb{R}$.

Proof. Suppose $H(\varepsilon,\alpha)$ has eigenvalue λ. This means there exists $\psi \in \mathscr{D}(H(\varepsilon,\alpha))$ such that

$$H(\varepsilon,\alpha)\psi = \lambda\psi. \tag{2.3}$$

This equation can be rewritten as

$$M^{-2}\psi = -q(x - i\alpha)\psi, \tag{2.4}$$

where $M = M(\varepsilon,\lambda + i\varepsilon\alpha)$. Here we use that $\mathscr{D}(H(\varepsilon,\alpha)) = \mathscr{D}(M^{-2}(\varepsilon,\lambda + i\varepsilon\alpha))$. Using (2.4) it is easy to see that $\phi = M^{-1}\psi$ satisfies

$$S(\varepsilon,\lambda + i\varepsilon\alpha, e^{\alpha})\phi = -\phi. \tag{2.5}$$

Conversely, if (2.5) holds for some $\phi \in L^2$, i.e. if $S(\varepsilon,\lambda + i\varepsilon\alpha, e^{\alpha})$ has eigenvalue -1, then it follows that $\psi = M\phi$ is in $\mathscr{D}(H(\varepsilon,\alpha))$ and satisfies (2.3).

For any $a \in \mathbb{R}$ we have the unitary equivalence

$$T(-a)S(\varepsilon,\mu,z)T(a) = S(\varepsilon,\mu + \varepsilon a, e^{ia}z),$$

where $T(a)$ is given by (1.1). Thus $S(\varepsilon,\lambda + i\varepsilon\alpha, e^{\alpha})$ has eigenvalue -1 iff $S(\varepsilon,\lambda + i\varepsilon\alpha + \varepsilon a, e^{\alpha + ia})$ has eigenvalue -1 for some $a \in \mathbb{R}$. The proof is complete.

Given Lemma 1, the following theorem implies Theorem 1.

Theorem 2. *There exists* μ_0 *and* ε_0 *with* $\operatorname{Im}\mu_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for every* $\varepsilon > \varepsilon_0$, $S(\varepsilon,\mu_0,z)$ *has eigenvalue* -1 *for some value of* z.

We conclude this section with an outline of the proof of Theorem 2. We will show that $S(\varepsilon,\mu,z) \in \mathscr{I}_4$, the fourth trace ideal class (see [8] for definitions and properties of trace ideals). It then follows that $S(\varepsilon,\mu,z)$ has eigenvalue -1 iff $\det_4(1 + S(\varepsilon,\mu,z))$ vanishes, where $\det_4$ is the regularized determinant. We will show that for some fixed μ_0 with $\operatorname{Im}\mu_0 > 0$, $\ln\det_4(1 + S(\varepsilon,\mu_0,z))$ is analytic in z

for $|z| < 1$. Thus for $|z| < 1$

$$\ln \det_4(1 + S(\varepsilon, \mu_0, z)) = \sum_{\ell=0}^{\infty} f_\ell(\varepsilon) z^\ell. \tag{2.6}$$

Let $\mathscr{E}$ denote the set of ε for which $S(\varepsilon, \mu_0, z)$ never has eigenvalue -1, as z ranges through $\mathbb{C}\backslash\{0\}$ If $\varepsilon \in \mathscr{E}$ then $\det_4(1 + S(\varepsilon, \mu, z))$ never vanishes, so the left side of (2.6) is entire in z and it will be shown that in that case that case the left side of (2.6) is in fact a polynomial of degree at most $4N_1$. Thus, $\mathscr{E}$ must lie in the zero set of f_ℓ for all $\ell > 4N_1$. A calculation showing f_{5N_1} does not vanish for large ε will complete the proof.

3. Proof of Theorem 2

Instead of dealing with $S(\varepsilon, \mu, z)$ directly we work with an operator unitarily equivalent to it. For $\varepsilon > 0$ let $U(\varepsilon) = \exp(ip^3/3\varepsilon)$. This operator is common in the theory of Stark Hamiltonians as it has the property that

$$U(\varepsilon)^*(p^2 + \varepsilon x)U(\varepsilon) = \varepsilon x.$$

Define

$$R(\varepsilon, \mu, z) = U(\varepsilon)^* S(\varepsilon, \mu, z) U(\varepsilon).$$

Using the identities

$$U(\varepsilon)^* M(\varepsilon, \mu) U(\varepsilon) = [\varepsilon x - \mu]^{-1/2},$$

$$e^{-i\alpha x} f(p) e^{i\alpha x} = f(p + \alpha), \quad \text{for } f \text{ bounded and measurable,} \tag{3.1}$$

and

$$\begin{aligned} U(\varepsilon)^* e^{inx} U(\varepsilon) &= e^{i(n/2)x} \exp\left(-i\left(p + \frac{n}{2}\right)^3 \Big/ 3\varepsilon\right) \exp\left(i\left(p - \frac{n}{2}\right)^3 \Big/ 3\varepsilon\right) e^{i(n/2)x} \\ &= \exp(-in^3/12\varepsilon) e^{i(n/2)x} \exp\left(-\frac{in}{\varepsilon} p^2\right) e^{i(n/2)x}, \end{aligned}$$

it is easy to see that

$$R(\varepsilon, \mu, z) = \sum_{n=-N}^{N} c_n z^n K_n(\varepsilon, \mu), \tag{3.2}$$

where

$$K_n(\varepsilon, \mu) = e^{-in^3/12\varepsilon} [\varepsilon x - \mu]^{-1/2} e^{i(n/2)x} \exp\left(i - \frac{n}{\varepsilon} p^2\right) e^{i(n/2)x} [\varepsilon x - \mu]^{-1/2}. \tag{3.3}$$

Since R has the same eigenvalues as S it will suffice to prove Theorem 2 with R in place of S. The following lemma uses the method of computing integral kernels introduced in [4].

Lemma 2. *For* $\varepsilon > 0$, $\operatorname{Im}\mu > 0$ *and* $n \neq 0$, $K_n(\varepsilon,\mu) \in \mathscr{I}_4$ *and*

$$\| K_n(\varepsilon,\mu) \|_4^4 \leqq \frac{\pi}{4|n|\varepsilon(\operatorname{Im}\mu)^2}.$$

Proof. Let $\|\cdot\|_p$ denote the $\mathscr{I}_p$ norm. Then $\|K\|_4^4 = \|K^*K\|_2^2$, so it suffices to show that K^*K is Hilbert–Schmidt for $K = K_n(\varepsilon,\mu)$. Since (see [6], p 496) $\exp(-itp^2)$ has integral kernel

$$\exp(-itp^2)(x,y) = (4\pi it)^{-1/2} \exp(i(x-y)^2/4t), \tag{3.4}$$

the integral kernel for $K = K_n(\varepsilon,\mu)$ is

$$K(x,y) = \left(\frac{4\pi in}{\varepsilon}\right)^{-1/2} e^{-in^3/12\varepsilon}[\varepsilon x - \mu]^{-1/2} e^{i(n/2)x} \cdot \exp(i\varepsilon(x-y)^2/4n) e^{i(n/2)y}[\varepsilon y - \mu]^{-1/2}.$$

Thus K^*K has integral kernel

$$K^*K(x,y) = \int_{-\infty}^{\infty} \overline{K(u,x)} K(u,y)\,du = \left|\frac{\varepsilon}{4\pi n}\right| \overline{[\varepsilon x - \mu]^{-1/2}} [\varepsilon y - \mu]^{-1/2}$$
$$\cdot \exp\left(i\left(\frac{\varepsilon}{4n}(y^2 - x^2) + \frac{n}{2}(y-x)\right)\right)$$
$$\cdot \int_{-\infty}^{\infty} |\varepsilon x - \mu|^{-1} \exp\left(i\frac{\varepsilon}{2n}(x-y)u\right) du.$$

Now set $f(x) = |\varepsilon x - \mu|^{-1}$ and let $\hat{f}$ denote the Fourier transform of f. Then

$$\|K^*K\|_2^2 = \frac{\varepsilon^2}{16\pi^2 n^2} \cdot 2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\hat{f}\left(\frac{\varepsilon}{2n}(x-y)\right)\hat{f}\left(\frac{\varepsilon}{2n}(y-x)\right) dx\,dy$$
$$\leqq \frac{\varepsilon^2}{8\pi n^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x)|^2 \left|\hat{f}\left(\frac{\varepsilon}{2n}(x-y)\right)\right|^2 dx\,dy = \frac{\varepsilon}{4\pi|n|}\|f\|^4.$$

Since $\|f\|^2 = \pi/\varepsilon \operatorname{Im}\mu$, this concludes the proof of Lemma 2.

The next lemma plays a technical role in our proof.

Lemma 3. *Let* $m \geqq 4$ *and let* $n_1 \ldots, n_m$ *be non-zero integers. Let* $K_{n_j} = K_{n_j}(\varepsilon,\mu)$ *for* $\varepsilon > 0$ *and* $\operatorname{Im}\mu > 0$ *be given by (3.2). Define* $\sigma = \sum_{j=1}^{n} n_j$.

Then

$$\operatorname{tr}\left(\prod_{j=1}^{m} K_{n_j}\right) = \begin{cases} 0 & if\, \sigma \leqq 0 \\ \dfrac{2\pi(i)^m}{(4\pi i)^{1/2}} \varepsilon^{-m+1/2} \quad \sigma^{m-3/2} \exp\left(i\dfrac{\sigma\mu}{\varepsilon}\right) & \\ \cdot \exp\left(-\dfrac{i}{12\varepsilon}\displaystyle\sum_{j=1}^{m} n_j^3\right) \displaystyle\int_{\Omega_m} \exp\left(-\dfrac{i}{\varepsilon} C(m;n_j;u_j)\right) du_1,\ldots,du_{m-1} & if\, \sigma \geqq 0, \end{cases}$$

where Ω_m is the region

$$\Omega_m = \{(u_1,\ldots,u_{m-1}): u_i \geqq 0 \text{ for } i=1,\ldots,m-1 \text{ and } \sum_{i=1}^{m-1} u_i \leqq 1\},$$

α_k and β_k are given by (3.6) below, and

$$C(m;n_j;u_j) = \sigma^2 \sum_{k=1}^{m} n_k\left(-\tfrac{1}{2}\beta_k + \tfrac{1}{2}\alpha_k + \sum_{j=1}^{k-1} \beta_j u_j - \sum_{j=k}^{m-1} \alpha_j u_j\right)^2.$$

Proof. From the definition of K_{n_j}, it follows that

$$\prod_{j=1}^{m} K_{n_j} = \exp\left(-\frac{i}{12\varepsilon}\sum_{j=1}^{m} n_j^3\right)[\varepsilon x - \mu]^{-1/2} \prod_{j=1}^{m-1}\left[e^{i(n_j/2)x} \cdot \exp\left(-\frac{i}{\varepsilon} n_j p^2\right) e^{i(n_j/2)x} (\varepsilon x - \mu)^{-1}\right]$$
$$\cdot e^{i(n_m/2)x} \exp\left(-\frac{i}{\varepsilon} n_m p^2\right) e^{i(n_m/2)x} [\varepsilon x - \mu]^{-1/2}.$$

Using the representation

$$(\varepsilon x - \mu)^{-1} = i\varepsilon^{-1} \int_0^\infty e^{-ix\xi} e^{i(\mu\xi/\varepsilon)} d\xi,$$

and then repeatedly applying formula (3.1) we obtain

$$\begin{aligned}
\prod_{j=1}^{m} K_{n_j} = {} & (i)^{m-1}\varepsilon^{-m+1} \exp\left(-\frac{i}{\varepsilon}\sum_{j=1}^{m} n_j^3/12\right)[\varepsilon x - \mu]^{-1/2} \\
& \cdot \int_0^\infty \cdots \int_0^\infty \exp\left(i\frac{\mu}{\varepsilon}\sum_{j=1}^{m-1}\xi_j\right) \exp\left(i\left(\sum_{j=1}^{m}\frac{n_j}{2} - \sum_{j-1}^{m-1}\alpha_j\xi_j\right)x\right) \qquad (3.5) \\
& \cdot \exp\left(-\frac{i}{\varepsilon}\left\{n_1\left(p + 0 + \frac{n_2}{2} + \frac{n_3}{2} \overset{\text{etc}}{\cdots} + \frac{n_{m-1}}{2} + \frac{n_m}{2} - \alpha_1\xi_1 \right.\right.\right. \\
& \left. - \alpha_2\xi_2 - \cdots - \alpha_{m-1}\xi_{m-1}\right)^2 \\
& + n_2\left(p - \frac{n_1}{2} + 0 + \frac{n_3}{2} + \cdots + \frac{n_{m-1}}{2} + \frac{n_m}{2} + \beta_1\xi_1 - \alpha_2\xi_2 \right. \\
& \left. - \cdots - \alpha_{m-1}\xi_{m-1}\right)^2 \\
& + \cdots \\
& \cdot \\
& \cdot \\
& \cdot
\end{aligned}$$

$$+n_m\left(p-\frac{n_1}{2}-\frac{n_2}{2}-\frac{n_3}{2}-\cdots-\frac{n_{m-1}}{2}+0\right.$$
$$\left.\left.\left.+\beta_1\xi_1+\beta_2\xi_2+\cdots+\beta_{m-1}\xi_{m-1}\right)^2\right\}\right)$$
$$\cdot\exp\left(i\left(\sum_{j=1}^{n}\frac{n_j}{2}-\sum_{j=1}^{m-1}\beta_j\xi_j\right)x\right)d\xi_1\ldots d\xi_{m-1}[\varepsilon x-\mu]^{-1/2},$$

where α_k, β_k are any pairs of real numbers satisfying $\alpha_k+\beta_k=1$ for each k. The expression inside the braces is quadratic in p. It can be written Ap^2+Bp+C with $A=\sigma$. Now assume $\sigma\neq 0$ and let

$$\alpha_k=\sigma^{-1}\sum_{i=k+1}^{m}n_i,$$
$$\beta_k=\sigma^{-1}\sum_{i=1}^{k}n_i. \tag{3.6}$$

With this choice of α_k and $\beta_k, B=0$ and

$$C=\sum_{k=1}^{m}n_k\left(-\frac{\sigma}{2}\beta_{k-1}+\frac{\sigma}{2}\alpha_k+\sum_{j=1}^{k-1}\beta_j\xi_j-\sum_{j=k}^{m-1}\alpha_j\xi_j\right)^2,$$

where all undefined sums, e.g. α_m and β_0, are equal to zero.

Using (3.4) and (3.5) one can compute an integral kernel for ΠK_{n_j}. Since the resultant kernel is continuous, the trace of ΠK_{n_j} is equal to the integral along the diagonal of this kernel. Thus

$$\operatorname{tr}\left(\prod_{j=1}^{m}K_{n_j}\right)=(i)^{m-1}\varepsilon^{m+1}\left(\frac{4\pi i\sigma}{\varepsilon}\right)^{-1/2}\exp\left(-\frac{i}{12\varepsilon}\sum_{j=1}^{m}n_j^3\right)$$
$$\cdot\int_{-\infty}^{\infty}(\varepsilon x-\mu)^{-1}\int_0^{\infty}\cdots\int_0^{\infty}\exp\left(i\frac{\mu}{\varepsilon}\sum_{j=1}^{m-1}\xi_j\right)$$
$$\cdot\exp\left(i\left(\sigma-\sum_{j=1}^{m-1}\xi_j\right)x\right)\exp\left(-\frac{i}{\varepsilon}C\right)d\xi_1\cdots d\xi_{m-1}dx. \tag{3.7}$$

We split the integral over $\xi_1\cdots\xi_{m-1}$ into two parts. Let

$$F_1(x)=\underset{\substack{\xi_j\geqq 0\\ \Sigma\xi_j\leqq\sigma}}{\int\cdots\int}I(\xi_i,x)d\xi_1\cdots d\xi_{m-1}, \tag{3.8}$$
$$F_2(x)=\underset{\substack{\xi_j\geqq 0\\ \Sigma\xi_j\geqq\sigma}}{\int\cdots\int}I(\xi_i,x)d\xi_1\cdots d\xi_{m-1}, \tag{3.9}$$

where $I(\xi_i,x)$ is the integrand from (3.7), i.e.

$$I(\xi_i,x)=\exp\left(i\frac{\mu}{\varepsilon}\sum_{j=1}^{m-1}\xi_j\right)\exp\left(i\left(\sigma-\sum_{j=1}^{m-1}\xi_j\right)x\right)\exp\left(-\frac{i}{\varepsilon}C\right).$$

We claim that

$$\int_{-\infty}^{\infty}(\varepsilon x-\mu)^{-1}F_2(x)dx=0. \tag{3.10}$$

First, note that $F_2(x)$ has an analytic extension to the half plane, $\operatorname{Im} x < \operatorname{Im}(\mu/\varepsilon)$. Since $\operatorname{Im}\mu > 0$, (3.10) will follow by Cauchy's theorem if we can show that

$$|F_2(x)| \leqq C|x|^{-1} \tag{3.11}$$

for large $|x|$ in the lower half-plane. If $\sigma < 0$, then

$$|F_2(x)| \leqq \int_{\xi_j \geqq 0} \cdots \int |I(\xi_j; x)| d\xi_1 \cdots d\xi_{m-1} = e^{-\sigma \operatorname{Im} x}\left(\operatorname{Im}\left(\frac{\mu}{\varepsilon} - x\right)\right)^{-m+1},$$

and (3.10) follows trivially.

We now consider the case $\sigma > 0$. In (3.9) the change of variables $\xi_j = \rho\omega_j$ for $1 \leqq j \leqq m-2$ and $\xi_{m-1} = \rho\left(1 - \sum_{j=1}^{m-2} \omega_j\right)$ yields

$$F_2(x) = \int_\sigma^\infty f(\sigma)\exp(i\sigma x)\exp\left(i\left(\frac{\mu}{\varepsilon} - x\right)\rho\right)d\rho, \tag{3.12}$$

where

$$f(\rho) = \rho^{m-2} \int_{\substack{\omega_j \geqq 0 \\ \Sigma\omega_j \leqq 1}} \cdots \int \exp\left(-\frac{i}{\varepsilon}C\right)d\omega_1 \cdots d\omega_{m-2}.$$

In this formula C is a quadratic polynomial in ρ. Thus,

$$|f'(\rho)| \leqq C_1 + C_2\rho^{m-1}.$$

Now integrate the right side of (3.12) by parts. This gives

$$F_2(x) = -\frac{f(\sigma)\exp(i\sigma\mu/\varepsilon)}{i(\mu/\varepsilon - x)} - \frac{\exp(i\sigma x)}{i(\mu/\varepsilon - x)} \int_\sigma^\infty f'(\rho)\exp(i(\mu/\varepsilon - x)\rho)d\rho.$$

Since for $\operatorname{Im} x < 0$,

$$\begin{aligned} &\left|\exp(i\sigma x)\int_\sigma^\infty f'(\rho)\exp(i(\mu/\varepsilon - x)\rho)d\rho\right| \\ &\leqq \exp(-\sigma \operatorname{Im} x)\int_\sigma^\infty (C_1 + C_2\rho^{m-1})\exp((-\operatorname{Im}(\mu/\varepsilon) + \operatorname{Im} x)\rho)d\rho \\ &\leqq C, \end{aligned}$$

Eq. (3.13) implies (3.11). This proves (3.10) when $\sigma \neq 0$. It also establishes Lemma 3 for the case $\sigma < 0$, since, if $\sigma < 0$, then $F_1(x) = 0$. To handle the case $\sigma > 0$ note that $F_1(x)$ extends to be entire in x. By an analysis similar to the one above we obtain

$$F_1(x) = \frac{f(\sigma)\exp(i\sigma\mu/\varepsilon)}{i(\mu/\varepsilon - x)} - \frac{\exp(i\sigma x)}{i(\mu/\varepsilon - x)}\int_0^\sigma f'(\rho)\exp(i(\mu/\varepsilon - x)\rho)d\rho.$$

Since for $\operatorname{Im} x > 0$,

$$\left| \exp(i\sigma x) \int_0^\sigma f'(\rho) \exp(i(\mu/\varepsilon - x)\rho) d\rho \right|$$

$$\leqq \exp(-\sigma \operatorname{Im} x)(C_1 + C_2\sigma^{m-1}) \int_0^\sigma \exp((-\operatorname{Im}\mu/\varepsilon + \operatorname{Im} x)\rho) d\rho$$

$$\leqq C,$$

it follows that $F_1(x)$ decreases rapidly enough in the upper half-plane to allow the evaluation of the integral of $(\varepsilon x - \mu)^{-1} F_1(x)$ by a contour in the upper half-plane. Since $(\varepsilon x - \mu)^{-1}$ has a simple pole there, this gives

$$\int_{-\infty}^{\infty} (\varepsilon x - \mu)^{-1} F_1(x) dx = 2\pi i \varepsilon^{-1} F_1(\mu/\varepsilon).$$

Using (3.7) through (3.10) and making the change of variables $\sigma u_i = \xi_i$ concludes the proof of Lemma 3 for the case $\sigma > 0$.

The case $\sigma = 0$ requires a separate argument. We omit the details, noting that the precise value of the trace when $\sigma = 0$ is not crucial in what follows.

From Lemma 2 it follows that for $\varepsilon > 0$, $\operatorname{Im}\mu > 0$ and $z \neq 0, R(\varepsilon, \mu, z) \in \mathscr{I}_4$. Thus $\det_4(1 + R)$ is well defined. It follows easily from (3.2) that

$$\| R(\varepsilon, \mu, z) \| \leqq C(|z|^N + |z|^{-N})(\operatorname{Im}\mu)^{-1}.$$

Now choose $M_0 > 0$ and a neighbourhood $\mathscr{N}$ of $|z| = 1$, and that for $\operatorname{Im}\mu > M_0$ and $z \in \mathscr{N}$, $\| R(\varepsilon, \mu, z) \| < 1$. Then for such μ and z, $\ln \det_4(1 + R(\varepsilon, \mu, z))$ is well defined and analytic in z.

Lemma 4. *If* $\varepsilon > 0$ *and* $\operatorname{Im}\mu > M_0$, *then* $\ln \det_4(1 + R(\varepsilon, \mu, z))$ *extends to be analytic in* $|z|$ *for* $|z| < 1$.

Proof. If $\operatorname{Im}\mu > M_0$ and $z \in \mathscr{N}$ then $\| R(\varepsilon, \mu, z) \| < 1$. Thus the eigenvalues $\{\lambda_k\}$ of R must satisfy $\sup_k |\lambda_k| < 1$. Using this fact and Theorem 9.2(a) from [8], it is not difficult to see that

$$\ln \det_4(1 + R(\varepsilon, \mu, z)) = \sum_{m=4}^{\infty} \frac{(-1)^{m+1}}{m} \operatorname{tr}(R(\varepsilon, \mu, z)^m), \tag{3.14}$$

where the convergence is uniform for $z \in \mathscr{N}$.

By Lemma 3, $\operatorname{tr}(R(\varepsilon, \mu, z)^m)$ extends to be analytic on $|z| < 1$, so the conclusion of Lemma 2 follows by the maximum modulus principle.

For the remainder of this section fix μ_0 with $\operatorname{Im}\mu_0 > M_0$. Define the functions $f_\ell(\varepsilon)$ for $\varepsilon > 0$ by

$$\ln \det_4(1 + R(\varepsilon, \mu_0, z)) = \sum_{\ell=1}^{\infty} f_\ell(\varepsilon) z^\ell \tag{3.15}$$

for $|z| < 1$. Let $\mathscr{E}$ denote the set of ε for which $R(\varepsilon, \mu_0, z)$ never has eigenvalue -1 as z ranges through $\mathbb{C} \setminus \{0\}$.

Let N_1 be the largest integer such that $c_{N_1} \neq 0$, where c_n denotes the coefficient in $q(x)$.

Lemma 5. *If* $\varepsilon \in \mathscr{E}$ *then* $\ln \det_4(1+R(\varepsilon,\mu,z))$ *is a polynomial in* z *of degree at most* $4N_1$, *i.e.* $f_\ell(\varepsilon)=0$ *for* $\ell > 4N_1$.

Proof. If $\varepsilon \in \mathscr{E}$ then $\ln \det_4(1+R(\varepsilon,\mu_0,z))$ extends to be entire in z. Using Lemma 2 and Theorem 9.2 (b) from [8] we see that

$$|\ln \det_4(1+R(\varepsilon,\mu_0,z))| \leqq C_1 \| R(\varepsilon,\mu_0,z)\|_4^4$$
$$\leqq C_2(|z|^{4N_1}+|z|^{-4N})\varepsilon^{-1}(\operatorname{Im}\mu_0)^{-2}.$$

These two facts imply the conclusion of Lemma 5.
Lemma 5 shows that $\mathscr{E}$ must lie in the zero set of $f_\ell(\varepsilon)$ for every $\ell > 4N_1$. Thus the following lemma will complete the proof of Theorem 2.

Lemma 6. *The function* $f_{5N_1}(\varepsilon)$ *satisfies*

$$\lim_{\varepsilon\to\infty} \varepsilon^{5-1/2} f_{5N_1}(\varepsilon) \neq 0.$$

Proof. From (3.2), (3.14), (3.15) and Lemma 3 it follows that

$$\varepsilon^{5-1/2} f_{5N_1}(\varepsilon) = \varepsilon^{5-1/2} \sum_{m=4}^{\infty} \sum_{\substack{-N\leqq n_1,\ldots,n_m\leqq N_1\\ \Sigma n_j=5N_1}} \frac{(-1)^{m+1}}{m}$$
$$\cdot c_{n_1}\cdots c_{n_m} \operatorname{tr}(K_{n_1}\cdots K_{n_m})$$
$$= \sum_{m=5}^{\infty} \sum_{\substack{-N\leqq n_1,\ldots,n_m\leqq N_1\\ \Sigma n_j=5N_1}} \frac{(-1)^{m+1}}{m}$$
$$\cdot c_{n_1}\cdots c_{n_m}\frac{2\pi(i)^m}{(4\pi i)^{1/2}}\varepsilon^{5-m}(5N_1)^{m-3/2}\exp(i5N_1\mu_0/\varepsilon)$$
$$\cdot\exp-\left(\frac{i}{12\varepsilon}\sum_{j=1}^{m} n_j\right)\int_{\Omega_m}\exp\left(-\frac{i}{\varepsilon}C(m;n_i;u_i)\right)du_1\ldots du_{m-1}.$$

The interchange of sums used to derive this formula can be justified by noting that the series obtained from the substitution of (3.2) in (3.14) converges absolutely for small $|z|$. To obtain the relevant estimate, use Lemma 3 to express $\operatorname{tr}(K_{n_1}\cdots K_{n_m})$ and the fact that $|\Omega_m| = ((m-1)!^{-1}$. We also used the fact that if $\sum_{j=1}^{m} n_j = 5N_1$ with $n_j < N_1$, then m must be at least 5. The series in m can be bounded by a fixed ℓ_1 sequence for ε large. Thus, we can take the limit inside the sum. Only the $m=5$ term survives:

$$\lim_{\varepsilon\to\infty} \varepsilon^{5-1/2} f(\varepsilon) = \frac{(-1)^6}{5} c_{N_1}^5 \frac{2\pi(i)^5}{(4\pi i)^{1/2}}(5N_1)^{5-3/2}(4!)^{-1} \neq 0.$$

This establishes Lemma 6.

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