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Effective Actions and Large-N Limits

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Abstract. The saddlepoint action for a large-N theory is given as an effective action for composite operators. This effective action is computed explicitly for $O(N)$ models and as a series in large-N invariants for matrix models. In the latter case, the use of the first term of the series is found to give good numerical agreement with the exact solutions of the solvable models.

Introduction

A very appealing approach to the solution of quantum field theories is the $\frac{1}{N}$ expansion [0]. One assumes that the number of fields is a power of a parameter N and that the coupling constants scale appropriately to give a large-N limit for correctly normalized expectations. It is then possible to expand the expectations as series in $\frac{1}{N}$. This nonperturbative approach to field theories gives rich physical information in the solvable examples. An interesting feature of the $N \to \infty$ limit is that it is given by extremizing a saddle point functional.

When the number of fields scales as N^1, this functional can be found by path integral methods. In the case of most physical interest, in which the number scales as N^2 (as in QCD), the $N \to \infty$ limit is unknown. We wish to explore this functional in both cases.

As an example of the N^1 case, consider the field space $\{\phi^i : \mathbb{R}^n \to \mathbb{R}^N\}$ and the (Minkowski space) Lagrangian

$$L(\phi^i) = \int \left[\frac{1}{2} (\partial \phi^i)^2 - N V\left(\frac{1}{N} \phi^i \cdot \phi^i \right) \right] d^n X .$$

We wish to find the normalized free energy

$$Z = -i \frac{1}{N} \int \exp(i\mathscr{L}(\phi^i)) \mathscr{D}\phi^i .$$

As in [1], we introduce a bilinear composite field $\chi(X, Y)$. Then

$$Z = -i\frac{1}{N}\int \exp(i\mathscr{L}(\phi^i))\,(\Pi\,\delta(N\chi(X,Y) - \phi^i(X)\cdot\phi^i(Y))\mathscr{D}\phi^i\mathscr{D}\chi$$

$$= -i\frac{1}{N}\int \exp i\Big[\int -\frac{N}{2}(\partial^2\delta)(X-Y)\,\chi(Y,X)d^nXd^nY - N\int V(\chi(X,X))d^nX$$

$$+\frac{1}{2}\int\lambda(X,Y)$$

$$\cdot(\phi^i(Y)\cdot\phi^i(X) - N\chi(Y,X))d^nX\,d^nY\Big]\mathscr{D}\phi^i\mathscr{D}\chi\mathscr{D}\lambda$$

$$= -i\frac{1}{N}\int \exp iN\Big[\int -\frac{1}{2}(\partial^2\delta)(X-Y)\chi(Y,X)d^nX\,d^nY - \int V(\chi(X,X))d^nX$$

$$-\frac{1}{2}\int\lambda(X,Y)\chi(Y,X)d^nXd^nY + \frac{1}{2}i\ln\det\lambda\Big]\mathscr{D}\chi\mathscr{D}\lambda,$$

the determinant being taken of the operator λ with operator kernel $\lambda(X, Y)$. For large N, Z is given by stationarizing

$$\int -\tfrac{1}{2}(\partial^2\delta)(X-Y)\chi(Y,X)d^nXd^nY - \int V(\chi(X,X))d^nX$$

$$-\tfrac{1}{2}\int\lambda(X,Y)\chi(Y,X)d^nXd^nY + \tfrac{1}{2}i\ln\det\lambda$$

with respect to $\chi(Y, X)$ and $\lambda(X, Y)$. The λ variation gives $\lambda(X, Y) = (i\chi^{-1})(X, Y)$, and so the saddle point functional is

$$S(\chi) = \int -\tfrac{1}{2}(\partial^2\delta)(X-Y)\,\chi(Y,X)d^nXd^nY - \int V(\chi(X,X))d^nX - \tfrac{1}{2}i\,\mathrm{Tr}\,I - \tfrac{1}{2}i\ln\det\chi.$$

The standard gap equation can now be obtained by stationarizing with respect to χ. The salient point is that $S(\chi)$ is exactly the sum of the original Lagrangian, written in terms of a composite field $\chi(Y, X) = \frac{1}{N}\phi^i(Y)\cdot\phi^i(X)$, and a universal term $-\frac{1}{2}i\,\mathrm{Tr}\,I - \frac{1}{2}i\ln\det\chi$. The field χ is itself a large-N invariant.

For matrix models, with the number of fields scaling as N^2, these path integral methods do not work, although attempts have been made to find an approximate solution by the judicious use of auxiliary fields [3]. We wish to show that again the saddle point functional is given by the sum of the original Lagrangian, written in terms of large-N invariants, and a universal term. In fact, this can be done for any quantum field theory, whether large-N or not, by using an effective action for composite fields. The universal term is shown in the large-N cases to be given by the relative fiber volume of a fibration of field space by the global symmetry group. This volume factor is computed explicitly for $O(N)$ models (with N^1 fields) and as a series in large-N invariants for matrix models (with N^2 fields). Using the first term of this series we compute approximate effective actions for large-N matrix models and find good numerical agreement with the solvable models in 0 and 1 dimensions. The outstanding problem is to find an explicit form of the universal term in the N^2 case.

I. Effective Action

An effective action $\Gamma(\phi, G)$ for both the fields of a quantum field theory and their bilinear composites was introduced in [1] as a tool to study dynamical symmetry breaking. $\Gamma(\phi, G)$ is the sum of 2-particle irreducible diagrams and the standard effective action is obtained by $\Gamma(\phi) = \min_G \Gamma(\phi, G)$. This has the technical advantage that a finite diagrammatic expansion of $\Gamma(\phi, G)$ corresponds to an infinite number of diagrams in the expansion of $\Gamma(\phi)$. The diagrammatic expansion of an effective action including up to quartic composite operators was given in [2]. We wish to give the expansion when all composite operators are variables. It is found that such an action is simply the sum of the Lagrangian and an entropy-type term.

On a measure space $(X, d\phi)$, let $\varrho(\phi) \in L^1(d\phi)$ satisfy $\varrho \geqq 0$, $\int \varrho d\phi = 1$ and $\ln \varrho \in L^1(\varrho d\phi)$. (Sufficient for the latter is that $\varrho \in L^p(d\phi)$, $p<1$). Take a Lagrangian $\mathscr{L}(\phi) \in L^\infty(d\phi)$ with $e^{-\mathscr{L}} \in L^1(d\phi)$. To define the standard effective action one adds to $\mathscr{L}$ a linear term in ϕ. In analogy, we add an arbitrary function on the field space X. For $J \in L^1(\varrho d\phi)$, define $W(J) = -\ln \int e^{-(L+J)} d\phi$ and $\Gamma(\varrho) = \sup_J (W(J) - \int J \varrho d\phi)$. By Jensen's inequality, $\Gamma(\varrho) \leqq \int (\mathscr{L}\varrho + \varrho \ln \varrho) d\phi$. Taking $J = -\mathscr{L}\text{-}\ln \varrho$ gives $\Gamma(\varrho) = \int (\mathscr{L}\varrho + \varrho \ln \varrho) d\phi$. This is the composite action. As is used in mean field theory, the inf of $\Gamma(\varrho)$ is at $\varrho_0 = e^{-\mathscr{L}} / \int e^{-\mathscr{L}} d\phi$, and ϱ_0 generates the Green's functions in the tautological sense that for $f \in L^\infty(d\phi)$, $\langle f \rangle = \int f \varrho_0 d\phi$. However, we wish to think of the entropy term as a functional of the Green's functions.

It is well-known that the standard effective action has the interpretation that if ϕ is static then $\Gamma(\phi_0)$ is the minimum of the energy expectation on the set of wave functions with the constraint that the expectation of ϕ is ϕ_0 [7]. There is also a spacetime version of this interpretation.

For a function $\mathscr{F} : L^1(d\phi) \to M$ mapping to some space M, define $\Gamma_{\mathscr{F}}(m) = \inf_{\varrho \in \mathscr{F}^{-1}(m)} \int (\mathscr{L}\varrho + \varrho \ln \varrho) d\phi$. To show that this includes the case of the standard effective action when X is a linear space, take $M = X$, and $\mathscr{F}(\varrho) = \int \phi \varrho d\phi$. Then

$$\Gamma_{\mathscr{F}}(\phi_0) = \inf_{\varrho \text{ such that } \int \phi \varrho d\phi = \phi_0} \int (\mathscr{L}\varrho + \varrho \ln p) d\phi = \sup_{J \in X^*} \inf_{\varrho} \int (\mathscr{L}\varrho + \varrho \ln \varrho$$
$$+ \langle J, \phi - \phi_0 \rangle \varrho) d\phi = \sup_{J \in X^*} -\ln (\int \exp -(\mathscr{L} + \langle J, \phi - \phi_0 \rangle) d\phi ,$$

which is the standard effective action. We can also write this as

$$\sup_{J \in X^*} -\ln \int \exp -(\mathscr{L}(\phi) + \langle J - \nabla \mathscr{L}|_{\phi_0}, \phi - \phi_0 \rangle) d\phi$$
$$= \sup_{J \in X^*} -\ln \int \exp -(\mathscr{L}(\phi + \phi_0) + \langle J - \nabla \mathscr{L}|_{\phi_0}, \phi \rangle) d\phi$$
$$= \mathscr{L}(\phi_0) + \sup_{J \in X^*} -\ln \int \exp -(\hat{\mathscr{L}}(\phi) + \langle J, \phi \rangle) d\phi$$

with $\hat{\mathscr{L}}(\phi) = \mathscr{L}(\phi + \phi_0) - \mathscr{L}(\phi_0) - \langle \nabla \mathscr{L}|_{\phi_0}, \phi \rangle$, which gives the background field method for computing $\Gamma(\phi_0)$. Equivalently,

$$\Gamma(\phi_0) = \inf_{\int \phi \varrho d\phi = 0} \int (\hat{\mathscr{L}}\varrho + \varrho \ln \varrho) d\phi ; \tag{1}$$

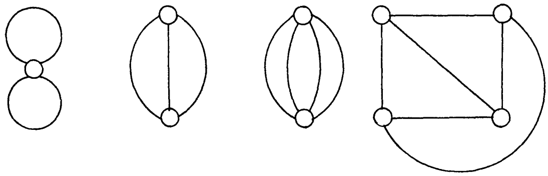

Fig. 1

the Hartree-Fock approximation is given by minimizing over ϱ's of the form $\varrho(\phi) = \det^{-1/2}(2\pi G_2)\exp(-\frac{1}{2}\langle\phi, G_2^{-1}\phi\rangle)$ with G_2 a positive operator on X. To get the composite action of [1], take $M = X \oplus (X \otimes X)$ and $\mathscr{F}(\varrho) = (\int \phi\varrho d\phi, \int (\phi\otimes\phi)\varrho d\phi)$. Then

$$\begin{aligned}\Gamma(\phi_0, G) &= \inf_{\substack{\varrho \text{ such that } \int\phi\varrho d\phi=\phi_0,\\ \int(\phi\otimes\phi)\varrho d\phi=\phi_0\otimes\phi_0+G}} \int (\mathscr{L}\varrho + \varrho\ln\varrho)d\phi\\ &= \sup_{J_2\in X^*\otimes X^*} \inf_{\varrho} \int (\mathscr{L}\varrho + \varrho\ln\varrho + \langle J_1, \phi-\phi_0\rangle\varrho + \langle J_2, \phi\otimes\phi - \phi_0 - G\rangle\varrho)d\phi\\ &= \sup_{J_1,J_2} -\ln\int \exp-(\mathscr{L} + \langle J_1, \phi-\phi_0\rangle + \langle J_2, \phi\otimes\phi - \phi_0\otimes\phi_0 - G\rangle)d\phi\,.\end{aligned}$$

To get the background field version, write this as

$$\begin{aligned}\Gamma(\phi_0, G) &= \sup_{J_1,J_2} -\ln\int\exp-(\mathscr{L}(\phi+\phi_0) + \langle J_1, \phi\rangle\\ &\quad + \langle J_2, \phi\otimes\phi + \phi\otimes\phi_0 + \phi_0\otimes\phi - G\rangle)d\phi\\ &= \sup_{J_1,J_2} -\ln\int\exp-(\mathscr{L}(\phi+\phi_0) + \langle J_1 - \nabla\mathscr{L}|\phi_0, \phi\rangle\\ &\quad + \langle J_2 - \tfrac{1}{2}\nabla\nabla\mathscr{L}|\phi_0 + \tfrac{1}{2}G^{-1}, \phi\otimes\phi - G\rangle)d\phi\\ &= \mathscr{L}(\phi_0) + \tfrac{1}{2}\langle\nabla\nabla\mathscr{L}|_{\phi_0} G\rangle - \tfrac{1}{2}\mathrm{Tr}\, I + \sup_{J_1,J_2} -\ln\int\exp-[\tilde{\mathscr{L}}(\phi)\\ &\quad + \langle J_1, \phi\rangle + \langle J_2, \phi\otimes\phi - G\rangle]d\phi\end{aligned}$$

with

$$\tilde{\mathscr{L}}(\phi) = \mathscr{L}(\phi+\phi_0) - \mathscr{L}(\phi_0) - \langle\nabla\mathscr{L}|_{\phi_0}, \phi\rangle + \tfrac{1}{2}\langle G^{-1} - \nabla\nabla\mathscr{L}|_{\phi_0}, \phi\otimes\phi\rangle\,.$$

The result of [1] is that the $\sup_{J_1,J_2}$ term is given by summing the 2PI diagrams for $\tilde{\mathscr{L}}$.

Finally, the term $\int \varrho\ln\varrho d\phi$ can be expressed in terms of the connected Green's functions $\{G_j\}_{j=1}^{\infty}$ of ϱ, defined by $\ln\int e^{i\langle\sigma,\phi\rangle}\varrho(\phi)d\phi = \sum_{j=1}^{\infty} i^j\frac{1}{j!}\langle G_j, \otimes^j\sigma\rangle$. The first few terms are given in Fig. 1. The circles denote connected vertex functions and the lines denote propagators. Their numerical factors are most easily found by computing $\int\varrho\ln\varrho d\phi$ for a zero-dimensional quantum field theory. In this case we obtain

$$\begin{aligned}\int\varrho\ln\varrho d\phi &= -\tfrac{1}{2} - \tfrac{1}{2}\ln(2\pi G_2) + \sum_{m=1}^{\infty}(-)^{m+1}\frac{1}{m}\sum_{\{p_i\}_{i=1}^{\infty}}\ \sum_{\substack{\{n_{i,j}\}_{i=1,j=1}^{\infty,\infty}\\ n\neq 0}}\\ &\quad\cdot C_{p,n}\frac{1}{p_3!}\frac{1}{n_{3,1}!}\cdots\frac{1}{n_{3,m}!}\frac{1}{p_4!}\frac{1}{n_{4,1}!}\cdots\frac{1}{n_{4,m}!}\cdots((2G_2)^{-3/2}G_3/3!)^{p_3+n_{3,1}+\cdots+n_{3,m}}\\ &\quad\cdot((2G_2)^{-4/2}G_4/4!)^{p_4+n_{4,1}+\cdots+n_{4,m}}\cdots,\end{aligned}$$

the combinatoric factor being given by

$$C_{p,n}=\left(\frac{\partial}{\partial z_0}\right)^{3p_3+4p_4+\cdots}\left(\frac{\partial}{\partial z_1}\right)^{3n_{3,1}+4n_{4,1}+\cdots}\cdot\left(\frac{\partial}{\partial z_m}\right)^{3n_{3,m}+4n_{4,m}+\cdots}\Bigg|_{z=0}\exp\left(2\sum_{0\le i<j\le m} z_i z_j\right). \tag{2}$$

For the bare Lagrangian part of $\Gamma(\varrho)$, if $\mathscr{L}(\phi)=\sum_{k=1}^{n} l_k\phi^k$, then $\int\mathscr{L}\varrho d\phi=\sum_{k=1}^{n} l_k\langle\phi^k\rangle$ with $\langle\phi^k\rangle$ being the standard expression for the Green's function in terms of the connected Green's functions $\{G_j\}_{j=1}^{\infty}$.

II. $O(N)$ Models

We have shown that a quantum field theory can be solved by minimizing the sum of the Lagrangian and an entropy-type term. This is similar to what happens in large-N models, in which one minimizes a saddle point action. We now show that the saddle point action is exactly the previous expression, the distinguishing feature of a large-N theory being that it goes like a power of N.

Suppose that we have N fields $\{\phi_i\}_{i=1}^{N}\in X=\bigoplus^{N} X_0$, and that the Lagrangian has the form $\mathscr{L}(\phi)=Nl\left(\frac{1}{N}\Sigma\phi_i\otimes\phi_i\right)$. (This formalism applies to the standard $O(N)$ model on $\mathbb{R}^n$.) Taking ϱ to have the form $\varrho(\phi)=a\exp\left(-Nr\left(\frac{1}{N}\Sigma\phi_i\otimes\phi_i\right)\right)$, we have

$$\int\varrho\ln\varrho d\phi=-\frac{\int Nr\left(\frac{1}{N}\Sigma\phi_i\otimes\phi_i\right)\exp\left(-Nr\left(\frac{1}{N}\Sigma\phi_i\otimes\phi_i\right)\right)d\phi}{\int\exp\left(-Nr\left(\frac{1}{N}\Sigma\phi_i\otimes\phi_i\right)\right)d\phi}-\ln\int\exp\left(-Nr\left(\frac{1}{N}\Sigma\phi_i\otimes\phi_i\right)\right)d\phi$$

$$=-\frac{\int NV(\xi)r(\xi)\exp(-Nr(\xi))d\xi}{\int V(\xi)\exp(-Nr(\xi))d\xi}-\ln\int V(\xi)\exp(-Nr(\xi))d\xi\,,$$

with $\zeta\in X_0\otimes X_0$ and $V(\zeta)=\mathrm{vol}\left\{\phi:\frac{1}{N}\sum_i\phi_i\otimes\phi_i=\zeta\right\}$.

With the assumption that $O(N)$ acts freely on almost all of X, $V(\zeta)$ is given as follows: Let $\{T_j\}_{j=1}^{N(N-1)/2}$ be a basis of $O(N)$ and let $T_j\phi\in T_\phi X$ denote the infinitesimal action of T_j on ϕ. Then up to a constant, $V(\zeta)=\det^{1/2}M$, with $M\in M_N(\mathbb{R})$ given by $M_{jk}=\langle T_j\phi,T_k\phi\rangle$. Now $\ln\det M=-\int_0^\infty\frac{1}{T}\mathrm{Tr}\left(e^{-\frac{TM}{N}}-e^{-\frac{T}{N}}\right)dT$ $=-\sum_{j=0}^{\infty}\int_0^\infty\frac{1}{T}\frac{1}{j!}\left(-\frac{T}{N}\right)^j(\mathrm{Tr}M^j-N)dT$. However, the leading term in N of $\mathrm{Tr}M^j$ is

$N^{j+1}\mathrm{Tr}\zeta^j$ and so $\ln\det M \sim N\ln\det\zeta + \mathrm{const}$. Thus

$$\int \varrho \ln \varrho d\phi \sim -N[\int r(\zeta)\exp(-N(r(\zeta)-\tfrac{1}{2}\ln\det\zeta))d\zeta / \int \exp(-N(r(\zeta)-\tfrac{1}{2}\ln\det\zeta)d\zeta)]$$
$$-\ln[\int \exp(-N(r(\zeta)-\tfrac{1}{2}\ln\det\zeta))d\zeta] \sim -\tfrac{1}{2}N\ln\det\zeta_0\,,$$

with ζ_0 being the minimum of $r(\zeta)-\frac{1}{2}\ln\det\zeta$. This could also be derived by summing the bubble diagrams of (2).
For large N we wish to minimize

$$\Gamma(\varrho)=\left[\int Nl\left(\frac{1}{N}\sum_i \phi_i\otimes\phi_i\right)\exp\left(-Nr\left(\frac{1}{N}\sum_i \phi_i\otimes\phi_i\right)\right)d\phi/\right.$$
$$\left.\cdot\int \exp\left(-Nr\left(\frac{1}{N}\sum_i \phi_i\otimes\phi_i\right)\right)d\phi\right]-\tfrac{1}{2}N\ln\det\zeta_0 \sim N\left[l(\zeta_0)-\frac{1}{2}\ln\det\zeta_0\right]$$

as a function of r. Note that this is the saddlepoint functional derived in the introduction. The minimization gives $\zeta_0\frac{\delta l}{\delta\zeta}|_{\zeta_0}=\frac{1}{2}I$, which is the well-known gap equation for the $O(N)$ model [4].

III. Matrix Models

Suppose we have N^2 fields $\{\phi_{ij}\}\in X=H\otimes X_0$, H being the $N\times N$ Hermitian matrices, and that $\mathscr{L}=N^2 l\left(\left\{N^{\frac{-j+2}{2}}\mathrm{Tr}\otimes^j\phi\right\}_{j=1}\right)$, the trace being over the $\otimes^j H$ part. As before, $-\int\varrho\ln\varrho d\phi$ will be the log of the relative volume of an orbit of the global $U(N)$ action $\phi\to U\phi U^{-1}$. With $\{T_j\}_{j=1}^{N^2}$ a basis of $U(N)$, for a field theory on $\mathbb{R}^n$, we have

$$M_{jk}=\sum_{abcd} T_{jab}T_{kcd}\int(2\phi_{bc}\phi_{da}-\phi^2_{da}\delta_{bc}-\phi^2_{bc}\delta_{da})(x)d^nx$$

and

$$\ln\det M=-\int_0^\infty \frac{1}{T}\mathrm{Tr}e^{-\frac{TM}{N}}dT+\mathrm{const}=-N^2\int_0^\infty\frac{1}{T}\left[-\int 2T\left(\frac{1}{N^2}\mathrm{Tr}\phi^2(X_1)\right.\right.$$
$$\left.-\frac{1}{N^3}(\mathrm{Tr}\phi)^2(X_1)\right)d^nX_1+T^2\iint\left(\frac{1}{N^2}\mathrm{Tr}\phi^2(X_1)\phi^2(X_2)\right.$$
$$+3\frac{1}{N^4}\mathrm{Tr}\phi^2(X_1)\mathrm{Tr}\phi^2(X_2)-2\frac{1}{N^4}\mathrm{Tr}\phi(X_1)\mathrm{Tr}\phi^3(X_2)$$
$$\left.\left.-2\frac{1}{N^4}\mathrm{Tr}\phi^3(X_1)\mathrm{Tr}\phi(X_2)\right)d^nX_1d^nX_2+\ldots\right]dT+\mathrm{const}. \qquad (3)$$

Although this series expresses the relative orbit volume in terms of large-N invariants, it will not be very useful unless it can be summed explicitly, and an alternative is to use the planar diagram expansion of the n-dimensional version of (2). We will now compute exactly using the first term of the planar expansion. To

Table 1

λ	g_{EXACT}	$g_{\text{H-F}}$	$(\Delta E)_{\text{EXACT}}$	$(\Delta E)_{\text{H-F}}$
0.01	0.9317	0.9307	0.01844	0.01858
0.1	0.6676	0.6559	0.1217	0.1248
0.5	0.4074	0.3904	0.3086	0.3179
1	0.3125	0.2965	0.4197	0.4320
50	5.389×10^{-2}	4.877×10^{-2}	1.241	1.273
1000	1.209×10^{-2}	1.112×10^{-2}	1.979	2.002

Table 2

λ	E_{EXACT}	$E_{\text{H-F}}$
0.01	0.505	0.505
0.1	0.542	0.543
0.5	0.651	0.653
1	0.740	0.743
50	2.217	2.236
1000	5.915	5.968

lowest order, $\int \varrho \ln \varrho d\phi = -\frac{N^2}{2}\ln \det g(X,Y)$ with

$$g(X,Y)=\frac{1}{N^2}\int \mathrm{Tr}\phi(X)\phi(Y)\varrho(\phi)d\phi-\frac{1}{N^3}\left(\int \mathrm{Tr}\phi(X)\varrho(\phi)d\phi\right)\left(\int \mathrm{Tr}\phi(Y)\varrho(\phi)d\phi\right).$$

Only varying $\int(\mathscr{L}\varrho+\varrho\ln\varrho)d\phi$ with respect to the first two connected Green's functions amounts to the Hartree-Fock (H-F) approximation (1), and we can compare this to exact solutions in 0 and 1 dimensions. For the zero-dimensional Lagrangian $\mathscr{L}=\frac{1}{2}\mathrm{Tr}\phi^2+\frac{\lambda}{N}\mathrm{Tr}\phi^4$ the H-F approximation gives the 2-point function $g_{\mathrm{HF}}=\frac{1}{16\lambda}(\sqrt{1+32\lambda}-1)$ and the free energy $E_{\mathrm{HF}}(\lambda)-E_{\mathrm{HF}}(0)=N^2[\frac{1}{4}g-\frac{1}{4}-\frac{1}{2}\ln g]$, which is compared with the exact results of [5] in Table 1. For the 1-dimensional Lagrangian $\mathscr{L}=\frac{1}{2}\mathrm{Tr}(\partial\phi)^2+\frac{1}{2}\mathrm{Tr}\phi^2+\frac{\lambda}{N}\mathrm{Tr}\phi^4$, H-F predicts a 2-point function

$$\langle\phi_{ab}(T_1)\phi_{cd}(T_2)\rangle=\frac{1}{2m}e^{-m(T_1-T_2)}\delta_{ad}\delta_{bc}$$

with $m^3-m=4\lambda$, and a ground state energy of $N^2\frac{1}{8}\left(3m+\frac{1}{m}\right)$, which is compared with the exact results of [5] in Table 2. For the n-dimensional Lagrangian

$$\mathscr{L}(\phi)=\frac{1}{2}\mathrm{Tr}(\partial\phi)^2+\frac{1}{2}\mu_1^2\mathrm{Tr}\phi^2+\frac{1}{2N}\mu_2^2(\mathrm{Tr}\phi)^2+\frac{\lambda}{N}\mathrm{Tr}\phi^4,$$

the H-F version of $\Gamma(\phi)$, using (1), is

$$\Gamma(\phi)=\mathscr{L}(\phi)-2\lambda N^2\int g^2(X,X)d^nX-\frac{N^2}{2}\ln\det g$$

with

$$g=\left(-\partial^2+\mu_1^2+8\lambda\frac{1}{N^2}\mathrm{Tr}\,\phi^2+4\lambda\frac{1}{N^3}(\mathrm{Tr}\,\phi)^2+8\lambda g(X,X)\right)^{-1}.$$

This can be renormalized as in [4]. In 4 dimensions the H-F approximation predicts, in analogy to [6], that there is no nontrivial large-N ϕ^4 matrix theory.

The main problem in this approach is to find the large-N volume factor exactly, whether by summing the series (3), adding the planar diagrams of (2) or using some other method. We only have the intriguing 0-dimensional result $-\int\varrho\ln\varrho \sim N^2\int d\mu d\lambda N^{-4}\varrho((N^{1/2}\lambda I)\ln|\lambda-\mu|\varrho(N^{1/2}\mu I)$ [5].

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