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A Remark on the Cluster Theorem

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Abstract. An improved version of the cluster theorem of relativistic quantum field theory with the correct exponential decay is derived by function theoretical methods.

In its simplest form the cluster theorem of relativistic quantum field theory states that in a massive theory the correlations of local observables in the vacuum state decrease exponentially with their spacelike separation. The standard proof of this fact [1] uses the method of the Jost–Lehmann–Dyson representation and leads to the estimate

$$|(\Omega, AB\Omega) - (\Omega, A\Omega)(\Omega, B\Omega)| \leqq c(\tau)e^{-m\tau}\{\|A^*\Omega\| \, \|HB\Omega\| + \|HA^*\Omega\| \, \|B\Omega\|\}, \quad (1)$$

where H is the Hamiltonian, m the mass gap, Ω the vacuum vector, A and B are local operators and τ is the spacelike distance of the origin from the spacelike complement of a wedge[1] W with the property that $[\alpha_x(A), B] = 0$ for $x \in W$, α_x denoting the translation of an observable by x. The function $c(\tau)$ depends on the volume of the spatial localization regions of A and B and is polynomially decreasing for finite volumes and at most polynomial increasing for unbounded localization regions (e.g. wedges).

This theorem gives no information on the correlations of local operators for more complicated geometrical situations, for instance if A is localized in a double cone O_1 and B is known to commute with all operators in a larger double cone O_2. A further unsatisfying feature of the estimate (1) is the occurrence of the Hamiltonian in the bound on the right-hand side. The first proof of the cluster theorem by Ruelle [2] avoids these drawbacks; however, it apparently leads only to a decay faster than any power of the distance,

$$|(\Omega, AB\Omega) - (\Omega, A\Omega)(\Omega, B\Omega)| \leqq c_N \tau^{-N}\{\|A^*\Omega\| \, \|B\Omega\| + \|A\Omega\| \, \|B^*\Omega\|\}, \quad (2)$$

where $\tau > 0$ such that $[\alpha_t(A), B] = 0$, $|t| \leqq \tau$, α_t denoting the time translations, and

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1 A wedge is the image of the set $\{|x^0| < x^1\}$ under a Poincaré transformation

where the constants c_N depend only on the mass gap. Actually, Haag and Swieca [3] estimated the constants c_N and showed that

$$\inf_N c_N \tau^{-N} \leqq c e^{-\lambda m \tau} \tag{3}$$

with $\lambda = \frac{1}{2}\sqrt{1/2}$; by improving their estimates one can find $\lambda = \frac{1}{2}$, but there remains a gap to the value $\lambda = 1$ which one expects from (1).

In this note I want to give a straightforward and quick proof of the cluster theorem in the spirit of the results of Ruelle, Haag and Swieca with the correct exponential decay. Since only little structure of quantum field theory is used I shall formulate it in an abstract mathematical way.

Theorem. *Let $\mathscr{H}$ be a Hilbert space, H a selfadjoint operator in $\mathscr{H}$ with* $\operatorname{sp} H \subset \{0\} \cup [m, \infty), m > 0, \Omega$ *the unique normalized eigenvector of H with eigenvalue* 0, *and let A, B be bounded operators in $\mathscr{H}$ such that*

$$[e^{iHt} A e^{-iHt}, B] = 0$$

for $|t| \leqq \tau$, $\tau > 0$. *Then*

$$|(\Omega, AB\Omega) - (\Omega, A\Omega)(\Omega, B\Omega)| \leqq e^{-m\tau} \{ \|A^*\Omega\| \, \|B\Omega\| \, \|A\Omega\| \, \|B^*\Omega\| \}^{1/2}.$$

Proof. It is sufficient to consider the case $(\Omega, A\Omega) = 0$. The general case can be obtained by inserting $A_0 = A - (\Omega, A\Omega)1$ into the estimate and using $\|A_0\Omega\| \leqq \|A\Omega\|$ and $\|A_0^*\Omega\| \leqq \|A^*\Omega\|$.

The function $z \to (\Omega, Ae^{izH}B\Omega)$ is analytic in the upper half plane $\operatorname{Im} z > 0$, the function $z \to (\Omega, Be^{-izH}A\Omega)$ in the lower half plane $\operatorname{Im} z < 0$. Both functions have continuous boundary values for $\operatorname{Im} z = 0$. According to the assumptions, these boundary values coincide for $|\operatorname{Re} z| \leqq \tau$. Using the Edge-of-the-Wedge-Theorem one concludes that there is a function f which is analytic on the twofold cut plane $\mathscr{G}_\tau = \{z \in \mathbb{C} \mid \operatorname{Im} z \neq 0 \text{ or } |\operatorname{Re} z| < \tau\}$ such that

$$f(z) = \begin{cases} (\Omega, Ae^{izH}B\Omega), & \operatorname{Im} z > 0, \\ (\Omega, Be^{-izH}A\Omega), & \operatorname{Im} z < 0, \\ (\Omega, Ae^{izH}B\Omega) = (\Omega, Be^{-izH}A\Omega), & \operatorname{Im} z = 0, |\operatorname{Re} z| < \tau. \end{cases}$$

$z = 2y\tau/(1 + y^2)$ defines a conformal mapping from the unit disc $|y| < 1$ onto the twofold cut plane $\mathscr{G}_\tau$. Therefore $g(y) = f(2y\tau/(1 + y^2))$ is an analytic function on the unit disc. We are interested in an estimate of $(\Omega, AB\Omega) = f(0) = g(0)$. This can be obtained from Jensen's formula [4], which states that for $r < 1$

$$\log|g(0)| = \frac{1}{2\pi} \int_0^{2\pi} d\vartheta \log|g(re^{i\vartheta})| + \sum_a \log\frac{|a|}{r},$$

where the sum in the second term on the right-hand side is over all zeros a of g with $|a| < r$. Therefore this term is nonpositive, and one obtains the estimate

$$|g(0)| \leqq \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} d\vartheta \log|g(re^{i\vartheta})|\right\}. \tag{$*$}$$

Using the assumption on the spectrum of H, the vanishing of $(\Omega, A\Omega)$ and the formula

$$\operatorname{Im} 2re^{i\vartheta}/(1+r^2e^{2i\vartheta}) = \lambda \sin\vartheta/(1+\lambda^2\cos^2\vartheta)$$

with $\lambda = 2r/(1-r^2)$, one finds

$$|g(re^{i\vartheta})| \leqq e^{-m\tau\lambda|\sin\vartheta|/(1+\lambda^2\cos^2\vartheta)} \times \begin{cases} \|A^*\Omega\|\,\|B\Omega\|, & 0 \leqq \vartheta \leqq \pi \\ \|A\Omega\|\,\|B^*\Omega\|, & \pi \leqq \vartheta \leqq 2\pi. \end{cases}$$

Inserting this estimate into inequality (*) gives

$$|g(0)| \leqq e^{-m\tau c(r)}\{\|A^*\Omega\|\,\|B\Omega\|\,\|A\Omega\|\,\|B^*\Omega\|\}^{1/2}$$

with $c(r) = (1/2\pi)\int_0^{2\pi} d\vartheta\lambda|\sin\vartheta|/(1+\lambda^2\cos^2\vartheta)$. The integration over ϑ can be carried through by substituting $u = \lambda\cos\vartheta$; one obtains $c(r) = (2/\pi)\arctan(2r/(1-r^2))$, thus $\lim_{r\to 1} c(r) = 1$, which proves the theorem.

q.e.d.

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