Commun. Math. Phys. 97, 443–452 (1985)

Communications in
Mathematical
Physics

Support Theorems for Random Schrödinger Operators

Shinichi Kotani*
Department of Mathematics, Kyoto University, Kyoto, Japan

Abstract. In the multi-dimensional case it is shown that the increase of the topological support of the probability measure describing the randomness of potentials implies the increase of the spectrum. In the one-dimensional case the converse statement for the absolutely continuous spectrum is valid. Especially the spectrum (in general dimension) and the absolutely continuous spectrum (in one-dimension) are determined only by the topological support of the random potentials.

1. Introduction

Let $\Omega = \{q; q: \mathbf{R}^d \to [0,1]$, measureable$\}$ and impose the Schwartz distribution topology on Ω. Denote by $\{T_x, x \in \mathbf{R}^d\}$ the shift on Ω defined by $T_x q(\cdot) = q(\cdot + x)$. For any shift invariant ergodic probability measure P on Ω it is known that there exists a unique closed subset $\Sigma(P)$ of $\mathbf{R}$ such that the spectrum of self-adjoint operator $L(q) = -\Delta + q(\cdot)$ on $L^2(\mathbf{R}^d, dx)$ coincides with $\Sigma(P)$ for almost every $q \in \Omega$ with respect to P. Suppose we are given two such measures P_1 and P_2 on Ω. Then we have in Sect. 2:

Theorem 1. Supp $P_1 \subset \operatorname{supp} p_2$ *implies* $\Sigma(P_1) \subset \Sigma(P_2)$. (Supp P *means the topological support of* P *in* Ω.)

This theorem is an extension of what they have observed in Kirsch–Martinelli [2] and Kunz–Souillard [4]. An easy consequence of this result is that if $0 \in \operatorname{supp} P$, then $\Sigma(P) = [0, \infty)$.

In the one-dimensional case, instead of the above Ω, we consider

$$\Omega = \{q; q\text{: Random measure on } \mathbf{R} \text{ satisfying } \int_x^{x+1} q_-(dy) \leq c \text{ for any } x \in \mathbf{R}\},$$

where q_- denotes the minus sign of the lower variation measure of q and a constant c may depend on q. A modified Schwartz topology is given to Ω and for any $q \in \Omega$ a self-adjoint operator $L(q)$ formally defined by $-d^2/dx^2 + q(\cdot)$ on $L^2(\mathbf{R}, dx)$ is introduced in Sect. 3. In the one-dimensional case Theorem 1 is valid in this

* Research partially supported by the USNSF under grant MCS-81-20833

more general space Ω. The absolutely continuous spectrum of a self-adjoint operator L is defined, following Simon [5], by an essential support of the resolution of the identity of L with respect to Lebesgue measure on $\mathbf{R}$. This spectrum is determined uniquely up to Lebesgue measure zero sets. Suppose we are given a shift invariant ergodic measure P on Ω satisfying $\int_\Omega |q(0,1)|P(dq) < \infty$. Then as in Kotani [3], in this case too, it is shown that there exists a Borel set $\Sigma_{\text{a.c.}}(P)$ in $\mathbf{R}$ such that the absolutely continuous (a.c.) spectrum of $L(q)$ coincides with $\Sigma_{\text{a.c.}}(P)$ for almost every $q \in \Omega$ with respect to P.

Theorem 2. *Let P_1 and P_2 be two shift invariant ergodic probability measures on Ω satisfying $\int_\Omega |q(0,1)|P_1(dq) < \infty$, $\int_\Omega |q(0,1)|P_2(dq) < \infty$. Then* $\operatorname{supp} P_1 \subset \operatorname{supp} P_2$ *implies* $\Sigma_{\text{a.c.}}(P_1) \supset \Sigma_{\text{a.c.}}(P_2)$.

As a corollary of this theorem there will be given in Sect. 3 deterministic random potentials having no absolutely continuous spectrum almost surely.

2. Support Theorem for Spectrum in Multi-dimensional Case

It is easy to see that Ω is a compact metrizable space. We need the following:

Lemma 1. *For any $f \in L^2(\mathbf{R}^d, dx)$ a map*

$$\Omega \ni q \to (G_\lambda(q)f, f) \in \bar{\mathbf{C}}_+ = \overline{\{z \in \mathbf{C}; \operatorname{Im} z > 0\}}$$

is continuous for any fixed $\lambda \in \mathbf{C}_+$, where $G_\lambda(q) = (L(q) - \lambda)^{-1}$.

Proof. Because of the uniform boundedness of the norms of $G_\lambda(q)$, we can assume that f is a smooth function with compact support. We show that if $q_n \to q$ in Ω, then for any fixed $t > 0$ and $x \in \mathbf{R}^d$,

$$u_n(t,x) = e^{tL(q_n)} f(x) \to e^{tL(q)} f(x) = u(t,x).$$

Denoting $T_t = e^{t\Delta}$, we see

$$u_n(t,x) = T_t f(x) - \int_0^t (T_{t-s} q_n u_n(s,\cdot))(x)\,ds = f(t,x) + G_n u_n(t,x).$$

Therefore u_n can be expanded in a Neumann series

$$u_n(t,x) = \sum_{p=0}^{\infty} (G_n^p f)(t,x).$$

However by induction we have

$$|G_n^p f(t,x)| \leqq \frac{\|f\|_\infty}{p!} t^p \qquad \text{for } p = 0,1,2,\ldots, \quad \text{where } \|f\|_\infty = \sup|f|.$$

Hence we have only to prove that for each fixed $p \geqq 1$, $G_n^p f(t,x) \to G^p f(t,x)$ as $n \to \infty$, where $Gf(t,x) = -\int_0^t (T_{t-s} q f(s,\cdot))(x)\,ds$. However

$$G_n^p f(t,x) = (-1)^p \int_{0<t_1<t_2<\cdots<t_p<t} \cdots \int T_{t-t_p} q_n \cdots T_{t_2-t_1} q_n f(t_1,x)\,dt_1\,dt_2 \cdots dt_p$$

holds and T_t has a bounded smooth kernel for $t > 0$, so it is not difficult to conclude from $q_n \to q$ in Ω that $G_n^p f(t,x) \to G^p f(t,x)$ as $n \to \infty$. This completes the proof.

Proof of Theorem 1. Suppose $q \in \operatorname{supp} P_1$. Since $\operatorname{supp} P_1 \subset \operatorname{supp} P_2$, there exists $\{q_n\}_{n=1}^{\infty} \subset \Omega$ such that $q_n \to q$ in Ω and for every q_n the spectrum of $L(q_n)$ is equal to $\Sigma(P_2)$. However from Lemma 1 we have the weak convergence of the Green operators $G_\lambda(q_n)$ to $G_\lambda(q)$ for every fixed $\lambda \in \mathbf{C}_+$. On the other hand, for any $f \in L^2(\mathbf{R}^d, dx)$

$$(G_\lambda(q_n)f, f) = \int_0^\infty \frac{(E(d\xi, q_n)f, f)}{\xi - \lambda}, \tag{2.1}$$

where $E(d\xi, q_n)$ is the resolution of identity of the self-adjoint operator $L(q_n)$. Since the weak convergence of the measures $(E(d\xi, q_n)f, f)$ follows from the convergence of the right-hand side of (2.1), we easily have $\Sigma(P_2) \supset$ the spectrum of $L(q)$. This implies $\Sigma(P_1) \subset \Sigma(P_2)$.

Remark. Obviously the above Ω can be generalized to much wider classes assuring the essential self-adjointness of $L(q)$.

3. Support Theorem for Absolutely Continuous Spectrum in One-Dimensional Case

First we have to realize the $L(q)$ as a self-adjoint operator in $L^2(\mathbf{R}, dx)$. For this sake a domain of $L(q)$ is defined by

$\mathscr{D}(L(q)) = \{\phi$; absolutely continuous function on $\mathbf{R}$ with compact support. Moreover its derivative ϕ' has a bounded variation modification satisfying $-d\phi' + \phi q(dx) = f\,dx$ for some $f \in L^2(\mathbf{R}, dx)$ with compact support.$\}$

If we introduce functions ϕ_1, ϕ_2 defined as solutions of $-d\phi' + \phi q(dx) = 0$, satisfying $\phi_1(0) = 1$ $(\phi_2(0) = 0)$ and $\phi_1'(0) = 0$ $(\phi_2'(0) = 1)$ respectively, then the above ϕ can be represented by

$$\phi(x) = \int_{-\infty}^{x} \{\phi_1(x)\phi_2(y) - \phi_1(y)\phi_2(x)\} f(y)dy,$$

where $f \in L^2(\mathbf{R}, dx)$ with compact support should satisfy $\int_{\mathbf{R}} f(x)\phi_i(x)dx = 0$ for $i = 1$ and 2. Therefore it is not difficult to see that the $\mathscr{D}(L(q))$ is dense in $L^2(\mathbf{R}, dx)$. Define an operator $L(q)$ by $L(q)\phi = f$ for $\phi \in \mathscr{D}(L(q))$. Then $L(q)$ turns out to be a symmetric operator in $L^2(\mathbf{R}, dx)$.

Lemma 2. *The $L(q)$ has a unique self-adjoint extension in $L^2(\mathbf{R}, dx)$ for any $q \in \Omega$. In other words, the two boundaries $\pm\infty$ are of limit point type. Moreover the self-adjoint extension $L(q)$ (we use the same notation as the original one) satisfies*

$$(L(q)\phi, \phi) \geqq (1 - \varepsilon c(q_-)) \int_{\mathbf{R}} |\phi'(x)|^2 dx - \delta c(q_-) \int_{\mathbf{R}} |\phi(x)|^2 dx \tag{3.1}$$

for any $\phi \in \mathscr{D}(L(q))$ with some positive constants ε, δ independent of q. $c(q_-)$ is the constant c appearing in the definition of Ω. We can choose ε arbitrarily small.

Proof. The estimate (3.1) is clear from an inequality

$$|\phi(x)|^2 \leqq \varepsilon \int_x^{x+1} |\phi'(y)|^2 dy + \delta \int_x^{x+1} |\phi(y)|^2 dy,$$

where ε, δ are constants independent of $x \in \mathbf{R}$ and ϕ. ε can be chosen arbitrarily small (for a proof see [1] p. 193). We show that the boundary $+\infty$ is of limit point type (the same thing can be shown also for the boundary $-\infty$). For this we prove that there does not exist two linearly independent solutions of $-du' + uq(dx) = \lambda u\, dx$ belonging to $L^2(\mathbf{R}_+, dx)$ for $\lambda < 0$ ($\mathbf{R}_+ = [0, \infty)$). Suppose a solution u belongs to $L^2(\mathbf{R}_+, dx)$. Observe

$$\begin{aligned} -\frac{u'(x)u(x)}{x^2} &= -u'(1)u(1) - \int_1^x \frac{u(x)}{x^2} u'(dx) - \int_1^x \frac{u'(x)^2}{x^2} dx + 2\int_1^x \frac{u'(x)u(x)}{x^3} dx \\ &\leqq c_1 + \int_1^x \frac{u(x)^2}{x^2} q_-(dx) - H(x) + c_2 H(x)^{1/2}, \end{aligned}$$

where $H(x) = \int_1^x \frac{u'(x)^2}{x^2} dx$ and $c_1 = -u'(1)u(1)$, $c_2 = 2\left(\int_1^\infty u(x)^2\, dx\right)^{1/2}$.

Set $Q(x) = q_-[1, x)$. Then the second term of the above inequality is

$$\frac{u(x)^2}{x^2} Q(x) - 2\int_1^x \frac{u'(x)u(x)}{x^2} Q(x)dx + \int_1^x 2\frac{u(x)^2}{x^3} Q(x)dx.$$

Since $q \in \Omega$, we have $Q(x) \leqq c(q_-)x$ for $x \geqq 1$. Hence

$$-2\int_1^x \frac{u'(x)u(x)}{x^2} Q(x)dx \leqq c(q_-)c_2 H(x)^{1/2}, \quad \int_1^x 2\frac{u(x)^2}{x^3} Q(x)dx \leqq \tfrac{1}{2} c(q_-)c_2^2.$$

Moreover $(u(x)^2/x^2)Q(x) \leqq c(q_-)(u(x)^2/x)$, and

$$\begin{aligned} \frac{u(x)^2}{x} &= u(1)^2 + 2\int_1^x \frac{u(x)u'(x)}{x} dx - \int_1^x \frac{u(x)^2}{x^2} dx \\ &\leqq u(1)^2 + c_2 H(x)^{1/2}. \end{aligned}$$

Therefore we have

$$-\frac{u'(x)u(x)}{x^2} \leqq c_3 + c_4 H(x)^{1/2} - H(x),$$

with some positive constants c_3, c_4. If $H(+\infty) = +\infty$, then the above estimate shows

$$\frac{u'(x)u(x)}{x^2} > 0$$

for all sufficiently large x. However this contradicts the fact that $u \in L^2(\mathbf{R}_+)$. Thus $H(+\infty) < +\infty$. Now assume that u_1 and u_2 are two linearly independent solutions of $L(q)u = \lambda u$ belonging to $L^2(\mathbf{R}_+, dx)$.We can assume that

$$u_1(x)u_2'(x) - u_1'(x)u_2(x) = 1$$

identically on $\mathbf{R}_+$. However this obviously contradicts the above argument. This completes the proof.

Now for $\lambda \in \mathbf{C}$ define ϕ_λ, ψ_λ as unique solutions of integral equations

$$\phi_\lambda(x) = 1 + \int_0^x (x-y)\phi_\lambda(y)(q(dy) - \lambda\, dy),$$

$$\psi_\lambda(x) = x + \int_0^x (x-y)\psi_\lambda(y)(q(dy) - \lambda\, dy).$$

Since the two boundaries $\pm\infty$ are of limit point type, we can show that

$$h_\pm(\lambda)^1 = \mp \lim_{x\to\pm\infty} \frac{\phi_\lambda(x)}{\psi_\lambda(x)}$$

exist as holomorphic functions on $\mathbf{C}\setminus[-\delta c(q_-), \infty)$. If we denote the Green function of $L(q) - \lambda$ by $g_\lambda(x, y, q)$, then

$$g_\lambda(0,0,q) = -(h_+(\lambda, q) + h_-(\lambda, q))^{-1}. \tag{3.2}$$

For later use, we estimate $h_\pm(\lambda, q)$ and $g_\lambda(0,0,q)$ for $\lambda \in (-\infty, -\delta c(q_-))$. Since we have (3.1),

$$0 < g_\lambda(0,0,q) \leqq c_1(c(q_-) - \lambda)^{-1/2}, \tag{3.3}$$

where c_1 depends only on $c(q_-)$. To estimate h_+, we observe that $-h_+^{-1}$ is the Green function evaluated at 0 of the operator $L(q) - \lambda$ with Neumann boundary condition at 0. To obtain an upper estimate of $-h_+^{-1}$, we check for any smooth function ϕ with compact support in $\mathbf{R}_+$ and $\phi'(0) = 0$,

$$\int_{\mathbf{R}^+} |\phi'(x)|^2 dx + \int_{\mathbf{R}^+} |\phi(x)|^2 q(dx) \leqq (1+\varepsilon) \int_{\mathbf{R}^+} |\phi'(x)|^2 dx + \delta \int_{\mathbf{R}^+} |\phi(x)|^2 |q[0,x)|\, dx, \tag{3.4}$$

where ε can be chosen arbitrarily small. (3.4) follows immediately from the integration by parts of the second term of the left-hand side and a trivial inequality $|xy| \leqq \varepsilon|x|^2 + \delta|y|^2$. Therefore $-h_+^{-1}(\lambda, q)$ is greater than the Green function evaluated at 0 of the operator $-(1+\varepsilon)(d^2/dx^2) + \delta|q[0,x)|$ in $L^2(\mathbf{R}_+, dx)$ with Neumann boundary condition at 0. However, generally

$$-h_+(\lambda, q) = \lim_{x\to\infty} \frac{\psi'_\lambda(x)}{\phi'_\lambda(x)} = \int_{\mathbf{R}^+} \frac{1}{\psi'_\lambda(x)^2}(q(dx) - \lambda\, dx) \tag{3.5}$$

holds for any $\lambda < -\delta c(q_-)$. Hence, if we note that the solution $\psi(x)$ of $-(1+\varepsilon)(d^2u/dx^2) + \delta|[0,x)|u = \lambda u$ satisfying $u(0) = 0$, $u'(0) = 1$ has an estimate

$$\psi'(x) \geqq \cosh\left(-\frac{\lambda}{1+\varepsilon}\right)^{1/2} x > \tfrac{1}{2}\exp\left(-\frac{\lambda}{1+\varepsilon}\right)^{1/2} x$$

for any $x > 0$ and $\lambda < 0$, then we obtain from (3.5),

$$0 < -h_+(\lambda, q) \leqq 4 \int_{\mathbf{R}^+} (\delta|q[0,x)| - \lambda)\exp\left(-2\left(-\frac{\lambda}{1+\varepsilon}\right)^{1/2} x\right) dx \tag{3.6}$$

for any $\lambda < -c(q_-)\delta$.

---

1 The dependence on $q \in \Omega$ will be denoted by $\phi_\lambda(x, q)$, $h_+(\lambda, q), \ldots,$.

Now we introduce a topology to Ω by giving a fundamental system of neighbourhoods at each point q of Ω:

$$U_q(k,m,n)=\left\{q'\in\Omega;\, d_k(q,q')<\frac{1}{m} \quad \text{and} \quad \|q'\|_k<n\right\},$$

where
$$d_k(q,q')=\sum_{p=1}^{\infty}\int_{|x|<k}\phi_p(x)(q-q')(dx)|1/2$$

for some countable dense set $\{\phi_p\}$ in $\mathbf{C}(|x|\leqq k)$, and $\|q'\|_k$ denotes the total variation of q' on $[-k,k]$. In this topology Ω turns out to be a Hausdorff space satisfying the first countability axiom. It should be remarked that a sequence $\{q_n\}$ in Ω converges to q in this topology if and only if on each compact interval of R $q_n\to q$ weakly preserving a uniform bound of their total variations on the interval. If we denote the shift on Ω by $\{T_x; x\in\mathbf{R}\}$, then T_x defines a one-parameter group of homeomorphisms on Ω. Now we can discuss (Borel) shift invariant ergodic probability measure P on Ω. If P satisfies

$$\int_{\Omega}|q[0,1)|P(dq)<\infty, \tag{3.7}$$

then (3.6) implies the finiteness of $\int_{\Omega}|h_+(\lambda,q)|P(dq)$ for $\lambda<-c(P)\delta$, where $c(P)=\sup_{x\in R}\int_x^{x+1}q_-(dy)$ which is independent of q a.e. because the right-hand side is a shift invariant function on Ω and we assume the ergodicity of P. Similarly we have the finiteness of the expectation of $h_-(\lambda,q)$. The finiteness of $\int_{\Omega}g_\lambda(0,0,q)P(dq)$ comes from Remark (3.3). Consequently it is not difficult to check the all theorems in [3] in this more general case. Among them we need

Lemma 3. $\operatorname{Re} g_{\xi+i0}(0,0,q)=0$ a.e. *on* $\Sigma_{\text{a.c.}}(P)$ *holds for almost every* $q\in\Omega$ *with respect to P. (See the definition of* $\Sigma_{\text{a.c.}}(P)$ *in Sect. 1.)*

Introduce for $c>0$ $\Omega_c=\{q\in\Omega;\, c(q_-)\leqq c\}$. Then we have

Lemma 4. *The correspondence*

$$\Omega_c\ni q\to g_\lambda(0,0,q)\in\mathbf{C}$$

is continuous for any fixed $\lambda\in\mathbf{C}\backslash[-c\delta,\infty)$.

Proof. Because of the identity (3.2) we have only to check the continuity of $h_+(h_-)$. Fix $\lambda<-2c\delta$. Set $f(x)=\phi_\lambda(x,q)$ and $\kappa=-\lambda-c>0$. First we claim that if $q\in\Omega_c$, then

$$f(x)\geqq f(2)+(f'(2)-c)(x-2)+\frac{\kappa}{2}f(2)(x-2)^2 \tag{3.8}$$

is valid for any $x\geqq 2$. To show this we observe that for any $a>0$

$$\int_{[0,a]}|\phi'(x)|^2dx+\int_{[0,a]}|\phi(x)|^2q(dx)\geqq(1-c\varepsilon)\int_{[0,a]}|\phi'(x)|^2dx - c\delta\int_{[0,a]}|\phi(x)|^2dx \tag{3.9}$$

holds for every smooth function ϕ on $[0, a]$ satisfying $\phi'(0) = 0$ and $\phi(a) = 0$. The proof can be done similarly to (3.1). However to prove an analogous estimate for ϕ such that $\phi'(0) = 0$ and $\phi'(a) = 0$, we have to restrict $a \geqq 2$ and replace ε by 2ε and δ by 2δ in (3.9). Actually we have only to divide the integral $\int_{[0,a]} |\phi(x)|^2 \times q_-(dx)$ into the integration of the same thing on the intervals $[0, a/2]$, $[a/2, a]$ and use a similar estimate

$$|\phi(x)|^2 \leqq \varepsilon \int_{x-1}^{x} |\phi'(x)|^2 dx + \delta \int_{x-1}^{x} |\phi(x)|^2 dx.$$

One conclusion from (3.9) is that $f(x)$ never vanishes on $\mathbf{R}_+$, and hence $f(x) > 0$ on $\mathbf{R}_+$. Similarly we can prove $f'(x) \neq 0$ on $[2, \infty)$. In order to prove (3.8) we need the monotone increasing property of $f(x)$ on $[2, \infty)$, that is

$$f'(x) > 0 \qquad \text{on } [2, \infty). \tag{3.10}$$

Since f is positive on $\mathbf{R}_+$, we have

$$\begin{aligned} f(x) &= f(2) + f'(2)(x-2) + \int_2^x dy \int_2^y f(z)(q(dz) - \lambda\, dz) \\ &\geqq f(2) + f'(2)(x-2) - \int_2^x dy \int_2^y f(z)(q_-(dz) - \lambda\, dz) \\ &= f(2) + f'(2)(x-2) - f(2) \int_2^x (Q(y) + \lambda(y-2))dy \\ &\quad + \int_2^x dy \int_2^y f'(z)(Q(z) - Q(y) + \lambda(z-y))dz, \end{aligned} \tag{3.11}$$

where $Q(x) = q_-[2, x)$. Now assume $f'(x) \leqq 0$ on $[2, \infty)$. Then noting

$$0 \leqq Q(z) - Q(y) \leqq c(z - y + 1) \tag{3.12}$$

for $2 \leqq y \leqq z$ if $q \in \Omega_c$, we have

$$\begin{aligned} f(x) &\geqq f(2) + f'(2)(x-2) + \kappa \int_2^x dy \int_2^y f(z)dz + c \int_2^x dy \int_2^y f'(z)dz \\ &\quad - f(2) \int_2^x (Q(y) - c(y-2))dy \\ &\geqq f(2) + (f'(2) - 2cf(2))(x-2) + c\int_2^x f(y)dy + \kappa \int_2^x dy \int_2^y f(z)dz, \end{aligned} \tag{3.13}$$

where we have used the fact $Q(y) \leqq c(y-1)$ for $y \geqq 2$. On the other hand, since λ is not in the spectrum, f does not belong to $L^2([2, \infty), dx)$. Therefore, together with $0 < f(x) \leqq f(2)$ on $[2, \infty)$, this implies $\int_2^\infty f(y)dy = \infty$. However, in view of (3.13), this shows the unboundedness of $f(x)$ on $[2, \infty)$, which contradicts the assumption $f'(x) \leqq 0$ on $[2, \infty)$. Hence we can conclude (3.10). Coming back to

(3.11), we have

$$f(x) \geqq f(2) + f'(2)(x-2) - f(2)\int_2^x Q(y)dy - \lambda \int_2^x dy \int_2^y f(z)dz$$

$$\geqq f(2) + f'(2)(x-2) + f(2)\int_2^x (-\lambda(y-2) - Q(y))dy$$

$$\geqq f(2) + f'(2)(x-2) - c(x-2) + \frac{\kappa}{2} f(2)(x-2)^2,$$

which yields the desired (3.8). Here we have used (3.10) and $Q(y) \leqq c(y-1)$.

From the definition of h_+, we see

$$-h_+(\lambda, q)^{-1} = \int_0^\infty \frac{dx}{\phi_\lambda(x,q)^2} = \int_0^\infty \frac{dx}{f(x)^2}, \tag{3.14}$$

for at least $\lambda < -2c\delta$ because of (3.8). It is not difficult to see that $\phi_\lambda(x,q)$ as a function of q on Ω is continuous for each fixed $\lambda \in \mathbf{C}$. Therefore the uniform bound (3.8) combined with (3.14) shows the continuity of $h_+(\lambda, q)$, and hence that of $g_\lambda(0,0,q)$ on Ω_c for $\lambda < -2c\delta$. For general $\lambda \in \mathbf{C} \backslash [-c\delta, \infty)$ we have only to note (3.3) and $g_\lambda(0,0,q)$ can be represented by a Stieltjes transformation of a non-negative Radon measure on $[-c\delta, \infty)$. This completes the proof.

The following fact was already pointed out in [3], however for the sake of completeness we give the proof here again.

Lemma 5. *Let* $\{h_n\}$ *be a sequence of holomorphic functions on* $\mathbf{C}_+$ *with non-negative imaginary part. For a Borel set* A *of* $\mathbf{R}$ *assume* $\operatorname{Re} h_n(\xi + i0) = 0$ a.e. *on* A *holds. Then this is true also for any limit point* h *of* $\{h_n\}$ *in the sense of point-wise convergence on* $\mathbf{C}_+$.

Proof. $z = \lambda - i/\lambda + i$ maps $\mathbf{C}_+$ onto $\mathbf{D} = \{z \in C; |z| < 1\}$ comformally. Set $f_n(z) = \log h_n(\lambda)$ and $f(z) = \log h(\lambda)$. Then h_n, $h \in \mathbf{C}_+$ implies $0 < \operatorname{Im} f_n(z) < \pi$, $0 < \operatorname{Im} f(z) < \pi$ for every $z \in \mathbf{D}$. The identity in Lemma is equivalent to

$$\operatorname{Im} f_n(e^{i\theta}) = \frac{\pi}{2} \text{ a.e.} \qquad \text{on } \hat{A},$$

where $\hat{A}$ is the image of A by the above fractional mapping. We can suppose $\operatorname{Im} f_n(z) \to \operatorname{Im} f(z)$ for every $z \in \mathbf{D}$ for simplicity. Here note that every bounded harmonic function on $\mathbf{D}$ can be represented by its boundary value through the Poisson kernel. Therefore the above convergence combined with the denseness of all linear combinations of the Poisson kernels in $L^2(\partial\mathbf{D}, d\theta)$ shows the weak convergence of $\operatorname{Im} f_n(e^{i\theta})$ to $\operatorname{Im} f(e^{i\theta})$ in $L^2(\partial\mathbf{D}, d\theta)$. Especially the property $\operatorname{Im} f_n(e^{i\theta}) = \pi/2$ a.e. on $\hat{A}$ is inherited by $\operatorname{Im} f(e^{i\theta})$, which proves the lemma.

Proof of Theorem 2. We remark that since the property $c(q_-) \leqq c$ is preserved in the limit, $\operatorname{supp} P \subset \Omega_{c(P)}$ is valid for any shift invariant ergodic probability measure P on Ω. Now choose any function q from $\operatorname{supp} P_1$. Then the assumption $\operatorname{supp} P_1 \subset \operatorname{supp} P_2$ together with Lemma 3 implies that there exists $\{q_n\}$ in Ω_c

($c = c(P_2)$) such that

$$\operatorname{Re} g_{\xi+i0}(0,0,q_n) = 0 \text{ a.e.} \quad \text{on } \Sigma_{\text{a.c.}}(P_2),$$
$$q_n \to q \quad \text{in } \Omega_c.$$

Therefore from Lemmas 4 and 5 we can conclude $\operatorname{Re} g_{\xi+i0}(0,0,q) = 0$ a.e. on $\Sigma_{\text{a.c.}}(P_2)$. However this implies in particular $\operatorname{Im} g_{\xi+i0}(0,0,q) \neq 0$ a.e. on $\Sigma_{\text{a.c.}}(P_2)$, which shows $\Sigma_{\text{a.c.}}(P_2) \subset$ the absolutely continuous spectrum of $L(q)$. This completes the proof.

Remark. In the proof of Theorem 2 we have not used $\int_\Omega |q[0,1)| P_1(dq) < \infty$. Moreover what we have really proved is the following:

"*Suppose that we are given a shift invariant ergodic probability measure P on Ω satisfying $\int_\Omega |q[0,1)| P(dq) < \infty$. Then for any $q \in \operatorname{supp} P$, $\Sigma_{\text{a.c.}}(P) \subset$ the absolutely continuous spectrum of $L(q)$.*"

Remark. In the lattice case also it is possible to establish theorems corresponding to Theorems 1 and 2 if we take as Ω the set of all bounded sequences $\mathbf{Z}^d$ with the point-wise convergence topology by applying [5].

Remark. Obviously the restrictions $c(P) < \infty$ and $\int_\Omega |q[0,1)| P(dq) < \infty$ are too strong to prove the theorems. It is natural to conjecture that in the one-dimensional case the two theorems are valid for any pair of shift invariant ergodic probability measures on Ω: Ω is the set of all potentials defining unique self-adjoint extension.

We close this section by giving deterministic random potentials with no absolutely continuous spectrum.

Let $\{X_x(\omega);\ x \in \mathbf{R}\}$ be an L^2-continuous ergodic stationary Gaussian process with mean zero. We assume that $\{X_x(\omega)\}$ is non-constant. Then the ergodicity is equivalent to the continuity of the spectral measure of its variance. This implies the strict positive definiteness of the bilinear form on $L^2([-T,T], dx)$ induced by the variance for any $T > 0$. Therefore the support of the probability measure on $L^2([-T,T], dx)$ induced by this Gaussian process coincides with the full space $L^2([-T,T], dx)$. Let F be a bounded continuous function on R. If we assume that F is non-constant, then the support of the probability measure on Ω induced by the process $\{F(X_x(\omega))\}$ contains a set $\{q; \inf F \leqq q(x) \leqq \operatorname{supp} F,\ q \text{ is continuous}\}$. Since the latter set coincides with the support of the stationary random process considered by the Russian school (a functional of a Brownian motion on a compact Riemanian manifold), applying Theorem 2, we easily see that the following random Schrödinger operator:

$$L(\omega) = -\frac{d^2}{dx^2} + F(X_x(\omega))$$

has no absolutely continuous spectrum almost surely. In [3] it was proved that any non-deterministic random potential gives the absence of absolutely continuous spectrum. The above examples show that there exist a lot of deterministic random potentials with this property.

In a forthcoming paper by Kirsch, Kotani and Simon, various examples with no absolutely continuous spectrum will be given by using Theorem 2.

Acknowledgement. The author appreciates B. Simon and W. Kirsch who gave him valuable comments, especially pointed out the importance of generalizing classes of potentials.

References

1. Kato, T.: Perturbation theory for linear operators. Berlin, Heidelberg, New York: Springer 1966
2. Kirsch, W., Martinelli, F.: On the spectrum of Schrödinger operators with a random potential. Commun. Math. Phys. **85**, 329–350 (1982)
3. Kotani, S.: Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators. Proc. Stoch. Analy. Kyoto 1982
4. Kunz, H., Souillard, B.: Sur le spectre des opérateurs aux différences finies aléatoires. Commun. Math. Phys. **78**, 201–246 (1980)
5. Simon, B.: Kotani theory for one-dimensional stochastic Jacobi matrices. Commun. Math. Phys. **89**, 227–234 (1983)

Communicated by B. Simon

Received May 25, 1984; in revised form June 19, 1984