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The Asymptotic Higgs Field of a Monopole

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Abstract. A simple formula is computed for the asymptotic Higgs field of an SU(2) monopole. This formula is derived from the twistor description of monopoles, and is applied to the study of boundary behaviour. It is found to be harmonic, and to have as its natural domain of definition a branched covering of $\mathbb{R}^3$. Explicit formulae are given in simple cases.

1. Introduction

The purpose of this paper is to compute a simple asymptotic formula for the Higgs field of an SU(2)-monopole, valid up to exponentially decreasing terms: the "algebraic" part of the Higgs field [11]. This formula is derived from the twistor description of monopoles, and is intimately linked with the geometry of the monopole's spectral curve [3], as follows: a point in $\mathbb{R}^3$ corresponds to a section of $T(\mathbb{P}_1(\mathbb{C}))$ over $\mathbb{P}_1(\mathbb{C})$; this section intersects the spectral curve in $2k$ points; choosing k out of the $2k$ (asymptotically this choice is canonical), one evaluates the slope of the curve at these points, and sums, adding 1, to obtain the asymptotic norm of the Higgs field. This expression is harmonic, and is seen, in particular cases, to arise from a "charge distribution" on a union of compact disk-like surfaces in $\mathbb{R}^3$. We also apply the formula to give a direct proof that the boundary conditions of a monopole are satisfied when the spectral curve satisfies the conditions given in [3].

Let P be a principal SU(2)-bundle over $\mathbb{R}^3$, p its associated su(2) bundle, Φ a section of p, ∇ a connection on P, with F its associated curvature. The couple (∇, Φ) is an SU(2) monopole if the following conditions are satisfied:

1) $*F = \nabla\Phi$, where $*$ is the Hodge star operator on two-forms over $\mathbb{R}^3$. (Bogomolny equations [1])

2) The boundary conditions, as $r \to \infty$:

a) $|\Phi| = 1 - k/2r + O(r^{-2})$,
b) $\partial|\Phi|/\partial\Omega = O(r^{-2})$,
c) $|\nabla\Phi| = O(r^{-2})$.

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The su(2) norm is given by $-\mathrm{tr}^2/2$. $\partial/\partial\Omega$ is the angular derivative, given in spherical coordinates by $\partial f/\partial\Omega = [(\partial f/\partial\theta)^2 + \sin^2\theta(\partial f/\partial\varphi)^2]^{1/2}$. The k in $k/2r$ is an integer, and is called the charge of the monopole.

We note in passing that the Bogomolny equations, the formula [8] $|\nabla\Phi|^2 = \Delta|\Phi|^2/2$, and our formula for $|\Phi|$ yield an asymptotic formula for $|F|$.

The first solutions produced [7] were of charge one, and depended on three spatial parameters.Subsequent solutions of higher charge were obtained using twistor methods; see Ward [8, 9], Prasad [6], and Corrigan and Goddard [2]. A systematic account of the twistor side of the theory is given in Hitchin [3]; we recall briefly the methods involved; this will serve the purpose of fixing notation.

Let $\tilde{T}$ be the space of oriented straight lines in $\mathbb{R}^3$; $\tilde{T}$ has a natural holomorphic structure under which $\tilde{T} = T(\mathbb{P}_1(\mathbb{C}))$, the holomorphic tangent bundle of $\mathbb{P}_1(\mathbb{C})$. $\tilde{T}$ has a natural real structure γ, with no fixed points, given by reversal of orientation of the lines. Also, fixing a point x, one obtains a section $C_x\colon \mathbb{P}_1(\mathbb{C}) \to \tilde{T}$, whose image is the set of lines through x; these sections are γ-invariant, and so are called real. Note that any two sections C_x, $C_{x'}$ intersect in two points; this corresponds to the two oriented lines in $\mathbb{R}^3$ passing through x and x'. Finally, let z' be an inhomogeneous coordinate on $\mathbb{P}_1(\mathbb{C})$; $(w,z) \to w\partial/\partial z'|_{z'=z}$ gives local coordinates on $\tilde{T}$, in which $\gamma(w,z) = (-\bar{w}/\bar{z}^2, -1/\bar{z})$. Also, if $x = (x_1, x_2, x_3)$, fix the correspondence between $\mathbb{R}^3$ and $\tilde{T}$ by setting the image of C_x to be $w = (x_1 + ix_2)z^2 - 2x_3 z - (x_1 - ix_2)$.

Let E be the standard rank 2 bundle associated to P; if r is an oriented line in $\mathbb{R}^3$, set $\tilde{E}_r = \{s \in \Gamma(r, E) \colon (\nabla_u - i\Phi)s = 0,\ u \text{ a positive unit vector}\}$. This defines a 2-bundle $\tilde{E}$ over $\tilde{T}$, with a quaternionic structure (an anti-linear lift σ of γ such that $\sigma^2 = -1$) and a symplectic structure (a nowhere degenerate skew form on $\tilde{E}$); if (∇, Φ) is a solution to the Bogomolny equations, $\tilde{E}$ has a natural holomorphic structure. Let L be the holomorphic line bundle on $\tilde{T}$ defined by the transition function $\exp(-w/z)$ from $\{z \neq \infty\}$ to $\{z \neq 0\}$; let $\mathcal{O}(n)$ be the lift to $\tilde{T}$ of the bundle $\mathcal{O}(n)$ on $\mathbb{P}_1(\mathbb{C})$; set $L(n) = L \otimes \mathcal{O}(n)$; then using the boundary conditions, one can show that $\tilde{E}$ can be written in two ways as a holomorphic extension:

$$0 \to L(-k) \to \tilde{E} \to L^*(k) \to 0, \qquad 0 \to L^*(-k) \to \tilde{E} \to L(k) \to 0. \tag{1}$$

These are permuted by the quaternionic structure. Let S be the curve over which $L(-k)$, $L^*(-k)$ coincide; S is a curve in the linear system $|\mathcal{O}(2k)|$, and is called the spectral curve of the monopole. One has:

Theorem 1 [3].

i) *S is compact.*

ii) *S can be written as $0 = w^k + a_1(z)w^{k-1} + \cdots + a_k(z)$, will a_i polynomial, of degree $2i$.*

iii) *L^2 is holomorphically trivial on S.*

iv) *S is preserved by γ, and $L(-k)$ has a quaternionic structure over S.*

Conversely, a curve S satisfying the above conditions (a spectral curve) yields back a (possibly singular) solution to the Bogomolny equations. This is done as follows. From the exact sequence on $\tilde{T}$, $0 \to L^2(-2k) \to L^2 \to L^2|_S \to 0$, one has the coboundary map, which here is an injection $\delta: H^0(S, L^2) \to H^1(\tilde{T}, L^2(-2k))$; as this last group describes extensions of $L^*(k)$ by $L(-k)$, choosing a non-zero section a in $H^0(S, L^2)$ gives a symplectic 2-bundle $\tilde{E}$ corresponding to $\delta(a)$, fitting into a sequence (1); condition iv) above then enables $\tilde{E}$ to possess a quaternionic structure. Alternatively, one can also obtain $\tilde{E}$ as an extension of $L(k)$ by $L^*(-k)$, using the fact that L^{*2} is also trivial on S.

From $\tilde{E}$ one obtains a solution to the Bogomolny equations on $\mathbb{R}^3$. First of all, assuming that $\tilde{E}$ is trivial on the real sections C_x, one obtains a 2-bundle E on $\mathbb{R}^3$ by setting $E_x = H^0(C_x, \tilde{E})$. We now define the Higgs field as an element of $\mathrm{End}(E)$, and also the connection, along an oriented line. For convenience, set r to be the line $x_1 = x_2 = 0$, with the positive direction being given by $x_3 \to -\infty$. r then corresponds to $(0,0)$ in $\tilde{T}$; setting $b = -2x_3$, the points $(0,0,x_3)$ correspond to real sections $w = bz$, passing through $(0,0)$ and $\gamma(0,0)$; along this pencil of sections, there are two natural connections ∇_0 and ∇_∞ over E, determined by evaluation of the corresponding elements of $H^0((w = bz), \tilde{E})$ at $(w,z) = (0,0)$ and $\gamma(0,0)$ respectively. $\Phi\, db = i(\nabla_0 - \nabla_\infty)$ and $\nabla_3 = (\nabla_0 + \nabla_\infty)/2$ then define the Higgs fields and the connection along the line.

2. The Asymptotic Higgs Field

Let $\tilde{E}$ be the bundle constructed from a spectral curve S; let $\mathbb{R}^3, \tilde{T}$ be parametrized as in the introduction. Cover $\tilde{T}$ by $V_1 = \{z \neq \infty\}$, $V_2 = \{z \neq 0\}$. We will be looking at the behaviour of $\tilde{E}$ on the sections $w = bz$.

Blow up $\tilde{T}$ at $(0,0)$ and $\gamma(0,0)$ to a surface $\tilde{T}'$ (for a complete treatment of blowing up, see [10]). The sections $w = bz$ lift to a ruling of lines; we compute a transition matrix for the lift of $\tilde{E}$ to this ruling; this corresponds to computing the transition matrix on $\tilde{T}$ from the set $V'_1 = V_1 - \{z = 0,\ w \neq 0\}$ to the set $V'_2 = \gamma(V'_1)$.

As $\tilde{E}$ is obtained as an extension of $L(k)$ by $L^*(-k)$ via the injection $\delta: H^0(S, L^{*2}) \to H^1(\tilde{T}, L^{*2}(-2k))$, setting f to be a non-zero element of $H^0(S, L^{*2}) = \mathbb{C}$, we first construct a Cech cocycle representative $\delta(f)_{12}$ for $\delta(f)$ relative to the covering V'_1, V'_2 [10].

One has coordinates $(z, b = w/z)$ on V'_1, $(1/z, b)$ on V'_2; represent the section f as $f_i(b,z)$ over $V'_i \cap S$; $f_2(b,z) = \exp(2b) f_1(b,z)$. To extend f_i to functions F_i on V'_i, we use Lagrange interpolation. More precisely, we suppose that S intersects $z = 0$ at $w = c_i$, $i = 1, \ldots, k$, with all c_i distinct, $\neq 0$; let S intersect the sections $w = bz$ at points $z = z_i(b)$, $i = 1, \ldots, 2k$; set $f_{ij}(b) = f_j(b, z_i(b))$, $i = 1, \ldots, 2k$, $j = 1, 2$. The $F_j(b,z)$ are constructed by interpolating the $f_{ij}(b)$ in the z-direction, with $F_1(b,0) = F_2(b,\infty) = 0$. One then has $\delta(f)_{12} = [F_1 - \exp(-2b)F_2]/\Pi(z - z_i(b))$; from this cocycle the transition matrix is then constructed:

$$T_{21}(z,b) = \begin{bmatrix} \exp(b) z^k & K(z,b) \\ 0 & \exp(-b) z^{-k} \end{bmatrix},$$

with

$$K(z,b) = \exp(b)z^k(\delta(f)_{12})$$
$$= \exp(b)\sum_{i=1}^{2k} a_i(b)(z^k + z_i(b)z^{k-1} + \cdots + z_i(b)^{2k}z^{-k}),$$
$$a_i(b) = \frac{f_{i1}(b)}{z_i(b)\prod_{j\neq i}(z_i(b) - z_j(b))}.$$

We use this transition matrix to prove:

Proposition 2. $\tilde{E}$ *defines a solution* (∇, Φ) *to the* SU(2) *Bogomolny equations, which is non-singular outside of a compact analytic set.*

Remark. This set has one defining equation; it is, generically, a surface.

We first prove a lemma:

Lemma 3. *Let* $a_i, c_i \in \mathbb{C}$, $i = 1, \ldots, k$. *Let* $N_{s,t} = \sum_{i=1}^{k} a_i c_i^{k-1-s+t}$, $s, t = 1, \ldots, k$. *Then* $\det N = \prod_{i=1}^{k} a_i \prod_{\substack{i,j=1\\ i\neq j}}^{k} (c_i - c_j)$.

Proof. Let V be the Vandermonde matrix $V_{i,j} = c_j^{k-i}$; let $W_{i,j} = V_{j,k-i+1} = c_i^{j-1}$; then the matrix above can be written as $V\operatorname{diag}(a_1, a_2, \ldots, a_k)\,W$; the lemma follows.

Proof of the proposition. A solution will be non-singular at a point x if $\tilde{E}$ is holomorphically trivial over the corresponding section C_x in $\tilde{T}$; if this is not so, we call C_x a jumping section. Using the jumping criterion of [2], we see that $w = bz$ is a jumping section iff the $k \times k$ determinant $\det(M_{ij}(b)) = 0$, where

$$M_{ij}(b) = \sum_{n=1}^{2k} a_n(b) z_n(b)^{k-i+j}.$$

Suppose now that $b \to +\infty$ in our expression for T_{12}; the reality of $w = bz$ and of S, and the compactness of the curve S imply that half of the points $z_i(b)$ tend to zero, with the other half going to infinity; renumbering, let $z_i \to 0$, $z_{k+i} = \gamma(z_i)$, $i = 1, \ldots, k$. Furthermore, bz_i, $i = 1, \ldots, k$, tends to a point of $S \cap (z = 0)$; again renumbering, set $bz_i \to c_i$; by reality, $z_{k+i}/b \to -1/\bar{c}_i$. Also, as $b \to +\infty$ and f_2 is bounded in a neighbourhood of $(z = \infty)$, $f_{i1} = \exp(-2b) f_{i2} = O(\exp(-2b))$ decreases exponentially with b, for $i = k+1, \ldots, 2k$. We get for $K(z, b)$,

$$K(z,b) = \exp(b)\left[\sum_{i=1}^{k} a_i(b)(z^k + z_i z^{k-1} + \cdots + z_i^{2k} z^{-k})\right] + O(\exp(-b)),$$

with

$$a_i(b) \to a_i(\infty) = \frac{f_1(c_i, 0)\prod_{j=1}(-\bar{c}_j)}{c_i \prod_{j\neq i}(c_i - c_j)}.$$

Returning now to M_{ij}, multiply row i by b^{k-i}, column j by b^j; as $b \to \infty$, the transformed matrix tends to

$$M'_{ij} = \sum_{n=1}^{k} a_n(\infty) c_n^{k-i+j},$$

using lemma 3, this has non-zero determinant, and so for $b > b_0$, say, $w = bz$ is a non-jumping section. In this way, varying the origin $z = 0$, and also the corresponding transition matrices, one can obtain a bound on an open patch of directions in $S_2 = \mathbb{P}_1(\mathbb{C})$.

We now must consider two cases left out: 1) S has multiple points at $z = 0$; this case goes through exactly as above; one still obtains in the limit matrices that are non-singular. 2) $(0,0) \in S$. This can be avoided by changing the origin in $\mathbb{R}^3$.

Thus one can cover the compact sphere of directions with patches on which the surface is bounded; the theorem is proved.

Remark. Hitchin [4] has shown that the additional condition for non-singularity of the monopole is $H^0(S, L^t(k-2)) = 0$, for $t \in (0,2)$.

We now turn to finding the asymptotic formula for the Higgs field. The principle will be the same as in the preceding proposition; a matrix contains sums indexed from 1 to $2k$; these degenerate asymptotically into sums indexed from 1 to k; the matrix becomes of Vandermonde type, and easily tractable.

Again, we compute along the line $w = bz$. Let ∇_0, ∇_∞ be defined as in the introduction. Suppose that $b > b_0$; let $s(b,z) = (s_1(b,z), s_2(b,z))$, $t(b,z) = (t_1(b,z), t_2(b,z))$ be sections of $\tilde{E}$ on $w = bz$ given in the V_1' trivialisation, with $s(b,0) = (1,0)$, $t(b,0) = (0,1)$; set $u(b,z)^{\mathrm{tr}} = T_{21}(b,z)s(b,z)^{\mathrm{tr}}$, $v(b,z)^{\mathrm{tr}} = T_{21}(b,z)t(b,z)^{\mathrm{tr}}$. (s,t) is a ∇_0-flat basis for E along $x_1 = x_2 = 0$ in $\mathbb{R}^3$; as the coordinates of s,t in a ∇_∞-flat basis are $u(b,\infty)$, $v(b,\infty)$, we have:

$$\Phi = -i\begin{bmatrix} v_2(b,\infty), & -v_1(b,\infty) \\ -u_2(b,\infty), & u_1(b,\infty) \end{bmatrix} \cdot \partial_b \begin{bmatrix} u_1(b,\infty), & v_1(b,\infty) \\ u_2(b,\infty), & v_2(b,\infty) \end{bmatrix},$$

remembering $\det T_{12} = 1$. Set

$$\begin{aligned} u_1(b,\infty) &= \exp(b)\tilde{u}_1(b), \quad u_2(b,\infty) = \exp(-b)\tilde{u}_2(b), \\ v_1(b,\infty) &= \exp(b)\tilde{v}_1(b), \quad v_2(b,\infty) = \exp(-b)\tilde{v}_2(b). \end{aligned}$$

s,t are determined by the condition that $T_{21}s^{\mathrm{tr}}$, $T_{21}t^{\mathrm{tr}}$ are continuous at $z = \infty$; writing $s_i(b,z)$ as $\Sigma s_{ij}(b)z^j$, etc., one can show that $s_{ij} = 0$ for $j > k$; one has $\tilde{u}_2(b) = s_{2k}(b)$, and also, substituting $\tilde{s}_{2j}(b) = s_{2j}(b)b^{-j}$, $u_1 * (b) = \tilde{u}_1(b)b^k$, one has:

$$\begin{bmatrix} \Sigma a_i(bz_i)^k, & \Sigma a_i(bz_i)^{k+1}, \ldots, \Sigma a_i(bz_i)^{2k} \\ \Sigma a_i(bz_i)^{k-1}, & \Sigma a_i(bz_i)^k, \quad \ldots, \Sigma a_i(bz_i)^{2k-1} \\ \vdots & \vdots \qquad\qquad \vdots \\ \Sigma a_i(bz_i), & \Sigma a_i(bz_i)^2, \quad \ldots, \Sigma a_i(bz_i)^{k+1} \\ \Sigma a_i, & \Sigma a_i(bz_i), \quad \ldots, \Sigma a_i(bz_i)^k \end{bmatrix} \cdot \begin{bmatrix} s_{20} \\ \tilde{s}_{21} \\ \tilde{s}_{22} \\ \vdots \\ \tilde{s}_{2k} \end{bmatrix} = \begin{bmatrix} u_{1*} \\ 0 \\ 0 \\ \vdots \\ 0 \\ s_{10} \end{bmatrix}, \tag{2}$$

where the summations are taken over i from 1 to $2k$. a_i and bz_i tend to finite, non-zero limits as $b \to \infty$, for $i \in (1, \ldots, k)$; for $i \in \{k+1, \ldots, 2k\}$, a_i decays exponentially. Using Lemma 3, the limit equation is non-singular; thus, upto exponentially decreasing terms, one may approximate solutions to (2) by solutions to the same equation, with the difference that the summations are taken from 1 to k; we also make the following remark:

(*) The exponentially decreasing terms are of the form $\exp(-nb)$ (meromorphic function in b); thus all their derivatives also decay exponentially; we can take derivatives of the approximations with impunity, and obtain approximations to the derivatives.

Row-reducing (2), remembering that $s_{10}=1$, $t_{10}=0$, we obtain:

$$\tilde{u}_1(b)=(-1)^k\prod_{i=1}^{k} z_i(b),$$

and similarly, $\tilde{v}_1(b)=0$. The symplectic structure then ensures that $\tilde{v}_2(b)=\tilde{u}_1(b)^{-1}$. Substituting in the expression for Φ, one obtains the asymptotic formula, valid up to an exponentially decaying term:

$$|\Phi|_{as}=1+\partial_b \ln\left[\prod_{i=1}^{k} z_i(b)\right]. \tag{3}$$

This formula extends to the (parallel) lines l_a of real sections $w=az^2+bz-\bar{a}$, b varying in $\mathbb{R}$; setting $\{z_i(a,b)\}=\{w=az^2+bz-\bar{a}\}\cap S$, numbering the z_i's as above, one has the same formula, with $z_i(a,b)$ substituted for $z_i(b)$.

We now give an intrinsic interpretation of (3). Defining S locally by $p(w,z)=0$, one has $p(az_i(a,b)^2+bz_i(a,b)-\bar{a},\ z_i(a,b))=0$. Taking the derivative of this with respect to b, one obtains:

$$\partial_b \ln(z_i(a,b))=\frac{-\partial p/\partial w}{(2az_i(a,b))+b)\,\partial p/\partial w+\partial p/\partial z}=\frac{-(\partial/\partial w)p}{i_*(\partial/\partial z)p}, \tag{4}$$

where $i\colon \mathbb{P}_1(\mathbb{C})\to\tilde{T}$ is the section whose image is $w=az^2+bz-\bar{a}$. One has the natural exact sequence for the tangent bundle of $\tilde{T}$:

$$0\to TF\to T(\tilde{T})\to\pi^*(T(\mathbb{P}_1(\mathbb{C})))\to 0,$$

where TF is the bundle of tangents to the fibers of $\tilde{T}$; the map i_* gives a splitting of the above along $w=az^2+bz-\bar{a}$. Equation (4) is then just the ratio of the components of a tangent vector to the spectral curve with respect to this splitting and the natural identification $TF=\pi^*(T(\mathbb{P}_1(\mathbb{C})))$.

One thus has the following picture of $|\Phi|_{as}$ at a point x far from the origin in $\mathbb{R}^3$, along a direction corresponding to $z=z_0$ in $\mathbb{P}_1(\mathbb{C})$; one looks at the k (out of $2k$) points in $(S\cap C_x)$ that are near $z=z_0$ in $\tilde{T}$; one evaluates (4) at these points, and then sums to obtain (3). Thus, the asymptotic Higgs field is determined by the choice of a set W of k points z_i out of $2k$, such that $(z_i\in W)\Rightarrow(\gamma(z_i)\notin W)$; asymptotically, this choice is natural, as the z_i's cluster into two distinct sets.

Formula (4) also shows where $|\Phi|_{as}$ can become singular: at sections tangent to the spectral curve. However, it is not all the tangents that give singularities; at those where two z_i's$\in W$ come together, $|\Phi|_{as}$ remains finite; it is only at sections where $z_i\in W$ and $z_j\notin W$ meet that singularities occur: this singularity, as we shall see, is generically a curve, and $|\Phi|_{as}$ is of order $d^{-1/2}$, where d is the distance to the curve in $\mathbb{R}^3$.

Note that the real structure of S and of the sections forces the tangents to come in pairs; thus the singular set is a subvariety of the variety of bitangent sections.

Proposition 4. *Let S be a compact curve in $\tilde{T}$. The set of bitangent sections to S is a union of algebraic curves in $\mathbb{C}^3$.*

Sketch of proof. The tangent sections to S are parametrised by a surface M in $\mathbb{C}^3$. It suffices to show that each component of M has a section that is simply tangent at one point only. Suppose that $w=0$ is tangent to S at $(w,z)=(0,0)$; then $w=cz^2$, $c\in\mathbb{C}$, is also tangent. Letting $c\to\infty$, one can show that there must be a simply tangent section.

Restricted to the $\mathbb{R}^3$ of real sections in $\mathbb{C}^3$, the bitangent variety is a set of algebraic curves C_n, $n=1,\dots,\alpha$ and of points P_m, $m=1,\dots,\beta$. If $|\Phi|_{as}$ becomes singular at C_n, branching around C_n has the effect of interchanging z_i''s in W and z_j''s not in W. Thus the appropriate domain of definition of $|\Phi|_{as}$ will be a branched covering of $\mathbb{R}^3$.

Theorem 5. *$|\Phi|_{as}$ is naturally defined on a branched cover $F:\tilde{\mathbb{R}}^3\to\mathbb{R}^3$ branching over a subset $(C_n,\ n\in A)$ of the curves of bitangents, with at most 2^k sheets. Away from $F^{-1}(C_n,\ n\in A,\ P_m,\ m=1,\dots,\beta)$, $|\Phi|_{as}$ is harmonic.*

Proof. Away from C_n, $\tilde{\mathbb{R}}^3$ is defined from pairs $(U, {}_U|\Phi|_{as})$, where U is an open subset of $\mathbb{R}^3$, and ${}_U|\Phi|_{as}$ is a single valued analytic continuation of $|\Phi|_{as}$ over U; $(U, {}_U|\Phi|_{as})$ and $(V, {}_V|\Phi|_{as})$ are identified over $U\cap V$ iff ${}_U|\Phi|_{as}={}_V|\Phi|_{as}$ over $U\cap V$. However, the different possible values of $|\Phi|_{as}$ are determined by the choice of k out of $2k$ points, in a way that respects the real structure; there are 2^k such choices.

To see that $|\Phi|_{as}$ is harmonic, we use the Penrose transform. This associates to each element f of $H^1(\tilde{T},\mathcal{O}(-2))=H^1(\tilde{T},\pi*(\Omega^1(\mathbb{P}_1(\mathbb{C}))))$ a harmonic function ψ_f over $\mathbb{R}^3$; covering $\tilde{T}$ by our open sets V_1, V_2 and representing f by a cocycle $f(w,z)dz$ over $V_1\cap V_2$, ψ_f is given by $\int i_x*(f(w,z)dz)$, where i_x: $\mathbb{P}_1(\mathbb{C})\to\tilde{T}$ is the section associated to x, and the integral is taken over an appropriate contour in $i_x^{-1}(V_1\cap V_2)$. It is then easy to see that the cocycle $(-(\partial p/\partial w)/p)dz$ gives $2\pi i(|\Phi|_{as}-1)$ by this transform; the different values of $|\Phi|_{as}$ correspond to different contours of integration.

Remark. We will see in the examples that it is possible there to define cut surfaces D_s so that $|\Phi|_{as}$ is defined and continuous over $\mathbb{R}^3\backslash[$(the union of a subset of the set of branching curves)$\cup\{P_m, m=1,\dots,\beta\}]$, and so that away from the surfaces D_s, $|\Phi|_{as}$ is also harmonic. $|\Phi|_{as}$ is then the field generated by a "charge" distribution on the surfaces D_s and on the points P_m. On the interior of the D_s, the distribution is represented by a function, i.e. a surface charge density, whose sign is opposite to that of the monopole. It is tempting to conjecture that this is possible in the general case.

3. Boundary Conditions

Proposition 6. *The solution (∇,Φ) defined by $\tilde{E}$ satisfies the boundary conditions of a monopole.*

Proof. We use formula (3), and note that $\prod_{i=1}^{k} z_i(b)$ has a zero of order k at $b=\infty$;

therefore its logarithmic derivative is $-kb^{-1}+O(b^{-2})=-kr^{-1}/2+O(r^{-2})$; this gives the first boundary condition. To get the two others, noting that $|\Phi|_{as}$ is analytic at $r=\infty$, expanding in spherical coordinates, one writes $|\Phi|_{as}=1-kr^{-1}/2+h_2(\theta,\varphi)r^{-2}+\cdots$ Remembering remark (*) above, and noting, as in Ward [8], that $|\nabla\Phi|^2=\Delta(|\Phi|^2)/2$, the second and third boundary conditions follow.

4. Examples

1. The Axisymmetric Case

i) k even, $k=2h$.

One has axisymmetric solutions of charge k with spectral curves [4]:

$$\prod_{n=1}^{h}\left[w^2+\left(\frac{(2n+1)\pi z}{2}\right)^2\right]=0.$$

One obtains, in $\mathbb{R}^3$:

$$|\Phi|_{as}(x_1,x_2,x_3)=1-\sum_{n=1}^{h}\mathrm{Re}\left[\left(r^2-\left(\frac{(2n+1)\pi}{4}\right)^2-\left(\frac{i(2n+1)\pi x_3}{2}\right)\right)^{-1/2}\right].$$

This is singular on the circles $C_n=\{x_3=0,\ r=(2n+1)\pi/4\}$. Defining the cut surface $D=\{x_3=0,\ r<(2h+1)\pi/4\}$, $|\Phi|_{as}$ is continuous and single valued on $\mathbb{R}^3\backslash(C_n, n=1,\ldots,h)$; on $D\cap\{(2j-1)\pi/4<r<(2j+1)\pi/4\}$,, $\partial|\Phi|_{as}/\partial x_3$ has a discontinuity ("charge density") of

$$\sum_{n=j}^{h}\frac{(2n+1)\pi}{2}\left[\left(\frac{(2n+1)\pi}{4}\right)^2-r^2\right]^{-3/2}.$$

ii) k odd, $k=2h+1$.

One has solutions with spectral curves:

$$w\prod_{n=1}^{h}(w^2+(n\pi z)^2)=0.$$

Then,

$$|\Phi|_{as}(x_1,x_2,x_3)=1-r^{-1}/2-\sum_{n=1}^{h}\mathrm{Re}\left[\left(r^2-\left(\frac{(n\pi)}{2}\right)^2-in\pi x_3\right)^{-1/2}\right].$$

This is singular on the circles $C_n=\{x_3=0,\ r=(n\pi/2)\}$, and at $r=0$. Again, setting the cut surface $D=\{x_3=0,\ r<h\pi/2\}$, $|\Phi|_{as}$ becomes single valued, and one has the charge density, on $D\cap\{(j-1)\pi/2<r<j\pi/2\}$:

$$\sum_{n=j}^{h}(n\pi)\left[\left(\frac{n\pi}{2}\right)^2-r^2\right]^{-3/2}$$

with a point charge superimposed at the origin.

2. *The Case of Charge 2*

After translation and rotation, one can reduce the spectral curve to the form [5]:

$$w^2 = k(z-s)(z+s)(sz+1)(sz-1), \quad k>0, \quad s\in[0,1).$$

By solving the appropriate quartic, one can obtain $|\Phi|_{as}$; this will have a singularity of type $r^{-1/2}$ over the ellipse

$$(s^2-1)^2 x_1^2 + (s^2+1)^2 x_2^2 = k(s^4-1)^2/4, \quad x_3 = 0.$$

Again, one can take a cut surface D as the interior of the ellipse in the plane $x_3=0$; one can see this directly, as follows. The monopole has a symmetry of reflection in the $x_3=0$ plane, corresponding to the symmetry $z\to 1/\bar{z}$ in $\mathbb{P}_1(\mathbb{C})$; near D, this symmetry interchanges z_i''s in W with z_i''s not in W; thus one obtains either two possible cut surfacs, images of each other by the symmetry, or one cut surface in the $x_3=0$ plane. However, algebraic considerations of degree in the deformation from the axisymmetric case (the moduli space is connected [5] preclude the first possibly.

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