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A Proof of the Axial Anomaly

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Abstract. The local form of the axial anomaly with both left and right-handed gauge fields and a metric present is given and proved using the families index theorem

The axial anomaly is the variation of the determinant of the Dirac operator $\not{D}$ under axial changes of the gauge field [1]. More specifically, the zeta-function definition of operator determinants gives that $\ln\det\not{D} = -\frac{1}{2}$(finite part at $s=0$) $(d/ds)\operatorname{Tr}(\not{D}^2)^{-s}$. Under a variation $\delta\not{D} = \{\alpha\gamma^5, \not{D}\}$, $\delta \ln\det\not{D} = 2\lim\limits_{s\to 0}\operatorname{Tr}\alpha\gamma^5(\not{D}^2)^{-s}$. Thus all we need know is $\lim\limits_{s\to 0}\operatorname{Tr}\gamma^5(\not{D}^2)^{-s}(x,x) = \lim\limits_{s\to 0}(1/\Gamma(s))\int_0^\infty \operatorname{Tr}\gamma^5 T^{s-1}\cdot e^{-T\not{D}^2}(x,x)\,dT$, which is given by a certain term in the asymptotic expansion of $e^{-T\not{D}^2}(x,x)$. It has become clear that the nontriviality of this variation is related to the topology of the space of gauge fields modulo gauge transformations [2]. We wish to show that in fact the exact form of the anomaly is given from topological arguments. In the physics literature the above heat kernel term is computed by expanding the kernel perturbatively around the flat kernel [1]. We believe that the following is the first nonperturbative (i.e. nondiagrammatic) proof of the axial anomaly in arbitrary dimension. The situation is similar to the special case of vector gauge fields, in which the heat kernel expansion of the square of the Dirac operator can be used to prove the index theorem [3]. With axial gauge fields present the direct analysis is surprisingly complicated. We work backwards and use the families index theorem along with invariance arguments to derive the expression for the local anomaly.

In the physics terminology, Lemma 2 below is the Wess-Zumino consistency condition and Lemma 4 amounts to showing the uniqueness of its nontrival solution.

Let A^+, A^- be connections on a principal G-bundle $G \to P \to M$ with G a compact Lie group and M an even n-dimensional closed Riemannian spin manifold with metric g. Let V be an associated bundle to P and let $S = S^+ \oplus S^-$ be the

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spinor bundle over M. With the Clifford algebra $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$, define $\varepsilon = i^{n(n-1)/2} \gamma_1 \ldots \gamma_n$, so that $(1/2)(1+\varepsilon)$ is projection from $\Gamma(S)$ to $\Gamma(S^+)$. Define $\partial\!\!\!/_A$: $\Gamma(S^+ \otimes V) \to \Gamma(S^- \otimes V)$ as $\partial\!\!\!/_A = -i\gamma^j(\nabla_j + A_j)$, and define $D\!\!\!\!/_{A^+,A^-} : \Gamma(S \otimes V) \to \Gamma(S \otimes V)$ as $D\!\!\!\!/_{A^+,A^-} = \partial\!\!\!/_{A^+} + \partial\!\!\!/^\dagger_{A^-}$. Let τ denote a spinor trace, Tr denote a matrix trace and Tr denote an operator trace.

Let $a_n(x) \in \mathrm{End}(V \otimes S)$ be the n^{th} term in the asymptotic expansion $e^{-TD\!\!\!\!/^2_{A^+,A^-}}(x,x) \sim T^{-n/2} \sum_{i=0}^{\infty} a_i(A^+, A^-)(x) T^{i/2}$.

Lemma 1. *Under the isomorphism between $\Lambda^0(M)$ and $\Lambda^n(M)$, $\tau(\varepsilon(a_n(A^+, A^-) + a_n(A^-, A^+)))$ is a differential form on M which is a polynomial in A^+, A^-, g, g^{-1} and their derivatives.*

Proof. For convenience write $V = (1/2)(A^+ + A^-)$ and $A = (1/2)(A^+ - A^-)$, so that $D\!\!\!\!/^2_{A^+,A^-} = [-i\gamma^j(\nabla_j + V_j + \varepsilon A_j)]^2 = -D_j D_j + E$ with $D_j = \nabla_j + W_j$, $W_j = V_j + 2\varepsilon\sigma^{jk} A_k$, and $-E = \sigma^{ij}(V_{ij} + \varepsilon A_{ij}) - (1/4)R + \varepsilon \tilde{D}_i A_i + (n-4)\sigma^{ij}[A_i, A_j] + (n-2)A_i A_i$ Here $V_{ij} \equiv (1/2)(F(A^+)_{ij} + F(A^-)_{ij})$, $A_{ij} \equiv (1/2)(F(A^+)_{ij} - F(A^-)_{ij})$, $\tilde{D}_i A_j \equiv \nabla_i A_j + [V_i, A_j]$ and $\sigma^{ij} \equiv (1/4)[\gamma^i, \gamma^j]$.

Then $a_n(A^+, A^-) + a_n(A^-, A^+)$ is $\sqrt{\det g}$ times a polynomial in W, E, g, g^{-1} and their derivatives [4] which is even in A. The ε terms in E and W always accompany one A, and so there is one net ε in $\varepsilon(a_n(A^+, A^-) + a_n(A^-, A^+))$. But $\tau(\varepsilon \Pi \gamma)$ always contains a factor $\varepsilon^{\mu_1 \mu_2 \ldots \mu_n} = (\sqrt{\det g})^{-1}[\mu_1 \ldots \mu_n]$ and the factors of $\sqrt{\det g}$ cancel. □

By a faithful SU(N) representation of G we can hereafter assume $G = \mathrm{SU}(N)$ with N arbitrarily large. For a polynomial ω in g-valued forms with value in the universal enveloping algebra of g we do not assume *a priori* that ω is symmetrized in its components. If ω is a sum of terms of the form $\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_j$, let $s\omega$ be the sum of $(1/j!) \sum_{\sigma \in S_j} \pm \phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(j)}$ with the signs chosen such that if $\{\phi_i\}_{i=1}^j$ were all $\mathbb{R}$-valued then $s\omega = \omega$.

Theorem. *$s\underset{\sim}{A} \equiv (\frac{1}{2}) s(\tau(\varepsilon(a_n(A^+, A^-) + a_n(A^-, A^+))))$ is given by the following algorithm: Construct the secondary characteristic class*

$$S_k(A^+, A^-) = \frac{1}{(2\pi i)^k} \frac{1}{k!} \mathrm{Tr} \int_0^1 (A^+ - A^-) \wedge (TF^+ + (1-T)F^- - T(1-T)(A^+ - A^-) \wedge (A^+ - A^-))^k dT.$$

Under an infinitesimal gauge transformation write $(d/d\beta)|_{\beta=0} S_k(A^+ + \beta D_{A^+} X, A^{-1}) = d\,\mathrm{Tr}(X B_k)$ for a local expression $B_k(A^+, A^-)$. Put $B(A^+, A^-) \equiv \sum_{k=0}^{\infty} B_k(A^+, A^-)$. Then $s\underset{\sim}{A} =$ the Λ^n part of $B(A^+, A^-)\hat{A}(g)$.

Notes. 1. In general $\tau(\varepsilon a_n(A^+, A^-)) \neq \tau(\varepsilon a_n(A^-, A^+))$.

2. The symmetrization s ensures Bose symmetry. An explicit calculation in $n = 6$ shows that it is necessary.

3. Let $U \in \Gamma(\mathrm{Ad}\, P)$ be a gauge transformation. We can write Witten's remarkable low energy anomalous Lagrangian L [5] as $dL = S_{n/2}(UA^+U^{-1} +$

$U\,dU^{-1}, A^-$). (This L differs from that of [5], which uses a different subtraction scheme and is not independent of the choice of trivialization over $M = \mathbb{R}^4$).

As in [3], the metric must enter in a polynomial in the Pontryagin forms. Let V and V' be vector bundles over M and M' with connections $A^\pm$, $A'^\pm$. Take $\{\gamma_M \otimes 1,\ \varepsilon_M \otimes \gamma_{M'}\}$ as generators of the $n+n'$ dimensional Clifford algebra. By separation of variables,

$$\tau(\varepsilon_{n+n'}(a_{n+n'}(A^+ \oplus A'^+, A^- \oplus A'^-))) = \sum_k \tau(\varepsilon_n(a_{n+k}(A^+, A^-)))\tau(\varepsilon_{n'}(a_{n'-k}(A'^+, A'^-))).$$

The proof of Lemma 1 shows

$$\tau(\varepsilon_n(a_i(A^+, A^-))) + \tau(\varepsilon_n(a_i(A^-, A^+))) = 0 \qquad \text{for } i < n.$$

Because $\frac{1}{2}\tau(\varepsilon_{n+n'}(a_{n+n'}(A^+ \oplus A'^+, A^- \oplus A'^-) + a_{n+n'}(A^- \oplus A'^-, A^+ \oplus A'^+)))$ is completely skew on $M \times M'$, it equals $\frac{1}{2}\tau(\varepsilon_n(a_n(A^+, A^-) + a_n(A^-, A^+))) \times \frac{1}{2}\tau(\varepsilon_n(a_{n'}(A'^+, A'^-) + a_{n'}(A'^-, A'^+)))$. Write $\underset{\sim}{A}_M = \sum_j \Sigma T_{n-j}(A^+, A^-, n) P_j(p_M, n)$ with $P_j \in A^j(M)$. We have $T_n(A^+, A^-, n) = \underset{\sim}{A}_{S^n}$ and $P_n(p_M, n) = \hat{A}_n(g)$. Taking a connection over $M = S^i \times N$ pulled back from

$$\begin{array}{ccc} (S_{S^i} \otimes S_N) \otimes V & \to & S_{S^i} \otimes V \\ \downarrow & & \downarrow \\ M = S^i \times N & \to & S^i \end{array}$$

with N an arbitrary $n-i$ dimensional spin manifold and applying the above, we obtain $T_{n-j}(A^+, A^-, n) = T_{n-j}(A^+, A^-, n-j)$ and $P_j(p_M, n) = \hat{A}_j(g)$. We can hereafter assume that M is isometrically S^n.

For large s, let $\mathfrak{A}$ denote the space of H^s connections on P and $\mathfrak{G}$ denote the group of H^{s+1} gauge transformations (base-preserving automorphisms of P) $\{\phi : \phi \in \Gamma(\mathrm{Ad}\,P),\ \phi(\infty) = I\}$ which are fixed at some $\infty \in M$. Then $\mathfrak{G} \times \mathfrak{G} \to \mathfrak{A} \times \mathfrak{A} \to (\mathfrak{A} \times \mathfrak{A})/(\mathfrak{G} \times \mathfrak{G})$ is a principal fiber bundle and $\underset{\sim}{A}$ defines a vertical 1-form ω on $\mathfrak{A} \times \mathfrak{A}$ by $\omega(V_X, V_Y) \equiv \omega((d/d\beta)|_{\beta=0}(A^+ + \beta D_{A^+}X),\ (d/d\beta)|_{\beta=0}(A^- + \beta D_{A^-}Y)) = \int_M \mathrm{Tr}(X - Y)\underset{\sim}{A}(A^+, A^-)$.

Lemma 2. *Upon restriction to a fiber,* $\omega \in H^1(\mathfrak{G} \times \mathfrak{G}, \mathbf{R})$.

Proof. In general let $O(\alpha, \beta)$ be a smooth 2-parameter family of elliptic operators on cross-sections of a vector bundle over M with positive-definite symbol. Let ω_T be the 1-form $\underset{\sim}{\mathrm{Tr}}(dO/d\alpha)d\alpha + (dO/d\beta)d\beta)e^{-TO}$. Then

$$\begin{aligned} d\omega_T &= \underset{\sim}{\mathrm{Tr}}\left(\frac{dO}{d\alpha}d\alpha + \frac{dO}{d\beta}d\beta\right) \wedge \int_0^T e^{-VO}\left(\frac{dO}{d\alpha}d\alpha + \frac{dO}{d\beta}d\beta\right)e^{(V-T)O}dV \\ &= \underset{\sim}{\mathrm{Tr}}\int_0^T \left(\frac{dO}{d\alpha}e^{-VO}\frac{dO}{d\beta}e^{(V-T)O} - \frac{dO}{d\beta}e^{-VO}\frac{dO}{d\alpha}e^{(V-T)O}\right) \\ &\quad \cdot dV\, d\alpha \wedge d\beta = 0. \end{aligned}$$

We apply this to $O = \not{D}^2_{A^+, A^-}$ with A^+, A^- lying on a fiber of $\mathfrak{A} \times \mathfrak{A}$. Let $(g(\beta),\ h(\beta))$ be a smooth 1-parameter family in $\mathfrak{G} \times \mathfrak{G}$ with $g(0) = h(0) = I$

and let $(A^+(\beta), A^-(\beta)) = (g^{-1}(\beta)A^+g(\beta) + g^{-1}(\beta)dg(\beta), h^{-1}(\beta)A^-h(\beta) + h^{-1}\cdot(\beta)dh(\beta))$. Then

$$\omega_T\left(\frac{d}{d\beta}\bigg|_{\beta=0}(A^+(\beta), A^-(\beta))\right) = \underset{\sim}{\mathrm{Tr}}\left\{\not{D}, \frac{d}{d\beta}\bigg|_{\beta=0}\not{D}\right\}e^{-T\not{D}^2}$$
$$= 2\underset{\sim}{\mathrm{Tr}}\frac{d}{d\beta}\bigg|_{\beta=0}\left(\frac{1+\varepsilon}{2}h^{-1} + \frac{1-\varepsilon}{2}g^{-1}\right)$$
$$\cdot\not{D}\left(\frac{1+\varepsilon}{2}g + \frac{1-\varepsilon}{2}h\right)\not{D}e^{-T\not{D}^2}$$
$$= 2\underset{\sim}{\mathrm{Tr}}\left(\frac{d}{d\beta}\bigg|_{\beta=0}(g-h)\right)\varepsilon\not{D}^2e^{-T\not{D}^2},$$

and so

$$\frac{1}{\Gamma(s+1)}\int_0^\infty T^s e^{-m^2T}\underset{\sim}{\mathrm{Tr}}\left(\frac{d}{d\beta}\bigg|_{\beta=0}(g-h)\right)\varepsilon\not{D}^2e^{-T\not{D}^2}dT$$
$$= \frac{1}{\Gamma(s)}\int_0^\infty T^{s-1}e^{-m^2T}\underset{\sim}{\mathrm{Tr}}\left(\frac{d}{d\beta}\bigg|_{\beta=0}(g-h)\right)\varepsilon e^{-T\not{D}^2}dT$$
$$-m^2\frac{1}{\Gamma(s+1)}\int_0^\infty T^s e^{-m^2T}\underset{\sim}{\mathrm{Tr}}\left(\frac{d}{d\beta}\bigg|_{\beta=0}(g-h)\right)\varepsilon e^{-T\not{D}^2}dT$$

defines an element of $H^1(\mathfrak{G} \times \mathfrak{G}, \mathbf{R})$ for $m \neq 0$. Taking $s \to 0$ and $m \to 0$ gives that

$$\lim_{s\to 0}(1/\Gamma(s))\int_0^1 T^{s-1}\underset{\sim}{\mathrm{Tr}}\left(\frac{d}{d\beta}\right)\bigg|_{\beta=0}(g-h))\varepsilon e^{-T\not{D}^2}dT$$
$$-\underset{\sim}{\mathrm{Tr}}|_{\mathrm{Ker}\,\not{D}^2}\left(\frac{d}{d\beta}\right)\bigg|_{\beta=0}(g-h)\varepsilon$$

also gives an element of $H^1(\mathfrak{G} \times \mathfrak{G}, \mathbf{R})$. The second term of the 1-form is clearly closed. Symmetrizing in A^+ and A^- gives that ω is closed. □

Because $\not{D}^2_{A^+,A^-}$ is a relatively compact perturbation of the self-adjoint operator $\not{D}^2_{A^+,A^+}$, its spectrum is not $\mathbf{C}$ and so it has discrete spectrum $\{\lambda_j\}$ with $|\lambda_j| \to \infty$. If P is the projection onto Ker $\not{D}^2$, define $\Delta = \not{D}^2 + P$. Let $A^+(\beta)$, $A^-(\beta)$ be a family as above with $(d/d\beta)|_{\beta=0}A^+(\beta) = V_X$, $(d/d\beta)|_{\beta=0}A^-(\beta) = V_Y$. Then with a fixed branching of $\mathbf{C}$,

$$\sum_{\lambda_i \neq 0}\lambda_i^{-s-1}\frac{d}{d\beta}\bigg|_{\beta=0}\lambda_i$$
$$= \underset{\sim}{\mathrm{Tr}}(1-P)\Delta^{-s-1}\left\{\not{D}, \not{D}\left(\frac{1+\varepsilon}{2}X + \frac{1-\varepsilon}{2}Y\right) - \left(\frac{1+\varepsilon}{2}Y + \frac{1-\varepsilon}{2}X\right)\not{D}\right\}$$
$$= \underset{\sim}{\mathrm{Tr}}(1-P)(X-Y)\varepsilon\Delta^{-s}$$

and

$$\lim_{s\to 0}\sum_{\lambda_i \neq 0} \lambda_i^{-s-1} \frac{d}{d\beta}\bigg|_{\beta=0} \lambda_i = -\mathrm{Tr}\, P(X-Y)\varepsilon + \lim_{s\to 0}\frac{1}{\Gamma(s)}\int_0^1 T^{s-1}\mathrm{Tr}(X-Y)\varepsilon e^{-T\not{D}^2}dT.$$

We can do the same with $(\not{D}^2)^\dagger$, having conjugate spectrum and, say, projection Q onto its kernel. Because $\mathrm{Index}(\not{D}-\lambda_j I)=0$, the multiplicities of λ_j for $\not{D}^2$ and $\bar{\lambda}_j$ for $(\not{D}^2)^\dagger$ are the same.

Let $A^{\pm}(\alpha,\beta)$ be two smooth families: $D^2 \to \mathfrak{A}$ with ∂D^2 mapping to a fiber of $\mathfrak{A}$. As $\not{D}^2_{A^+,A^-}$ varies over ∂D^2 only a finite number of λ_i will wind around the origin. For these λ_i the conjugate $\bar{\lambda}_i$ will wind in the opposite direction, giving $\lim_{s\to 0}\int_{\partial D^2}(\lambda_i^{-s-1}d\lambda_i + \bar{\lambda}_i^{-s-1}d\bar{\lambda}_i)=0$. Thus

$$0=\int_{\partial D^2}(-\mathrm{Tr}\, P(X-Y)\varepsilon - \mathrm{Tr}\, Q(X-Y)\varepsilon + 2\omega(A^+,A^-)(V_X,V_Y)).$$

Now $\mathrm{Tr}\, P(X-Y)\varepsilon + \mathrm{Tr}\, Q(X-Y)\varepsilon = \mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}_{A^+}}(X-Y) - \mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}^\dagger_{A^-}}(X-Y) + \mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}_{A^-}}(X-Y) - \mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}^\dagger_{A^+}}(X-Y)$. Let $Z_A = \mathrm{Ker}\,\not{\partial}_A - \mathrm{Ker}\,\not{\partial}^\dagger_A + [m]$ be the index bundle for $\not{\partial}$ over D^2, with $[m]$ a trivial bundle pulled back from $\mathfrak{A}/\mathfrak{G}$. Let P_{Z_d} denote projection from $\Gamma(S\otimes V)$ to the fiber of Z over $d\in D^2$, and define a connection ϕ on Z by $\phi(d/ds)(\xi)=P_Z(d\xi/ds)$ for $\xi\in\Gamma(Z)$.

Lemma 3. $\int_{D^2} c_1(\phi)(Z_{A^+}) - \int_{D^2} c_1(\phi)(Z_{A^-}) = (1/i\pi)\int_{\partial D^2}\omega(A^+,A^-)$.

Proof.

$$\int_{D^2} c_1(\phi)(Z_{A^+}) = \frac{1}{2\pi i}\int_{\partial D^2}\mathrm{Tr}\,\phi(Z_{A^+}) = \frac{1}{2\pi i}\int_{\partial D^2}(\mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}_{A^+}}X - \mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}^\dagger_{A^+}}X).$$

Doing this also for Z_{A^-} gives

$$\int_{D^2} c_1(\phi)(Z_{A^+}) - \int_{D^2} c_1(\phi)(Z_{A^-}) = \int_{\partial D^2}(\mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}_{A^+}}X - \mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}^\dagger_{A^+}}X - \mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}_{A^-}}Y + \mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}^\dagger_{A^-}}Y).$$

Let $\gamma:[0,1]\to\mathfrak{A}\times\mathfrak{A}$ be

$$\gamma(t)=\begin{cases}(A^+(2t),A^-(0)) & \text{for } 0\leqq t\leqq \frac{1}{2},\\ (A^+(1),A^-(1-2t)) & \frac{1}{2}\leqq t\leqq 1.\end{cases}$$

Then (image γ) is a deformation of $(A^+,A^-)(\partial D^2)$ and

$$\int_{\partial D^2}\omega(A^+,A^-) = \int_{\mathrm{Image}\,\gamma}\omega(A^+,A^-) = \frac{1}{2}\int_{\partial D^2}(\mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}_{A^+}}X - \mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}^\dagger_{A^-}}X - \mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}_{A^-}}Y + \mathrm{Tr}|_{\mathrm{Ker}\,\not{\partial}^\dagger_{A^-}}Y). \qquad \square$$

Let ξ_i be an orthonormal basis for $\operatorname{Index} \partial\!\!\!/_A \to \mathfrak{A}$. Then $\phi_{ij}(d/ds) = \langle \xi_i, (d/ds)\xi_j \rangle$. Under a gauge transformation we can assume $\xi_i \to g\xi_i$ and $\phi_{ij}(d/ds) \to \phi_{ij}(d/ds) + \langle \xi_i, g^{-1}(dg/ds)\xi_j \rangle$. Thus $\operatorname{Tr}\phi$ is $\mathfrak{G}$-invariant and the $\mathfrak{G}$-induced vector fields on $\operatorname{Index} \partial\!\!\!/_A \to \mathfrak{A}$ are horizontal for $\operatorname{Tr}\phi$, giving a $U(1)$ connection on a line bundle over $\mathfrak{A}/\mathfrak{G}$. Now $c_1(\operatorname{Tr}\phi) \in H^2(\mathfrak{A}/\mathfrak{G}, \mathbf{R}) = H^1(\mathfrak{G}, \mathbf{R}) \simeq (\pi_{n+1}(G)/[\pi_{n+1}(G), \pi_{n+1}(G)]) \otimes \mathbf{R}$, which is $\mathbf{R}$ for large N [7]. The families index theorem gives an alternative form of this cohomology element [8–10]. Consider

$$\begin{array}{ccc} \mathfrak{A} \times P & \to & (\mathfrak{A} \times P)/\mathfrak{G} \\ \pi\downarrow & & \downarrow \\ \mathfrak{A} \times M & \to & \mathfrak{A}/\mathfrak{G} \times M \end{array}$$

and put a $\mathfrak{G}$-equivariant connection η on $\mathfrak{G} \to \mathfrak{A} \times P \xrightarrow{\pi} \mathfrak{A} \times M$. Then $c_1(\phi) = (\operatorname{ch} \eta \hat{A}(g))$ in $H^2(\mathfrak{A}/\mathfrak{G}, \mathbf{R})$.

We compute the image of $c_1(\phi)$ in $H^1(\mathfrak{G}, \mathbf{R})$. Take

$$\eta(A,p)\left(\frac{d}{d\beta}\bigg|_{\beta=0}(A+\beta B), V\right) = ((D_A^* D_A)^{-1} D_A^* B)(p) + A(V).$$

Put $\tilde{B} = B - D_A(D_A^* D_A)^{-1} D_A^* B$ and $\tilde{C} = C - D_A(D_A^* D_A)^{-1} D_A^* C$. The curvature of η is

$$\Omega_{(A,p)}\left(\left(\frac{d}{d\beta}\bigg|_{\beta=0}(A+\beta B), V\right), \left(\frac{d}{d\beta}\bigg|_{\beta=0}(A+\beta C), W\right)\right)$$
$$= -(D_A^* D_A)^{-1} \sum_j [\tilde{B}^j, \tilde{C}_j](p) + F_A(V,W) + \tfrac{1}{2}\tilde{B}(W) - \tfrac{1}{2}\tilde{C}(V).$$

Take $M = S^n$, so $\hat{A}(g) = 1$ and fix the connection $\eta_0 = 0 + A_0$. Because $i(M)\operatorname{Tr}\Omega_0^{n/2+1} = 0$, the image of $c_1(\phi)$ is given up to a constant factor by the restriction of the secondary characteristic class

$$i(M)\operatorname{Tr}\int_0^1 (\eta - \eta_0) \wedge (T\Omega + (1-T)\Omega_0 - T(1-T)(\eta - \eta_0) \wedge (\eta - \eta_0))^{n/2} dT$$

to a fiber. Acting on the tangent vector V_X, this is

$$i(M)\operatorname{Tr}\int_0^1 (X(TF + (1-T)F_0 - T(1-T)(A-A_0) \wedge (A-A_0))^{n/2}$$
$$+ \sum_{j=0}^{n/2-1} (A-A_0) \wedge (TF + (1-T)F_0 - T(1-T)(A-A_0) \wedge (A-A_0))^j$$
$$\wedge (T(1-T)(X(A-A_0) - (A-A_0)X)) \wedge (TF + (1-T)F_0 - T(1-T)(A-A_0)$$
$$\wedge (A-A_0))^{n/2-j-1}) dT \sim \int_M \operatorname{Tr} X B_{n/2}(A, A_0).$$

We now show that $H^1(\mathfrak{G} \times \mathfrak{G}, \mathbf{R})$ is the only obstruction to integrating the anomaly.

Lemma 4. *For $(A^+, A^-) \in \mathfrak{A} \times \mathfrak{A}$, let $\sigma \in \Lambda^n(M) \otimes g$ be a polynomial in $(A^+ - A^-)$,*

F^+ and F^-. Define a vertical 1-form on $\mathfrak{A}\times\mathfrak{A}$ by $(\tilde{\omega}(V_X, V_Y) = \int_M \mathrm{Tr}(X-Y)\cdot \sigma(A^+, A^-)$. If $\tilde{\omega}$ is trivial in $H^1(\mathfrak{G}\times\mathfrak{G}, \mathbf{R})$ when restricted to a fiber then $s\sigma = 0$.

Proof. Let $\mathfrak{H}_1$ denote the connected component of the identity of the H^{s+1} automorphisms of P (not necessarily base-preserving) which are the identity at ∞. Let $\mathfrak{C}$ be $\{A\in\mathfrak{A}: h\in\mathfrak{H}_1,\ h\cdot A \neq A\}$ and $\mathfrak{H}_2 = \{(\phi_1, \phi_2)\in\mathfrak{H}_1\times\mathfrak{H}_1 : \pi\phi_1 = \pi\phi_2\}$. There is an action of $\mathfrak{H}_1$ on $\mathfrak{C}\times\mathfrak{C}$ induced from its action on $\mathfrak{C}$. Arguments such as those in [11] show that for large N, $\mathfrak{C}$ is open and dense in $\mathfrak{A}$ and $\mathfrak{C}\times\mathfrak{C}\underset{\pi_1}{\overset{\mathfrak{H}_1}{\to}}(\mathfrak{C}\times\mathfrak{C})/\mathfrak{H}_1\underset{\pi_2}{\overset{\mathfrak{G}}{\to}}(\mathfrak{C}\times\mathfrak{C})/\mathfrak{H}_2$ are fibrations with $(\mathfrak{C}\times\mathfrak{C})/\mathfrak{H}_1$ and $(\mathfrak{C}\times\mathfrak{C})/\mathfrak{H}_2$ being H^s Hilbert manifolds. Because $\tilde{\omega}$ restricted to $\mathfrak{C}\times\mathfrak{C}$ is $\mathfrak{H}_1$ invariant it descends to $\mathfrak{C}\times\mathfrak{C}/\mathfrak{H}_1$. On a local trivialization $\mathfrak{G}\to((\mathfrak{C}\times\mathfrak{C})/\mathfrak{H}_2)\times\mathfrak{G}\to(\mathfrak{C}\times\mathfrak{C})/\mathfrak{H}_2$ we can find a smooth f such that $(V_X, V_Y)\cdot \pi_i^* f = \tilde{\omega}(V_X, V_Y)$. Using a smooth partition of unity (these exist for Hilbert manifolds) on $\mathfrak{C}\times\mathfrak{C}/\mathfrak{H}_2$ we can extend f to a smooth function $\bar{f}$ on $\mathfrak{G}\times\mathfrak{G}/\mathfrak{H}_1$ and hence to a smooth $f = \pi_1^*\bar{f}$ on $\mathfrak{C}\times\mathfrak{C}$ which is $\mathfrak{H}_1$ invariant and satisfies $(V_X, V_Y)f = \tilde{\omega}(V_X, V_Y)$.

Fix $A_0\in\mathfrak{C}$ and consider the Taylor series

$$f = \sum_{k+l\leqq p} Q_{k,l} + R_p \equiv \sum_{k+l\leqq p} \langle f_{k,l}, (\bigotimes^k (A^+ - A_0))\otimes(\bigotimes^l (A^- - A_0))\rangle + R_p$$

with $f_{k,l}\varepsilon(\bigotimes^{(|k|+|l|)} H^s(A^1\otimes g))^*$. The infinitesimal automorphism of P given by parallel transporting with respect to A_0 over a vector field V changes A by

$$\delta A_\nu = (A_\mu - A_{0\mu})\nabla_\nu V^\mu + V^\omega(D_{A_\omega}(A - A_0)_\nu + F(A_0)_{\omega\nu} - [(A-A_0)_\omega, (A-A_0)_\nu]).$$

We can treat the $\nabla_\nu V^\mu$ part of this as an independent variation. Then

$$\begin{aligned}0 = \delta_{\nabla V} f = \sum_{k+l\leqq p} \langle f_{k\,l}, & \sum_{0\leqq m\leqq k-1} (\bigotimes^m (A^+ - A_0))\otimes\nabla_\nu V^\mu (A_\mu^+ - A_{0\mu}) \\ & \otimes(\bigotimes^{(k-m-1)} (A^+ - A_0))\otimes(\bigotimes^l (A^- - A_0)) \\ & + \sum_{0\leqq m\leqq l-1} (\bigotimes^k (A^+ - A_0))\otimes(\bigotimes^m (A^- - A_0)) \\ & \otimes\nabla_\nu V^\mu (A_\mu^- - A_{0\mu})\otimes(\bigotimes^{(l-m-1)} (A^- - A_0))\rangle + \delta_{\nabla V} R_p.\end{aligned}$$

As this is true regardless of $\nabla_\nu V^\mu$, $f_{k,l}$ must be supported on $\{(m_1, \ldots, m_{k+l}): i, m_i = m_j \text{ for some } j\neq i\}$ in M^{k+l}. As $f_{k,l}$ is a vector-valued distribution, it must locally be $\Sigma T(\Pi\delta(m_j - m_k)D(m_j))$ with each D a vector-valued distribution on M and each T a differential operator, each m_i occurring in at least one δ in each term of the sum [12]. That is, $Q_{k,l}$ has the form $\sum_r \prod_{i=1}^r \int_M P_i$ with each P_i a local function of $A^+ - A_0$, $A^- - A_0$ and their derivatives, and possibly of $m\in M$. Again by automorphism invariance there can be no explicit m dependence and the spatial indices must be contracted with $\varepsilon_{a_1 \ldots a_n}$. Because $(V_X, V_Y)f$ has only one M integration we may remove any term with $r \neq 1$ and preserve $(V_X, V_Y)f$. Each factor of $A^+ - A_0$ or $A^- - A_0$ has a spatial index and so we may take $f = \int_M P$ with P a polynomial of degree $\leqq n$. Because A_0 was arbitrary we can assume P is independent of A_0. Using the Bianchi identity and synchronous frames first for A^+ and then for A^-, we can write P as a polynomial in A^+, A^-, F^+ and F^-. Because P is H_1 invariant, P must actually be a poynomial in $A^+ - A^-$, F^+ and F^-.

As $\int_M \mathrm{Tr}(X-Y) s\sigma(A^+, A^-) = (V_X, V_Y) \int_M sP(A^+, A^-)$

we can take

$$sP(A^+, A^-) = s\,\mathrm{Tr} \sum_{j+2k+2l=n} c_{jkl} (A^+ - A^-)^j \wedge (F^+)^k \wedge (F^-)^l.$$

Then

$$\begin{aligned}(V_X, V_Y) \int_M sP(A^+, A^-) = \int_M \mathrm{Tr} \sum_{jkl} s \Bigg(& \sum_{m=0}^{j-1} (A^+ - A^-)^m \wedge (D_{A^+} X - D_{A^-} Y) \\ & \wedge (A^+ - A^-)^{j-m-1} \wedge (F^+)^k \wedge (F^-)^l + \sum_{m=0}^{k-1} (A^+ - A^-)^j \\ & \wedge (F^+)^m \wedge [F^+, X] \wedge (F^+)^{k-m-1} \wedge (F^-)^l + \sum_{m=0}^{l-1} (A^+ - A^-)^j \\ & \wedge (F^+)^k \wedge (F^-)^m \wedge [F^-, Y] \wedge (F^-)^{l-m-1} \Bigg) = 0. \qquad \square\end{aligned}$$

Proof of theorem. By Lemma 1, T_n is a polynomial in A^+, A^- and their derivatives. By Lemma 2 and Lemma 3, it is proportionate to $B_{n/2}$ in $H^1(\mathfrak{G} \times \mathfrak{G}, \mathbf{R}) = \mathbf{R} \oplus \mathbf{R}$. By Lemma 4, sT_n is proportionate to $sB_{n/2} = B_{n/2}$. The normalization is fixed by requiring that when $A^+ = A^-$ one recovers the integrand for the spinor index theorem. $\square$

Note. After this work was completed I learned of reference [13] which contains a more complete topological discussion, along with the extension to the gravitational anomaly. For a different treatment of the subject I recommend the paper of Alvarez-Gaumé and Ginsparg [14].

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References

1. Bardeen, W. A.: Anomalous ward identities in spinor field theories. Phys. Rev. **184**, 1848 (1969) and references therein
2. Singer, I. M.: (Private communication)
3. Ativah, M., Bott, R., Patodi, V. K.: On the heat equation and the index theorem. Inventions Math. **19**, 279 (1973)
4. Gilkey, P.: The spectral geometry of a Riemannian manifold. J. Diff. Geom. **10**, 601 (1975)
5. Witten, E.: Global aspects of current algebra. Nucl. Phys. **B223**, 422 (1983)

6. Atiyah, M., Hirzebruch, F.: Spin manifolds and group actions. In: Essays on Topology. Berlin, Heidelberg, New York: Springer, 1970
7. Singer, I. M.: Some remarks on the Gribov ambiguity. Commun. Math. Phys. **60**, 7 (1978)
8. Atiyah, M., Singer, I. M.: The index of elliptic operators IV. Ann. Math. **93**, 119 (1971)
9. Atiyah, M., Jones, J. D. S.: Topological aspects of Yang–Mills theory. Commun. Math. Phys. **61**, 97 (1978)
10. Quillen, D.: Lectures at Harvard, 1983
11. Ebin, D.: The manifold of Riemannian metrics. Proc. Symp. Pure Math. AMS **XV**, 11 (1970)
12. Schwartz, L.: Théorie des distributions. Paris: Hermann, 1966
13. Atiyah, M., Singer, I. M.: Dirac operators coupled to vector potentials. U. C. Berkeley Preprint, 1984
14. Alvarez-Gaumé, L., Ginsparg, P.: Topological meaning of non-Abelian anomalies. Harvard preprint HUTP-83/A081, Nucl. Phys. B. (to appear)

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