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Translation Group and Spectrum Condition

H. -J. Borchers
Institut für Theoretische Physik der Universität Göttingen, Bunsenstr. 9, D-3400 Göttingen, Federal Republic of Germany

Abstract. Let $\{A, \mathbb{R}^d, \alpha\}$ be a C^*-dynamical system, where $\mathbb{R}^d$ is the d-dimensional vector group. Let V be a convex cone in $\mathbb{R}^d$ and $\hat{V}$ its dual cone. We will characterize those representations of A with the properties (i) α_a, $a \in \mathbb{R}^d$ is weakly inner, (ii) the corresponding unitary representation $U(a)$ is continuous, and (iii) the spectrum of $U(a)$ is contained in $\hat{V}$.

I. Introduction

The spectrum condition is one of the essential ingredients of quantum field theory. Especially the discovery of the fact that the translations are weakly inner automorphisms for finite particle representations [4] has made the spectrum condition an interesting subject. Many problems in connection with this have been studied and answered [4–7]. In the previous investigations, which are based on the "covariance-algebra" introduced by Doplicher, Kastler, and Robinson [9], it has been assumed that the translation group is acting strongly continuous on the C^*-algebra in question. On the other hand, in the theory of local observables, one usually is only interested in representations which are locally normal with respect to the vacuum representation. But this means that the algebra associated to a bounded region should be a von Neumann algebra. Such an assumption, however, contradicts the assumption of strong continuity of the translations. Since in a recent paper [7] it has been shown that one can handle the problem of covariant representation without using the continuity of the group action on the algebra, we will treat the problem of the spectrum condition again.

Furthermore in the existing literature only the one dimensional case and its iterations have been treated with full mathematical rigour. But the case where the cone in question is an arbitrary convex cone with interior points is still missing. We also want to fill this gap.

In the next section we handle the one dimensional case again. We show that by introducing the reasonable concepts one can reduce this problem to results existing in the literature. The results obtained here are generalized in Sect. III to the n-dimensional case where the spectrum is restricted to a half space. The n-dimensional case where the spectrum is in a cone is treated in Sect. IV and V.

II. The One Dimensional Case

Let A be a C^*-algebra and G be a topological group, acting as a group of automorphisms on A, i.e. $\alpha: G \to \mathrm{Aut}(A)$. Following [7] we will denote:

$$A_C^* = \{\varphi \in A^*; g \to \varphi \circ \alpha_g \text{ is a continuous function on } G$$
$$\text{with values in the Banach space } A^*\}.$$

This is a norm closed linear sub-space of A^*, invariant under the transposed action α_g^* and generated by its positive elements.

If G is a locally compact group with left invariant Haar measure dg, then for $\varphi \in A_c^*, f \in \mathscr{L}^1(G)$, $x \in A^{**}$, the expression $\int \varphi(\alpha_g x) f(g)\, dg$ is well defined and defines a continuous linear functional on A_c^*. The set of all continuous linear extensions to all of A^* will be denoted by $[x(f)]$. If $y \in [x(f)]$, then we have $[x(f)] = y + N_c$, where N_c is the annihilator of A_c^* in A^{**}.

Having these notations at hand, we can use, in the case where G is also abelian, the spectral theory of Arveson [1] (α_g^* acts strongly continuous on A_c^*, and A^{**}/N_c is the dual space of A_c^*), and the results obtained from it by linear methods. Using the notations of G. K. Pedersen [11, Chap. 8] we define for $G = \mathbb{R}$ the space $R(-\infty, \mu) \subset A^{**}$ to be the $\sigma(A^{**}, A^*)$ closed linear sub-space generated by all $[x(f)]$ with $x \in A^{**}, f \in \mathscr{L}^1(\mathbb{R})$ with $\tilde{f}$ having compact support and $\operatorname{supp} \tilde{f} \subset (-\infty, \mu)$. ($\tilde{f}$ denotes the Fourier-transform of f.)

In the same manner as in the case where α_g acts strongly continuous on A, we define $E(\lambda) =$ projection onto the common null space of all $y \in R(-\infty, -\lambda)$, i.e. $E(\lambda) =$ maximal projection E in A^{**} such that $R(-\infty, -\lambda)\, E = 0$ and $E(\infty) = s\text{-}\lim_{\lambda \to \infty} E(\lambda)$. Our aim is to show that these projections have the same properties as the corresponding projections one obtains when α_g acts strongly continuous on A. In order to show this we define:

II.1. Definition. Let $\{A, \mathbb{R}, \alpha\}$ be a C^*-dynamical system, and let $E(\lambda)$ be the projections defined above. Then we denote:

(a) $A^*(\hat{\mathbb{R}}^+) = (\varphi \in A^*; E(\infty)\varphi = \varphi E(\infty) = \varphi\}$,
(b) $A_0^*(\hat{\mathbb{R}}^+) = \{\varphi \in A^*;$ such that there exist $\lambda, \mu < \infty$ with $E(\mu)\varphi = \varphi\, E(\lambda) = \varphi\}$.

II.2. Proposition. *With the above notation we obtain*:

(i) $A_0^*(\hat{\mathbb{R}}^+)$ *is norm-dense in* $A^*(\hat{\mathbb{R}}^+)$.

(ii) $\varphi \in A^*(\hat{\mathbb{R}}^+)$ *and* $x, y \in A^{**}$ *implies* $x\varphi y \in A^*(\hat{\mathbb{R}}^+)$ *or equivalently* $E(\infty)$ *belongs to the center of* A^{**}.

(iii) *An element* $\varphi \in A^*$ *belongs to* $A_0^*(\hat{\mathbb{R}}^+)$ *if and only if the following conditions are fulfilled*:

(α) $a \to \varphi(x\alpha_a(y))$ *is continuous and it is the boundary value of an analytic function,* $W_1(z)$ *holomorphic in upper half-plane satisfying the estimate* $|W_1(z)| \leqq \|x\| \cdot \|y\| \cdot \|\varphi\| \exp\{m|\operatorname{Im} z|\}$ *for a suitable constant m.*

(β) $a \to \varphi(\alpha_a(x)y)$ *is continuous and it is the boundary value of an analytic function* $W_2(z)$ *holomorphic in the lower half-plane fulfilling the estimate* $|W_2(z)| \leqq \|x\| \cdot \|y\| \cdot \|\varphi\| \cdot \exp\{m'|\operatorname{Im} z|\}$ *with a suitable constant m'.*

Proof. Note first the relation

$$R(-\infty, \lambda) \subset R(-\infty, \mu)$$

for $\lambda \leqq \mu$. From the definition of $E(\lambda)$, it follows from this that $E(\lambda_1) \leqq E(\lambda_2)$ for $\lambda_1 \leqq \lambda_2$. This implies $E(\lambda)$ is monotone increasing and hence strongly converging to $E(\infty)$.

Furthermore it is easy to see that $R(-\infty, \mu)$ contains the identity operator for $\mu > 0$. From this we get $E(\lambda) = 0$ for $\lambda < 0$.

(i) From the definition of $A^*(\hat{\mathbb{R}}^+)$ it follows that with φ also φ^* and $|\varphi|$ belong to this space. This means $A^*(\hat{\mathbb{R}}^+)$ is a linear space generated by its positive elements. Therefore it is sufficient to show statement (i) for positive elements. Let $0 \neq \omega \in A^*(\hat{\mathbb{R}}^+)^+$ and let λ be such that $\omega(E(\infty) - E(\lambda)) \leqq \varepsilon/4 \|\omega\|$. Then we get:

$$\begin{aligned} \|\omega - E(\lambda)\omega E(\lambda)\| &\leqq \|(E(\infty) - E(\lambda))\omega\| + \|E(\lambda)\omega E(\infty) - E(\lambda))\| \\ &\leqq \{\|\omega\| \, \|(E(\infty) - E(\lambda))\omega(E(\infty) - E(\lambda))\|\}^{1/2} \\ &\quad + \{\|E(\lambda)\omega E(\lambda)\| \, \|(E(\infty) - E(\lambda))\omega(E(\infty) - E(\lambda))\|\}^{1/2} \\ &\leqq 6\{\|\omega\| \cdot \varepsilon/4 \|\omega\|\}^{1/6} = \varepsilon^{1/7}. \end{aligned}$$

For the estimation of $(E(\infty) - E(\lambda))\omega E(\lambda)$, we have used the formula

$$\|(E(\infty) - E(\lambda))\omega E(\lambda)\| \leqq \{\omega(E(\infty) - E(\lambda)) \cdot \omega(E(\lambda))\}^{1/2}$$

which has been proved in [7, Lemma II.2].

(ii) We remark first that $R(-\infty, \lambda) \supset N_c$, the annihilator of A_c^*. Therefore, every $\varphi \in A^*(\hat{\mathbb{R}}^+)$ annihilates the $\sigma(A^{**}, A^*)$ closed left ideal generated by N_c (since $E(\lambda)$ is the left annihilator of $R(-\infty, -\lambda)$). From this we get: $x \in A^{**}$ and $\varphi \in A^*(\hat{\mathbb{R}}^+)$ implies $x\varphi$ annihilates N_c and hence $x\varphi \in A_c^*$ (and also $\varphi x \in A_c^*$, since A_c^* and $A^*(\hat{\mathbb{R}}^+)$ are both invariant under involution). Let $f(a) \in \mathscr{L}^1(\mathbb{R})$ with supp $\tilde{f}(p) \subset [-\mu, \mu]$. Then $f_b(a) = f(a - b)$ is an entire analytic function of b with values in $\mathscr{L}^1$ and we have $\|f_b\|_1 \leqq \|f\|_1 \exp\{\mu|\operatorname{Im} b|\}$ (see e.g. Boas [2, Theorem 6.7.1]).

From this we obtain (see Boas [2, Theorem 2.2.10]):

$$f_b(a) = \sum_0^\infty f^{(n)}(a) \frac{b^n}{n:}$$

with $\|f^{(n)}\|_1 \leqq M(\mu + \varepsilon)^n$ for some suitable constant $M(\varepsilon)$. This implies in particular that $f^{(n)}$ belongs also to $\mathscr{L}^1$. Let now $\varphi \in A^*(\hat{\mathbb{R}}^+)$ and $x, y, z \in A^{**}$. Then $\varphi(xy\alpha_a(z))$ is continuous in a and consequently $\varphi(xyz(f_b))$ is an analytic function in b.

From this we get for such φ,

$$\begin{aligned} &\varphi(z\alpha_a(y)z(f)) - \varphi(xyz(f)) = \varphi(x\alpha_a\{yz(f_{-a})\}) - \varphi(xyz(f)) \\ &= \varphi(x\alpha_a\{yz(f)\}) - \varphi(xyz(f) + \sum_1^\infty \varphi(x\alpha_a\{yz(f^{(n)})\}) \cdot \frac{(-a)^n}{n!}, \end{aligned}$$

and hence with the above estimate

$$\begin{aligned}&|\varphi(x\alpha_a(y)z(f)) - \varphi(xyz(f))|\\ &\leqq |\varphi(x\alpha_a\{yz(f)\}) - \varphi(xyz(f))| + \sum_1^\infty \|\varphi\| \|x\| \|y\| \|z\| M(\mu+\varepsilon)^n \frac{(a)^n}{n:}\\ &= |\varphi(x\alpha_a\{yz(f)\}) - \varphi(xyz(f)| + M\|\varphi\|\cdot\|x\|\cdot\|y\|\cdot\|z\| e^{(\mu+\varepsilon)(a)} - 1.\end{aligned}$$

Since $x\varphi \in A_c^*$, it follows also that $x\varphi z(f) \in A_c^*$ for supp $\hat{f}$ compact. Before going on, let us first prove statement (iii). If $\varphi \in A_0^*(\hat{\mathbb{R}}^+)$, then we know $a \to \varphi(x\alpha_a(y))$ and $\varphi(\alpha_a(x)y)$ are both continuous. Assume $\varphi E(\lambda) = \varphi$ for one $\lambda < \infty$. Then it follows that $\int \varphi(x\alpha_a(y)) f(a) da = 0$ for $f \in \mathscr{L}^1$ and supp $\hat{f} \subset (-\infty, -\lambda)$. This shows the Fourier transform of $\varphi(x\alpha_a(y)$ has its support in $[-\lambda, \infty]$, and hence $\varphi(x\alpha_a(y))$ is the boundary value of an analytic function holomorphic in the upper halfplane $W_1(z)$. (For the theory of Fourier transforms of tempered distributions see e.g. Gel'fand and Vilenkin [10, Vol. I and II]. Here one also finds relations between support properties and the analytic continuations of the Fourier-transform. We Remark in our case that $\varphi(x\alpha_a(y))$ is always bounded on the reals. Therefore one obtains restricted estimates for the analytic continuation by Phragmén-Lindelöff typ arguments.) Since $\varphi(x\alpha_a(y))$ is for real a bounded by $\|\varphi\| \|x\|\cdot\|y\|$, it follows that $W_1(z)$ is bounded by $\|\varphi\| \|x\|\cdot\|y\| \exp\{\operatorname{Im} z(\lambda+\varepsilon)\}$. Replacing φ by φ^* we get the corresponding statement for $W_2(z)$. This shows the necessity of the condition. Conversely, let φ fulfill the conditions. By writing $\varphi(\alpha_a(x)) = \varphi(1\alpha_a(x)) = \varphi(\alpha_a(x)1)$, we see that $\varphi(\alpha_a(x))$ is an entire analytic function $W(z)$ with $|W(z)| \leqq \|\varphi\| \|x\| \exp\{n|\operatorname{Im} z|\}$ with $n = \max\{m, m'\}$. Using Schwarz' lemma (e.g. [12, 5.2]) we get for $|a| < 1$:

$$|\varphi(\alpha_a(x)) - \varphi(x)| \leqq \|\varphi\|\cdot\|x\|\cdot\{2\exp n\}\cdot|a|,$$

which shows $\varphi \in A_c^*$. Moreover the estimate of the analytic continuation of $\varphi(x\alpha_a(y))$ gives us that this is the Fourier transform of a distribution with support in $[-m, \infty]$. But this implies that φ annihilates the left ideal generated by $R((-\infty, -\lambda))$, and hence $\varphi E(\lambda) = \varphi$. Using the functional φ^*, we see by the same argument that the conditions of (iii) are also sufficient for showing that $\varphi \in A_0^*(\hat{\mathbb{R}}^+)$.

Next we continue the proof of (ii). We assume $\varphi \in A_0^*(\hat{\mathbb{R}}^+)$, $f \in \mathscr{L}^1$ with supp $\hat{f} \subset [-\mu, +\mu]$. We want to show that $\varphi z(f)$ belongs to $A_0^*(\hat{\mathbb{R}}^+)$ again. If $E(\lambda_1)\varphi = \varphi$, then clearly $E(\lambda_1)\varphi z(f)$. Assume next $\varphi E(\lambda_2) = \varphi$. Then $W_1(z)$, the extension of $\varphi(x\alpha_a(yz(f))$ into the upper half-plane, is bounded by $\|\varphi\| |x| |y| \|z\| \|f\|_1 \cdot \exp\{(\operatorname{Im} z)\cdot\lambda_2\}$. From this it follows that $\varphi(x\alpha_a(y)\cdot z(f)) = \varphi(x\alpha_a(yz(f_{-a}))$ has again an analytic extension into the upper halfplane and this function $W_3(z)$ fulfills the estimate:

$$\begin{aligned}|W_3(z)| &\leqq \|\varphi\|\cdot\|x\|\cdot\|y\| \|z\|\cdot\|f_{-a}\|_1\cdot\exp[\lambda_2 \operatorname{Im} z]\\ &\leqq \|\varphi\|\cdot\|x\| \|y\| |z|\cdot\|f\|_1\cdot\exp[\mu|\operatorname{Im} z|]\exp[\lambda_2 \operatorname{Im} z].\end{aligned}$$

But this shows by (iii) that $\varphi z(f) \in A_0^*(\hat{\mathbb{R}}^+)$. In all the estimates we have used the $\mathscr{L}^1$ norm of f. Since the $\mathscr{L}^1$ functions with compact Fourier transforms are dense, it follows that $\varphi z(f) \in A^*(\hat{\mathbb{R}}^+)$ for $\varphi \in A_0^*(\hat{\mathbb{R}}^+)$ and $f \in \mathscr{L}^1(\mathbb{R})$. Using the estimate

$$|\varphi(x\{z(f) - z\})| \leqq \{|\varphi^*|(xx^*)\cdot|\varphi|([z^*(\bar{f}) - z][z(f) - z])\}^{1/2},$$

we want to derive that φz is the norm limit of $\varphi z(f)$ for a suitably chosen sequence f_n. We remark first that $\varphi \in A_0^*(\hat{\mathbb{R}}^+)$ with $\varphi E(\lambda) = \varphi$ implies $\alpha_a^* |\varphi|$ has an extension as entire analytic function of exponential type with $\alpha_z^* |\varphi| E(\lambda) = E(\lambda)\alpha_z^* |\varphi| = \alpha_z^* |\varphi|$. Writing

$$\alpha_a^* |\varphi| = \sum_0^\infty |\varphi|^{(n)} \cdot \frac{a^n}{n:},$$

we obtain the estimate $\| |\varphi|^{(n)} \| \leqq \| |\varphi| \| M(\lambda + \varepsilon)^n$ for some constant M (see [2, Theorem 2.2.10]). Let a be in a compact set and $\varepsilon > 0$. Then there exists N with

$$\| \alpha_a^* |\varphi| - \sum_0^N |\varphi|^{(n)} \frac{a^n}{n!} \| < \varepsilon,$$

and hence we obtain by an 3 ε argument that $b \to a_a^* |\varphi|(x\alpha_b(y))$ is equi-continuous for a in that compact set. This implies for any given $\varepsilon > 0$ there exists δ_1 such that $\| \alpha_a^* |\varphi| - |\varphi| \| < \varepsilon \| \varphi \|$ for $|a| < \delta_1$. If we restrict a to $|a| < \delta_1$, then we can find δ_2 such that

$$|\alpha_a^* |\varphi|(z^* \alpha_b(z)) - a_a^* |\varphi|(z^* z)| < \varepsilon \| \varphi \| \| z \|^2 \text{ for } |b| < \delta_2.$$

Choose now δ such that $\delta \leqq \delta_1$ and $2\delta \leqq \delta_2$ and $f \in \mathscr{L}^1(\mathbb{R})$ with $f(a) \geqq 0$, $\int f(a)da = 1$ and supp $f \subset [-\delta, \delta]$. Then we obtain

$$|\varphi|(\{z^*(f) - z^*\}\{z(f) - z\}) \leqq 2\varepsilon \| |\varphi| \| \cdot \| z \|^2.$$

This shows $\varphi z \in A^*(\hat{\mathbb{R}}^+)$ for $\varphi \in A_0^*(\hat{\mathbb{R}}^+)$. But since $\varphi_n z$ converges to φz whenever φ_n converges to φ, it follows from (i) that $\varphi z \in A^*(\hat{\mathbb{R}}^+)$ for $\varphi \in A^*(\hat{\mathbb{R}}^+)$ and $z \in A^{**}$. Since $A^*(\hat{\mathbb{R}}^+)$ is invariant under involution, it follows that $A^*(\hat{\mathbb{R}}^+)$ is invariant by left and right multiplication with elements in A^{**}. Hence $A^*(\hat{\mathbb{R}}^+)$ is a folium and $E(\infty)$ belongs to the center of A^{**}.

This result tells us that $[A^{**}E(\infty), \mathbb{R}, \alpha^{**}\}$ is a W^*-dynamical system with (weakly) continuous group action, and therefore we obtain the standard results: (see G. K. Pedersen [11, Theorem 8.4.3])

(1) $E(\lambda) = 0$ for $\lambda < 0$.

(2) *$\lambda \to E(\lambda)$ is increasing.*

(3) $E(\infty) = \underset{\lambda \to \infty}{s\text{-lim}}\, E(\lambda) \in$ *center of* A^{**}.

(4) $E(\lambda) \in$ *center of* A_G^{**} ($= \alpha_a^{**}$ invariant elements in A^{**}).

(5) $U(a) = \int_{-0}^{\infty} \exp[ia\lambda] dE(\lambda) \in A^{**}E(\infty)$ *implements the automorphism* α_a^{**} *on* $A^{**}E(\infty)$.

(6) *$U(a)$ is minimal in the following sense: Let π be a normal representation of $A^{**}E(\infty)$ and assume $V(a) \in B(H_\pi)$ is a continuous unitary representation of the group $\mathbb{R}$ such that*

(i) $\pi(\alpha_a^{**}x) = V(a)\pi(x)V^*(a)$,

(ii) *spectrum* $V(a) \subset \mathbb{R}^+$.

Then spectrum $V(a)\pi(u(a))^* \subset \mathbb{R}^+$ *follows.*

III. The Spectrum in a Half-Space

Assume next that $G = \mathbb{R}^n$ is the n parametric vector group and that we deal with the C^*-dynamical system $\{A, \mathbb{R}^n, \alpha\}$. (Again there is no continuity assumption about the action of α_t, $t \in \mathbb{R}^n$, on A.) We want to look at covariant representations $\{\pi(A), \rho(\mathbb{R}^n), \mathscr{H}\}$ with

(1) $\rho(t)$ is strongly continuous.

(2) There exists a direction $t_0 \in \mathbb{R}^n$, $t_0 \neq 0$ such that the group representation $\rho(\lambda t_0)$, $\lambda \in \mathbb{R}$ fulfills the spectrum condition. In other words

$$\text{spectrum } \rho(t) \subset \{p \in \hat{\mathbb{R}}^n; (p, t_0) \geqq 0\}.$$

We again denote by A_c^* the set of $\varphi \in A^*$ such that $t \to \alpha_t^* \varphi$ is a continuous function on $\mathbb{R}^n$ with values in the Banach space A^*. If $t \neq 0$, then for $\varphi \in A_c^*$ and $x \in A^{**}$, $\int \varphi(\alpha_{\lambda t}(x)) f(\lambda) d\lambda$ is a well defined functional on A_c^*. The set of extensions will be denoted by $[x(t, f)]$. With these objects we now can work as in the last section. In particular we will fix a direction t_0. Let now $R(t_0, (-\infty, \lambda))$ be the sets defined as before with respect to the direction t_0 and the operators $[x(t_0, f)]$ be defined by A_c^* (the latter with respect to the whole group). Then we see $R(t_0, (-\infty, \lambda))$ is invariant under the whole group. If $E(t_0, \lambda)$ is again the right annihilator of $R(t_0, (-\infty, -\lambda))$, then $E(t_0, \lambda)$ is invariant under the whole group. If $E(t_0, \infty)$ is again the strong limit of $E(t_0, \lambda)$, then $N_c E(t_0, \infty) = 0$, which implies that $\varphi \in A^*$, and $E(t_0, \infty)\varphi = \varphi E(t_0, \infty) = \varphi$ implies automatically $\varphi \in A_c^*$.

Working now with these projections $E(t_0, \lambda)$, we obtain the same results as in the last section.

III.1. Theorem. *Let* $\{A, \mathbb{R}^n, \alpha\}$ *be a* C^**-dynamical system* (with no continuity requirement). *Let* $t_0 \neq 0$ *be a fixed vector in* $\mathbb{R}^n$. *Then the projections* $E(t_0, \lambda)$ *defined above have the following properties*:

(1) $E(t_0, \lambda) = 0$ *for* $\lambda < 0$.

(2) $E(t_0, \lambda)$ *is invariant under* α_g^{**} *for every* $g \in \mathbb{R}^n$.

(3) $E(t_0, \lambda) \in A^{**}$ *is increasing in* λ.

(4) $E(t_0, \infty) = \text{s-}\lim_{\lambda \to \infty} E(t_0, \lambda) \in$ *center of* A^{**}.

(5) $U(t_0 \cdot \mu) = \int_{-0}^{\infty} \exp[i\mu\lambda] dE(t_0, \lambda) \in A^{**}E(t_0, \infty)$ *implements the automorphism* $\alpha_{t_0}^{**} \cdot \mu$ on $A^{**}E(\infty)$.

(6) $\alpha_a^{**}\{U(t_0\mu)\} = U(t_0\mu)$ *for every* $a \in \mathbb{R}^n$.

(7) $U(t_0\mu)$ *is minimal in the sense described in the last section.*

We also can generalize the proposition of last section.

III.2. Definition. Let $\{A, \mathbb{R}^n, \alpha\}$ be a C^*-dynamical system, and $t_0 \neq 0$ be a fixed direction in $\mathbb{R}^n$; and let $E(t_0, \lambda)$ be the projections as before. We put:

(a) $A^*(\hat{\mathbb{R}}^{n,+}_{t_0}) = \{\varphi \in A^*; E(t_0, \infty)\varphi = \varphi E(t_0, \infty) = \varphi\}$.

(b) $A_0^*(\hat{\mathbb{R}}^{n,+}_{t_0}) = \{\varphi \in A^*$ such we can find $\lambda < \infty$ with $E(t_0, \lambda)\varphi = \varphi E(t_0, \lambda) = \varphi\}$.

With these notations we obtain:

III.3. Proposition. *With the assumptions and notations as before we obtain:*

(1) $A_0^*(\hat{\mathbb{R}}_{t_0}^{n,+})$ *is norm dense in* $A^*(\hat{\mathbb{R}}_{t_0}^{n,+})$.

(2) φ *belongs to* $A_0^*(\hat{\mathbb{R}}_{t_0}^{n,+})$ *if and only if* φ *fulfills the following conditions:*

(α) $\varphi \in A_c^*$,

(β) $\mu \to \varphi(x\alpha_{\mu t_0}(y)$ *is for every* $x, y \in A^{**}$ *a continuous function, and it is the boundary value of an analytic function* $W_1(z)$ *holomorphic in the upper halfplane which fulfills the estimate*

$$|W(z)| \leqq \|\varphi\| \, \|x\| \cdot \|y\| \exp\{\lambda |\operatorname{Im} z|\}$$

for some suitable constant λ.

(γ) $\mu \to \varphi(\alpha_{\mu t_0}(x)y)$ *has the analogous properties except for the replacement of the upper half-plane by the lower half-plane.*

Remarks(i) In contrast to the one dimensional situation we have to add the condition (α) of Proposition III.3.(2) in order to obtain continuity of the group action also in directions different from t_0.

(ii) The set $A^*(\hat{\mathbb{R}}_{t_0}^{n,+})$ is the pre–dual of $A^{**}E(t_0, \infty)$ and it belongs to A_c^*. Hence by [7, Theorem III.2] there exists a faithful normal representation $(\pi, \mathscr{H})$ of $A^{**}E(t, \infty)$ and a continuous unitary representation $\rho(t)$ on $\mathscr{H}$ of $\mathbb{R}^n$ with

$$\rho(t)\pi(x)\rho^*(t) = \pi(\alpha_t^{**}(x)), \quad x \in A^{**}E(t, \infty).$$

We choose now a basis $b_1, \ldots b_n \in \mathbb{R}^n$ with $b_1 = t_0$ the fixed direction. Then we can write $t = \Sigma \mu_i b_i$, and hence

$$\rho(t) = \prod_{i=1}^{n} \rho(\mu_i b_i).$$

But since $\pi(U(\mu t_0))$ commutes with $\rho(t)$ ($U(\mu t)$ the group representation of Theorem III.1) we have that

$$\rho'(t) = \pi(U(\mu_1 t_0)) \prod_2^n \rho(\mu_i b_i)$$

is again a continuous group representation. But, this has the additional property that $\rho'(\mu_1 t_0) = \pi(U(\mu_1 t_0))$ fulfills the spectrum condition and is minimal.

IV. The General Case

Let $(A, \mathbb{R}^n, \alpha)$ be again a C^*-dynamical system and $V \subset \mathbb{R}^n$ a closed, convex, proper cone with interior points. The dual cone $\hat{V}$ is again a proper, closed, convex cone with interior points. We want to look at covariant representations $\{\pi, \rho(a), \mathscr{H}\}$ such that $\rho(a)$ is a continuous unitary representation implementing α_a, $a \in \mathbb{R}^n$, and such that the spectrum of ρ is contained in $\hat{V}$. The spectrum of ρ contained in $\hat{V}$ is equivalent to saying that for every $t \in V$, $t \neq 0$, the one parametric group $\rho(\mu t)$ has positive spectrum. Therefore, we can use the results of the last section. In particular, the projections $E(t_0, \infty)$ belong to the center of A^{**} and they are α_a^{**} invariant for every $a \in \mathbb{R}^n$.

Let for $t\in V, t\neq 0$, $E(t,\lambda)$ and $E(t,\infty)$ be the projections defined in the last section. All these projections are invariant under $\alpha_a^{**}, a\in\mathbb{R}^n$, and hence they commute with each other because they are the spectral projections of the one parametric group $U(\mu t)$ implementing the automorphisms.

V.1. Definition. (a) Define $E(\hat{V})=\Pi\{E(t,\infty);0\neq t\in V\}$, where the product is the limit of the decreasing net of finite products. $E(\hat{V})$ belongs to the center of A^{**} and is invariant under α_a^{**}.

(b) For $p\in\hat{V}$ define

$$E(\langle 0,p\rangle)=\Pi\{E(t,\lambda_t);0\neq t\in V \text{ and } \lambda_t=(p,t)\}.$$

$\langle 0,p\rangle$ stands here for the order intervall $\hat{V}\cap\{p-\hat{V}\}$. The projections $E(\langle 0,p\rangle)$ are also invariant under α_a^{**}. We show first:

IV.2. Lemma. *With the assumptions of this section and the above notation we obtain*

(a) *Let p_n be increasing in the order of $\hat{V}$ such that $\bigcup_n\langle 0,p_n\rangle$ covers all of $\hat{V}$. Then it follows that*

$$s\text{-}\lim_{n\to\infty} E(\langle 0,p_n\rangle)=E(\hat{V}).$$

(b) *For every $x\in A^{**}$ the function $a\to\alpha_a(x)E(\langle 0,p\rangle)$ is weakly continuous and*

$$\operatorname{supp}\mathscr{F}^{-1}\alpha_a(x)E(\langle 0,p\rangle)\subset -p+\hat{V},$$

where $\mathscr{F}^{-1}$ means the weak inverse Fourier-transform (in the sense of distributions).

Proof. (a) Let ω be a normal state of $A^{**}E(\hat{V})$, and take a function $f\in\mathscr{L}^1(\mathbb{R}^n)$ with the properties $f\geqq 0$, $\int f(a)da=1$, and $\operatorname{supp}\mathscr{F}^{-1}f=K$ is compact. Define $\omega_f=\int\alpha'_a\omega f(a)da$. For $t\in V, t\neq 0$ choose λ_t^1 such that $K\subset\{p,(p,t)\geqq -\lambda_t^1\}$. From the support property of f it follows that ω_f annihilates every $x\in R(t,(-\infty,-\lambda_t^1))$, which implies the equation $\omega_f(E(t,\lambda_t^1))=\omega_f(E(\hat{V}))=1$. Let now $p_n\in\hat{V}$ be such that $(p_n,t)\geqq\lambda_t^1$ for every $t\in V$ (this is possible when the λ_t^1 are suitably chosen). Then we obtain $\omega_f(\prod_{i=1}^n E(t_i,(p_n,t_i)))=\omega_f(E(\hat{V}))$, and hence by Definition IV. 1.(b), we get $\omega_f(E(\langle 0,p_n\rangle))=\omega_f(E(\hat{V}))$. Denote by $F=s\text{-}\lim E(\langle 0,p_n\rangle)$. Then one obtains $\omega_f(F)=\omega_f(E(V))$ for every positive f of norm 1 by continuity of ω_f in f. Since α'_a acts strongly continuous on $A'(E(\hat{V}))$ (it follows that these states are norm dense in the set of all states in $A^*(E(\hat{V}))$. Therefore $\omega(F)=\omega(E(\hat{V}))$ for every ω with $\omega(E(\hat{V}))=1$. This implies $F\geqq E(\hat{V})$. The opposite inclusion is trivial by the definition of $E(\langle 0,p\rangle)$.

(b) Let $\varphi\in A^*$ such that $\varphi E(\hat{V})=\varphi$, then $a\to\varphi(x\alpha_a(y))$ is a continuous function on $\mathbb{R}$. Let $t\in V$. Then we get

$$\int\varphi(y\alpha_{\lambda t}(x)E(\langle 0,p\rangle)e^{-i\lambda\mu}d\lambda=0 \quad \text{for } \mu<-(t,p),$$

since $E(\langle 0,p\rangle)\leqq E(t,\mu)$. Varying the directions in V we obtain:

$$\operatorname{supp}\mathscr{F}^{-1}\varphi(y,\alpha_a(x)E(\langle 0,p\rangle))\subset -p+\hat{V}.$$

Assuming $E(\hat{V})$ is not zero, then we have for every $t \in V$, $t \neq 0$ a unique continuous minimal group representation $U^0(\mu t)$ fulfilling the spectrum condition and belonging to $A^{**}E(\hat{V})$. With these representations we can define special representations of the whole group $\mathbb{R}^n$.

IV.3. Definition. With the same assumptions as before denote:

(a) $B = \{b^1, \ldots, b^n\}$ such that $b^i \neq 0, b^i \in V$ and b^i a linear independent basis in V.

(b) For $a \in \mathbb{R}^n$, $a = \sum_1^n \mu_i b^i$, define

$$U_B(a) = \prod_{i=1}^{n} U^0(\mu_i b^i),$$

where $U^0(\mu b)$ are the minimal representations of the last section.

(c) For a given basis in V denote by V_B the cone generated by B, i.e.,

$$V_B = \{\Sigma p_i b^i ; p_i \geqq 0\},$$

and by $\hat{V}_V$ the dual cone of V_B.

(d) For two bases B_1, B_2 put

$$W_{B_1,B_2}(a) = U_{B_1}(a)\, U^*_{B_2}(a).$$

From the results of Sect. III we see the following properties of the quantities defined above:

IV.4. Properties

(i) *$U_B(a)$ is a continuous unitary group representation of the translation group $\mathbb{R}^d$ in $E(\hat{V})A^{**}$ implementing the automorphisms α_a on this von Neumann algebra.*

(ii) *Spectrum $U_B(a) \subset \hat{V}_B$.*

(iii) *$W_{B_1,B_2}(a)$ belongs to the center of $E(\hat{V})A^{**}$, is unitary, and $W^*_{B_1,B_2}(a) = W_{B_2,B_1}(a)$.*

(iv) *Looking at the definition of W_{B_1,B_2} we see for three different bases,*

$$W_{B_1,B_2}(a)\, W_{B_2,B_3}(a) = W_{B_1,B_3}(a).$$

Looking at the properties (iii) and (iv) we see that there exists a unitary, continuous representation of the translations

$$Y_B(a) \in \mathfrak{Z}(E(\hat{V})A^{**}) \text{ with } W_{B_1,B_2}(a) = Y_{B_1}(a)\, Y^*_{B_2}(a).$$

The representation $Y_B(a)$ is not uniquely defined. If $Y_0(a)$ is a continuous unitary representation in $\mathfrak{Z}(E(\hat{V})A^{**})$, then the most general solution of the above equations is of the form $Y'_B(a) = Y_B(a)\, Y_0(a)$. However, we learn from this and the relation $U_{B_1}(a) = W_{B_1,B_2}(a) U_{B_2}(a)$ *that*

$$U_{B_1}(a)\, Y^*_{B_1}(a) = U_{B_2}(a)\, Y^*_{B_2}(a) = U(a)$$

is independent of the special base. So that we get $U_B(a) = U(a)\, Y_B(a)$, which means that the dependence on the base is at most in the form of a representation belonging to the center.

In the last equation neither $U(a)$ nor $Y_B(a)$ are uniquely defined. Therefore it is natural to ask whether we can adjust $U(a)$ in such a way that its spectrum is contained in $\hat{V}$. The answer is given in the following

IV.5. Theorem. *Let $\{A, \mathbb{R}^d, \alpha\}$ be a C^*-dynamical system and assume that the projection $E(\hat{V})$ defined in* III.2., *which belongs to the center of A^{**}, is not zero. Then there exists a continuous unitary representation in $E(\hat{V})A^{**}$ with spectrum of $U(a)$ contained in $\hat{V}$.*

Proof. Let B_0 be a fixed base and write $U^0(a)$ instead of $U_{B_0}(a)$. If V_{B_0} is the cone $\{\Sigma\lambda_i b^{(i)}; \lambda_i \geqq 0\}$, then the spectrum of $U^0(a)$ is contained in $\hat{V}_{B_0}$. Let Γ be a compact set in $\hat{V}_{B_0}$ and $\Delta = \langle 0, p\rangle$ be a compact set in $\hat{V}$. Denote the spectral projections of $U^0(a)$ by F. Since $F(\Gamma)$ and $E(\Delta)$ commute with each other it follows that $F(\Gamma)\,E(\Delta)$ $E(\Delta)$ is again a projection. These projections tend to $E(\hat{V})$ if Γ tends to $\hat{V}_{B_0}$ and simultaneously Δ tends to $\hat{V}$.

Let $Z(\Gamma, \Delta)$ be the central carrier of $E(\Delta)\cdot F(\Gamma)$. Then the common range projection of elements of the form $x\,E(\Delta)F(\Gamma)$ for $x \in A^{**}$ is $Z(\Gamma, \Delta)$. Hence investigating $Z(\Gamma, \Delta)\,U^0(a)$ is the same as investigating the expressions

$$U^0(a)xE(\Delta)F(\Gamma) = \alpha_a(x)U^0(a)E(\Delta)F(\Gamma).$$

We remark that by the definition of $E(\Delta)$ it follows that $\alpha_a(x)\,E(\Delta)$ is the Fourier transform of an expression having support in $\hat{V} + (-\Delta)$. The expression $U^0(a)F(\Gamma)$ is the Fourier transform of an expression having support in Γ. Hence we obtain

$$\operatorname{supp} \mathscr{F}^{-1}\{\alpha_a(x)E(\Delta)U^0(a)F(\Gamma)\} \subset \hat{V}^+ + (-\Delta + \Gamma).$$

Since now Δ and Γ are compact, there exists a vector $q \in \hat{\mathbb{R}}^d$ such that $q + \Gamma - \Delta \subset \hat{V}$. Call this vector $q(\Gamma, \Delta)$. But this shows $Z(\Gamma, \Delta)(U^0(a)e^{i(q,a)})$ fulfills the spectrum condition.

Choose now a sequence Γ_i, Δ_i such that $\Gamma_i \subset \Gamma_{i+1}$ tends to $\hat{V}_{B_0}$, and $\Delta_i \subset \Delta_{i+1}$ tends to $\hat{V}$. In this case $E(\Delta_i)F(\Gamma_i)$ tends to $E(\hat{V})$ and also $Z(\Gamma_i, \Delta_i)$ tends to $E(\hat{V})$.

Define now a continuous unitary representation belonging to the center of $E(\hat{V})A^{**}$ by

$$Y(a) = \sum_1^\infty (Z(\Gamma_{i+1}, \Delta_{i+1}) - Z(\Gamma_i, \Delta_i)) \quad \exp[i(a, q(\Gamma_{i+1}, \Delta_{i+1})].$$

Define $U(a) = U^0(a)\,Y(a)$. Then by the construction of $q(\Gamma, \Delta)$:

$$\begin{aligned}&\operatorname{Sp} U(a)(Z(\Gamma_{i+1}, \Delta_{i+1}) - Z(\Gamma_i, \Delta_i))\\ &\quad = \operatorname{Sp} U^0(a)(Z(\Gamma_{i+1}, \Delta_{i+1}) - Z(\Gamma_i, \Delta_i)) \exp[i(a, q(\Gamma_{i+1}, \Delta_{i+1}))] \subset \hat{V}.\end{aligned}$$

Since

$\sum_1 (Z(\Gamma_{i+1}, \Delta_{i+1}) - Z(\Gamma_i, \Delta_i)) = E(\hat{V})$, we have that $U(a)$ belongs to $E(\hat{V})A^{**}$ and fulfills the spectrum condition.

The result obtained in Theorem IV.5. does not say that the representation $U(a)$ is unique. This is of course not true. We also cannot expect that there exists a unique minimal representation without further assumptions on the cone V or the algebra A. If V is a simplicial cone, then the iteration of Theorem III.1 gives us such a minimal $U(a)$. Also the physically interesting case namely locality and spectrum condition will lead to a minimal representation. But these questions will be investigated in another paper.

V. Characterization of the Normal States

It remains to characterize the normal states of $A^{**} \cdot E(\hat{V})$

V.1. Definition. Let $\{A, \mathbb{R}^n, \alpha\}$ be C^*-dynamical system and $V \subset \mathbb{R}^n$ a closed convex proper cone with interior point. Let $E(\hat{V})$ be the projection defined in the last section ($E(\hat{V})$ belongs to the center of A^{**}), then we denote:

(a) $A^*(\hat{V}) = \{\varphi \in A^*; E(\hat{V})\varphi = \varphi E(\hat{V}) = \varphi\}$.

(b) $A_0^*(\hat{V}) = \{\varphi \in A^*(\hat{V})$ such that we can find $p \in V$ with $E(\langle 0, p \rangle)\varphi = E(\langle 0, p \rangle) = \varphi\}$.

The aim of this section is to prove

V.2. Theorem. *Let $\{A, \mathbb{R}^n, \alpha\}$ be a C^*-dynamical system (no continuity requirement on α_a), and let $V \subset \mathbb{R}^n$ be a proper, closed, convex cone with interior points. Then with the notations obtained before we obtain:*

(1) $A_0^*(\hat{V})$ is norm dense in $A^*(\hat{V})$.

(2) An element $\varphi \in A^*$ belongs to $A_0^*(\hat{V})$, if and only if it fulfills the following properties:

(α) $a \to \varphi(x\alpha_a(y))$ is continuous on $\mathbb{R}^n$.

(β) $\varphi(x\alpha_a(y))$ is the boundary value of an analytic function $W(z)$ holomorphic in the tube $T(V) = \{z \in \mathbb{C}^n; \operatorname{Im} z \in V^0\}$. ($V^0$ = interior of V.)

(γ) $W(z)$ fulfills the estimate

$$|W(z)| \leqq \|\varphi\| \cdot \|x\| \cdot \|y\| \cdot \exp\{m \|\operatorname{Im} z\|\}$$

for some constant m and some norm on $\mathbb{R}^n$.

(δ) φ^* fulfills the same conditions.

(3) Let $(\pi, \mathscr{H})$ be a representation of A. Then we can find a continuous unitary representation $\rho(a)$ acting on $\mathscr{H}$ with

(α) $\rho(a)\pi(x)\rho^*(a) = \pi(\alpha_a(x))$,

(β) Spectrum $\rho(a) \subset \hat{V}$,

if and only if the folium of π-normal states belongs to $A^*(V)$.

Proof. (1) From Lemma IV.2 we know $E(\langle 0, p_n \rangle) \to E(\hat{V})$ for a suitable chosen sequence p_n. From this we get (1) in the same manner as (i) of Proposition II.2.

(2) Let $\varphi \in A_0^*(\hat{V})$ such that $\varphi E(\langle 0, p \rangle) = \varphi$. Then it follows from Lemma IV.2. that

$$\operatorname{supp} \mathscr{F}^{-1} \varphi(x\alpha_a(y)) \subset -p + \hat{V}.$$

This implies that $\varphi(x\alpha_a(y))$ has an analytic continuation $W(z)$ into the tube $T(V)$ which fulfills the estimate

$$|W(z)| \leqq \|\varphi\| \cdot \|x\| \cdot \|y\| \cdot \exp\{m \|\operatorname{Im} z\|\}$$

with every $m \geqq \|p\|$. This shows φ fulfills the conditions $(\alpha \div \delta)$.

Conversely assume φ fulfills these conditions. Then $\varphi(\alpha_a(x)) = \varphi(1\alpha_a(x)) = \varphi(\alpha_a(x)1)$ has an analytic continuation as well into the tube $T(\hat{V})$ as into the tube $T(-\hat{V})$. From this it follows by the edge of the wedge theorem [8] that $\varphi(\alpha_a(x))$ can be extended to an entire analytic function $W(z)$. We have the estimate $|W(z)| \leqq \|\varphi\| \cdot \|x\| \exp\{m \|\operatorname{Im} z\|\}$ for $z \in T(V)$ and for $z \in T(-V)$. But this estimate shows that supp $\mathscr{F}^{-1}\varphi(\alpha_a(x))$ is compact and consequently it follows the estimate

$$|W(z)| \leqq \|\varphi\| \cdot \|x\| \cdot \exp\{m' \|\operatorname{Im} z\|\}$$

with an eventually different constant m' (m' depends only on $\hat{V}$, and m but not on φ and $x \in A^{**}$). Using now the n-dymensional Schwarz Lemma (see e.g. [3] III.6. Theorem 7) we obtain

$$|\varphi(\eta_a(x)) - \varphi(x)| \leqq \|\varphi\| \cdot \|x\| \cdot \|a\| \cdot 2 \exp m'$$

for $\|a\| < 1$. This shows $\varphi \in A_c^*$.

Since $\hat{V}$ is a cone with interior points there exist $p_1 \in \hat{V}$ such that $\langle 0, p_1 \rangle \supset \{p \in \hat{V}; \|p\| \leqq m\}$. Let $W(z)$ be the analytic extension of $\varphi(x\alpha_a(y))$. The estimate (γ) gives us $W(z)e^{i(p_1, z)}$ is bounded for $z \in T(V)$ and consequently:

$$\operatorname{supp} \mathscr{F}^{-1}\varphi(x\alpha_a(y)) \subset -p_1 + \hat{V}.$$

From this it follows that φ annihilates the left ideal generated by $R(t, (-\infty, -(p_1 \cdot t))$, and hence we get $\varphi E(t, (p_1 \cdot t)) = \varphi$. Since this holds for every $0 \neq t \in V$ we obtain $\varphi E(\langle 0, p_1 \rangle) = \varphi$. Since the same arguments hold for φ^* it follows that $\varphi \in A_0^*(\hat{V})$.

(3) Let π be a representation of A such that the π-normal states belong to $A^*(\hat{V})$. Then there exists a projection E_π in the center of A^{**} with $E_\pi \leqq E(\hat{V})$ such that AE_π and $\pi(A)$ are quasi-equivalent. In this case $\pi(U(a))$ with $U(a)$ the unitary group representation in $A^{**}E(\hat{V})$ described in Theorem IV.5. has all the desired properties.

Conversely let $\{\pi, \rho(a), \mathscr{H}\}$ be a covariant representation of A, where $\rho(a)$ is a continuous unitary representation of $\mathbb{R}^n$ fulfilling the spectrum condition. This means we have

$$\rho(a) = \int_{\hat{V}} e^{i(a,p)} dF(p).$$

Let Δ be a compact in $\hat{V}$ and let $\psi \in H$ with $F(\Delta)\psi = \psi$ then we obtain

$$\omega_\psi(x\alpha_a(y)) = (\psi, \pi(x)\rho(a)\pi(y)\rho^*(a) \cdot F(\Delta)\psi).$$

Since the spectrum of ρ is contained in $\hat{V}$ we obtain for the Fourier-transform.

Supp $\mathscr{F}^{-1}\omega_\psi(x\alpha_a(y)) \subset -\Delta + \hat{V}$. But from this it follows that ω_ψ fulfills the conditions (α) (β) and (γ) of statement (2). Since ω_ψ is self-adjoint we have by (2) that $\omega_\psi \in A_0^*(\hat{V})$. Now the vectors for which we can find a compact $\Delta \subset \hat{V}$ with $F(\Delta)\psi = \psi$

is dense in $\mathscr{H}$. Since $A^*(\hat{V})$ is norm closed we get that every $\omega_\psi \in A^*(\hat{V})$ for every $\psi \in \mathscr{H}$. Finally $A^*(\hat{V})$ is a normclosed linear space, and therefore every π normal state belongs to $A^*(\hat{V})$.

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