Commun. Math. Phys. 95, 277–287 (1984)

Communications in Mathematical Physics

The $1/N_F$ Expansion of the γ and β Functions in Q.E.D.

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Abstract. The Callan-Symanzik γ- and β-functions are calculated analytically for Q.E.D. in the limit of a large number of leptons ($N_F \to \infty$) up to terms of order $1/N_F$ inclusive. We give closed analytic expressions for the coefficients of these terms in their series expansion in powers of $K \equiv \alpha N_F/\pi$. We have been able to sum these series and to obtain some striking results.

1. Introduction

Apart from the first few perturbative terms of the renormalization group functions, not much more is known about them. To our knowledge in four-dimensional field theories one has gone beyond that only in $\lambda\Phi^4$ and in Q.E.D. in the limit of a large number of fermions, $N_F \to \infty$. In $\lambda\Phi^4$ Khuri [1] has been able to find a zero of the Callan-Symanzik β-function [2] by using a Borel resummation technique, and in Q.E.D. ($N_F \to \infty$) Coquereaux [3] has computed the first nontrivial coefficient in the $1/N_F$ expansion of the same function. In a recent paper [4] we have computed the Callan-Symanzik γ-function [2], which governs the dependence on the renormalization point of the renormalized fermion mass, in Q.E.D. ($N_F \to \infty$). The main results obtained there were:

i) The numerical computation of the first 19 terms strongly suggests a series with a finite radius of convergence. This is not what one expects in four-dimensional field theories.

ii) There are at least two zeros within the region of convergence.

iii) The analytical computation of the first 7 terms clearly hints on a factorization of the series into two factors. These are two series, one of them of only rational coefficients, and the other of coefficients which are sums of products of Riemann ζ-functions. To our knowledge this is the first time such a factorization has been found in field theory.

While trying to extend these results to the Callan-Symanzik β-function in Q.E.D. ($N_F \to \infty$) we have found that the results of our previous work can be considerably improved and extended to the calculation of the β-function. The main results of the present paper are:

i) Due to the fact that the γ- and β-functions must be momentum independent we have been able to give a closed analytical expression for the coefficients of the expansion of these functions in power series of $K \equiv \alpha N_F/\pi$ for the $1/N_F$ term and to sum analytically the obtained expansion.

ii) We have been able to prove, analytically, that the γ-function presents, in the limit $N_F \to \infty$, two zeros at $K=9/2$ and $K=6$, while the $1/N_F$ term of the β-function is a positive and increasing function of K.

iii) For the γ-function we have been able to prove that its series expansion can be factorized, to all orders, into two factors. These are two series, one of them of only rational coefficients, and the other of coefficients which are sums of products of Riemann ζ-functions. A similar factorization has been found for the series expansion of the $1/N_F$ term of $Kd\beta/dK-\beta$.

The next section is devoted to the analytic calculation of the γ-function. In Sect. 3 the β-function is considered. In the last section the main conclusions are drawn.

2. The γ-Function

In order to obtain an expansion in powers of $1/N_F$ for the γ- and β-functions the coupling constant, held fixed in the large N_F limit, has to be taken as $K \equiv \alpha N_F/\pi$, where $\alpha \equiv e^2/4\pi$, and e is the coupling constant of Q.E.D. We will work in the minimal subtraction (MS) scheme [5]. Since both functions are gauge parameter independent [6] we will work in the Landau gauge so that one does not have to consider gauge parameter renormalization. Since the MS scheme is mass independent we will put all fermion masses equal to zero. All these results are valid in Q.C.D. if $e \to g$ and the number of leptons N_F is substituted by the number of flavours N_f.

The γ-function is defined as

$$\gamma = -\frac{v}{m}\frac{dm}{dv}, \tag{2.1}$$

where m is the renormalized fermion mass and v is the renormalization scale. From the definition of the mass renormalization constant $m_0 = Z_m m$, m_0 being the bare mass, one can write for the leading term in $1/N_f$

$$\gamma = -v\frac{dZ_m^{-1}}{dv} + O(1/N_f^2). \tag{2.2}$$

As was done in [4] we will calculate this function in Q.C.D. since, in the order that we are interested in, both calculations are identical up to small changes due to the algebra associated to the gauge group that will be commented on later on.

Recalling that Z_m depends on v only through α, one obtains

$$\gamma = -\beta(K,\varepsilon)K\frac{dZ_m^{-1}}{dK} + O(1/N_f^2), \tag{2.3}$$

where the β-function for $n=4+2\varepsilon$ is given by

$$K\beta(K,\varepsilon) \equiv K[2\varepsilon + \beta(K)] = v\frac{dK}{dv}. \tag{2.4}$$

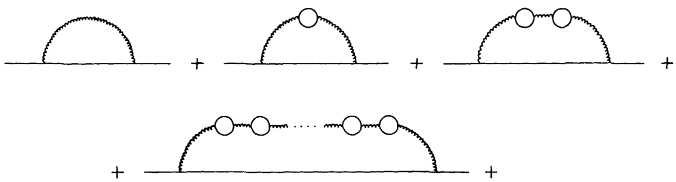

Fig. 1. Diagrams which contribute to the dominant term of the γ-function in the $1/N_F$ expansion

To leading order in the $1/N_f$ expansion, the only one we need here is

$$\beta(K)=\frac{K}{3}. \tag{2.5}$$

If the fermionic self-energy is decomposed according to (see [4])

$$\Sigma_0(\not{p}, K_0, \varepsilon, v)=m_0\tilde{\Sigma}_{1_0}\left(\frac{p^2}{v^2}, K_0, \varepsilon\right)+(\not{p}-m_0)\Sigma_{2_0}\left(\frac{p^2}{v^2}, K_0, \varepsilon\right), \tag{2.6}$$

one finds to leading order

$$Z_m^{-1}=1+\operatorname{div}\left[\tilde{\Sigma}_{1_0}\left(\frac{p^2}{v^2}, Z_k K, \varepsilon\right)\right], \tag{2.7}$$

where $K_0=Z_K K$, $Z_K=(1+K/6\varepsilon)^{-1}$ and $\operatorname{div}[\tilde{\Sigma}_{1_0}]$ means the poles in ε of $\tilde{\Sigma}_{1_0}$. What is left is the calculation of the dominant contribution to $\tilde{\Sigma}_{1_0}(p^2/v^2, K_0, \varepsilon)$. The diagrams one has to compute are shown in Fig. 1. For Q.C.D. the result is [4]

$$\begin{aligned}
\tilde{\Sigma}_{1_0}\left(\frac{p^2}{v^2}, K_0, \varepsilon\right) &= -\frac{1}{N_f}\sum_{n=1}^{\infty} K_0^n \frac{1}{6^{n-1} n\varepsilon^n} G\left(\frac{p^2}{v^2}, n, \varepsilon\right),\\
G\left(\frac{p^2}{v^2}, n, \varepsilon\right) &\equiv \left(-\frac{p^2}{4\pi v^2}\right)^{n\varepsilon}\left\{\frac{[\Gamma(1+\varepsilon)]^2\Gamma(1-\varepsilon)(1+\varepsilon)}{\Gamma(1+2\varepsilon)(1+2\varepsilon)\left(1+\frac{2\varepsilon}{3}\right)}\right\}^n\\
&\quad\cdot\frac{\Gamma(1+n\varepsilon)\Gamma(1-n\varepsilon)\Gamma(1+2\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)\Gamma(1+\varepsilon-n\varepsilon)\Gamma(1+\varepsilon+n\varepsilon)}\\
&\quad\cdot\frac{(1+2\varepsilon)\left(1+\frac{2\varepsilon}{3}\right)^2\left(1+\varepsilon-\frac{n(n-1)\varepsilon^2}{2}\right)}{(1+\varepsilon)\left(1+\frac{\varepsilon}{2}+\frac{n\varepsilon}{2}\right)(1+\varepsilon-n\varepsilon)(1+\varepsilon+n\varepsilon)}.
\end{aligned} \tag{2.8}$$

Inspection of $G(p^2/v^2, n, \varepsilon)$ allows us to see that we can write

$$G(p^2/v^2, n, \varepsilon)=\sum_{j=0}^{\infty} G_j(\varepsilon)(n\varepsilon)^j, \tag{2.9}$$

where the coefficients $G_j(\varepsilon)$, except $G_0(\varepsilon)$, depend on p^2/v^2 and all of them are n-independent series expansions of positive powers of ε.

Renormalizing the coupling constant and reexpanding in K one finds

$$\tilde{\Sigma}_{1_0}(p^2/v^2, Z_K K, \varepsilon) = -\frac{1}{N_f}\sum_{n=1}^{\infty}\frac{K^n}{6^{n-1}}\sum_{j=0}^{\infty}\frac{G_j(\varepsilon)}{\varepsilon^{n-j}}\sum_{i=0}^{n-1}\binom{n-1}{i}(n-i)^{j-1}(-1)^i. \tag{2.10}$$

Since we need only the poles in ε of this function, for a given n we can limit the sum over j from $j=0$ to $j=n-1$. The surprising thing is that

$$\sum_{i=0}^{n-1}\binom{n-1}{i}(n-i)^{j-1}(-1)^i=0, \qquad 1 \leqq j \leqq n-1, \tag{2.11}$$

and therefore

$$\operatorname{div}\left[\tilde{\Sigma}_{1_0}\left(\frac{p^2}{v^2}, Z_k K, \varepsilon\right)\right] = \frac{1}{N_f}\sum_{n=1}^{\infty}(-1)^n\frac{K^n}{n6^{n-1}}\operatorname{div}\left\{\frac{G_0(\varepsilon)}{\varepsilon^n}\right\}. \tag{2.12}$$

The relation (2.11) is the reason for the strong cancellations that we found in the direct calculation carried out in [4], without noticing their origin. Furthermore due to this relation it is clear that Z_m^{-1} and γ are independent of p^2/v^2 as it must be.

From (2.8) and (2.9) we obtain, using some well known properties of the $\Gamma(z)$-function (see [4]), that

$$\begin{aligned}
&G_0(\varepsilon) \equiv G(p^2/v^2, 0, \varepsilon) \equiv C(\varepsilon)D(\varepsilon), \\
&C(\varepsilon) = \frac{\Gamma(1+2\varepsilon)}{[\Gamma(1+\varepsilon)]^3\Gamma(1-\varepsilon)} = \exp\left\{\sum_{n=3}^{\infty}(-1)^n\frac{\zeta(n)}{n}\varepsilon^n[2^n-3-(-1)^n]\right\}, \\
&D(\varepsilon) = \frac{(1+2\varepsilon)(1+2\varepsilon/3)^2}{(1+\varepsilon)^2(1+\varepsilon/2)} = 1+\frac{5}{6}\varepsilon-\frac{35}{36}\varepsilon^2 \\
&\qquad + \sum_{n=3}^{\infty}\left[\frac{n+5}{3^2}-\frac{7}{2^n}-\frac{1}{3}\sum_{j=0}^{n-3}\frac{7+j}{2^{n-2-j}}\right](-\varepsilon)^n.
\end{aligned} \tag{2.13}$$

Now it is a straightforward calculation to obtain the power series expansion of $G_0(\varepsilon)$:

$$G_0(\varepsilon) = \sum_{i=0}^{\infty} G_i\varepsilon^i, \qquad G_0 = 1, \tag{2.14}$$

which is convergent for values of ε small enough. In terms of these coefficients we obtain immediately

$$Z_m^{-1} = 1 + \frac{1}{N_f}\sum_{n=1}^{\infty}\frac{(-1)^n K^n}{n6^{n-1}}\sum_{i=0}^{n-1}\frac{G_i}{\varepsilon^{n-i}}, \tag{2.15}$$

and using (2.3)

$$\gamma(K) = \frac{2K}{N_f}\sum_{n=0}^{\infty}\left(-\frac{K}{6}\right)^n G_n, \tag{2.16}$$

which is the desired result. Since (2.14) is convergent, this is also true for this series for K small enough. This expression allows us to compute the coefficients of the γ-function expansion as far as we like, since no important cancellations appear.

Comparison of (2.14) and (2.16) and the use of (2.13) gives

$$\gamma(K)=\frac{2K}{N_f}\cdot\frac{\Gamma(1-K/3)}{[\Gamma(1-K/6)]^3\Gamma(1+K/6)}\cdot\frac{(1-K/3)(1-K/9)^2}{(1-K/6)^2(1-K/12)}+O(1/N_f^2)$$

$$=\frac{2K}{3N_f}\frac{(1-K/9)\Gamma(4-K/3)}{\Gamma(1+K/6)[\Gamma(2-K/6)]^2\Gamma(3-K/6)}+O(1/N_f^2). \qquad (2.17)$$

The second of these expressions tells us that the radius of convergence of the γ-expansion is $K=15$, and therefore only for $K<15$ all these formulae are true. Furthermore it shows clearly that the γ-function has a zero at $K=9$ and another one at $K=12$. The first of these expressions, together with (2.13) shows clearly that the γ-function expansion in powers of K can be written as a product of two series. One of them has only rational coefficients, its convergence radius is $K=6$ and has a zero at $K=3$, while the other has coefficients which are positive and sums of products of Riemann ζ-functions [notice that $\zeta(2)$ does not appear], with convergence radius $K=3$.

With minor changes one obtains in Q.E.D.

$$\gamma(K)=\frac{K}{2N_F}\frac{(1-2K/9)\Gamma(4-2K/3)}{\Gamma(1+K/3)[\Gamma(2-K/3)]^2\Gamma(3-K/3)}+O(1/N_F^2), \qquad (2.18)$$

and its series expansion, which allows us to obtain the desired result, has a convergence radius of $K=15/2$ and presents zeroes at $K=9/2$ and $K=6$.

3. The β-Function

Let us now proceed to the calculation of the β-function in Q.E.D., which is defined in (2.4), being

$$\beta(K)=-\frac{v}{Z_k}\frac{\partial Z_k}{\partial v}. \qquad (3.1)$$

Since all the v-dependence on Z_K is due to its dependence on the renormalized coupling constant, we can write

$$\beta(K)\left[Z_K^{-1}-K\frac{\partial Z_K^{-1}}{\partial K}\right]=2\varepsilon K\frac{\partial Z_K^{-1}}{\partial K}. \qquad (3.2)$$

The leading terms of the $1/N_F$ expansion of Z_K^{-1} can be written as

$$Z_K^{-1}=1+\frac{K}{\varepsilon}B_0^{(1)}+\frac{1}{N_F}\sum_{n=2}^{\infty}K^n\sum_{i=0}^{n-1}B_i^{(n)}\frac{1}{\varepsilon^{n-i}}, \qquad (3.3)$$

and from (3.2) one obtains for the β-function the following expansion

$$\beta(K)=2KB_0^{(1)}+\frac{2}{N_F}\sum_{n=2}^{\infty}nK^nB_{n-1}^{(n)}. \qquad (3.4)$$

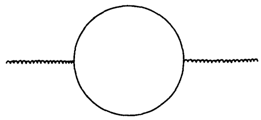

Fig. 2. Lowest order photon self-energy diagram

Since $\beta(K)$ is finite, we get the conditions

$$
\begin{aligned}
&B_0^{(n)}=0\,, \qquad n\geqq 2\\
&B_k^{(n+1)}=-\frac{n-1}{n+1}B_0^{(1)}B_k^{(n)}\,, \qquad n\geqq 2\,, \qquad k\leqq n-1
\end{aligned}
\tag{3.5}
$$

for the coefficients appearing in expansion (3.3).

If we write the photon self-energy as

$$\Pi_0^{\mu\nu}(q, K_0, \varepsilon, \nu)=(q^\mu q^\nu - q^2 g^{\mu\nu})\Pi_0(q^2/\nu^2, K_0, \varepsilon)\,, \tag{3.6}$$

then the equation that determines Z_K is

$$Z_K^{-1}=1-\operatorname{div}\{Z_K^{-1}\Pi_0(q^2/\nu^2, Z_K K, \varepsilon)\}\,. \tag{3.7}$$

Furthermore we can write

$$
\begin{aligned}
&\Pi_0(q^2/\nu^2, K_0, \varepsilon)=K_0\Pi^{(1)}(q^2/\nu^2, \varepsilon)+\frac{1}{N_F}\sum_{n=2}^{\infty}K_0^n\Pi^{(n)}(q^2/\nu^2, \varepsilon)\,,\\
&\Pi^{(n)}(q^2/\nu^2, \varepsilon)=\sum_{i=0}^{\infty}\frac{1}{\varepsilon^{n-i}}\Pi_i^{(n)}\,.
\end{aligned}
\tag{3.8}
$$

Substituting (3.3) and (3.8) in Eq. (3.7) we find

$$
\begin{aligned}
&B_0^{(1)}=-\Pi_0^{(1)}\,,\\
&\sum_{i=0}^{n-1}B_i^{(n)}\frac{1}{\varepsilon^{n-i}}=-\operatorname{div}\left\{\sum_{k=0}^{n-2}\binom{n-2}{k}[\Pi_0^{(1)}]^k\,\Pi^{(n-k)}\frac{1}{\varepsilon^k}\right\}, \qquad n\geqq 2\,,
\end{aligned}
\tag{3.9}
$$

which are the expressions needed in order to compute Z_K^{-1}. In order to calculate $B_0^{(1)}$ we need only the diagram given in Fig. 2, and we obtain the well known result,

$$\Pi^{(1)}=\left(-\frac{q^2}{4\pi\nu^2}\right)^{\varepsilon}2\frac{[\Gamma(2+\varepsilon)]^2\Gamma(-\varepsilon)}{\Gamma(4+2\varepsilon)}\,, \tag{3.10}$$

and hence

$$B_0^{(1)}=\tfrac{1}{3}\,. \tag{3.11}$$

Furthermore a straightforward calculation gives

$$
\begin{aligned}
\Pi^{(2)}=&-\left(-\frac{q^2}{4\pi\nu^2}\right)^{2\varepsilon}[\Gamma(2+\varepsilon)]^2\left\{\frac{3-2\varepsilon}{2}\frac{[\Gamma(1+\varepsilon)\Gamma(-\varepsilon)]^2}{\Gamma(2+2\varepsilon)\Gamma(4+2\varepsilon)}\right.\\
&+\frac{\Gamma(-\varepsilon)\Gamma(\varepsilon)\Gamma(-2\varepsilon)\Gamma(-3-2\varepsilon)}{\Gamma(1+3\varepsilon)\Gamma(-3\varepsilon)}+\frac{\Gamma(-2\varepsilon)\Gamma(-3-2\varepsilon)}{\varepsilon}\\
&\left.+2(1+2\varepsilon)(2+2\varepsilon+\varepsilon^2)\frac{\Gamma(\varepsilon)\Gamma(-3-2\varepsilon)}{\varepsilon\Gamma(3+3\varepsilon)}\right\}.
\end{aligned}
\tag{3.12}
$$

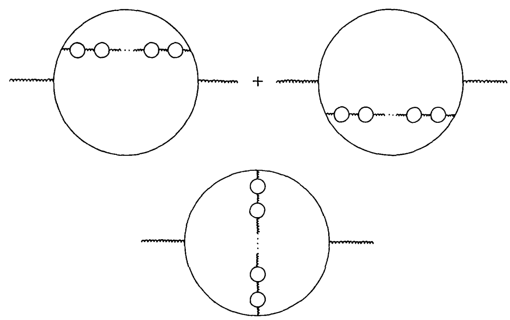

Fig. 3. Higher order photon self-energy diagrams

The calculation of $\Pi^{(n)}$, $n \geqq 3$, corresponding to the diagrams of Fig. 3, is very lengthy and Gegenbauer's integration techniques [7] are useful. The final result is

$$\begin{aligned}\Pi^{(n)} = & -\left(-\frac{q^2}{4\pi v^2}\right)^{n\varepsilon}\left\{-2\frac{\Gamma(-\varepsilon)[\Gamma(2+\varepsilon)]^2}{\Gamma(4+2\varepsilon)}\right\}^{n-2} \\ & \cdot\frac{1+\varepsilon}{3+2\varepsilon}\cdot\frac{[\Gamma(1+\varepsilon)]^2\Gamma(n\varepsilon-\varepsilon)}{\Gamma(1+2\varepsilon-n\varepsilon)\Gamma(3+n\varepsilon+\varepsilon)} \\ & \cdot\Big\{(6-2\varepsilon-2\varepsilon^3+8n\varepsilon-6n\varepsilon^2+n\varepsilon^3+4n^2\varepsilon^2-n^2\varepsilon^3)\Gamma(-2-n\varepsilon) \\ & +\frac{\Gamma(1+\varepsilon)}{\Gamma(2+2\varepsilon)}[(2+n\varepsilon+\varepsilon)(1+n\varepsilon+\varepsilon)F(0) \\ & -(2-\varepsilon)(n-1)(n-2)\varepsilon^2(2+\varepsilon+n\varepsilon)F(1) \\ & +(1+n\varepsilon-\varepsilon)(1-2\varepsilon+n\varepsilon)(n-1)(n-2)\varepsilon^2F(2)]\Big\},\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}F(s) \equiv & \frac{1}{\Gamma(-\varepsilon-n\varepsilon-s)}\sum_{m,k=0}^{\infty}\frac{\Gamma(m-s-\varepsilon-n\varepsilon)\Gamma(k+2+2\varepsilon)\Gamma(m+k+1-s-n\varepsilon)}{m!k!\Gamma(m+k+2+\varepsilon)} \\ & \cdot\left\{\frac{1}{(m+k+1-s+\varepsilon-n\varepsilon)(k+1-s+2\varepsilon-n\varepsilon)}-\frac{1}{(m+k+1)(k+1+s+n\varepsilon)}\right\}.\end{aligned} \tag{3.14}$$

A careful study of all these results shows that we can write

$$\begin{aligned}\Pi^{(n)} &= -\frac{1}{2^2\cdot 3^{n-2}}\frac{1}{n}\frac{1}{\varepsilon^{n-1}}F(q^2/v^2,n,\varepsilon), \\ F(q^2/v^2,n,\varepsilon) &= \sum_{j=0}^{\infty}F_j(\varepsilon)(n\varepsilon)^j\end{aligned} \tag{3.15}$$

where the coefficients $F_j(\varepsilon)$, except $F_0(\varepsilon)$, depend on q^2/v^2 and all of them are

n-independent series expansions of positive powers of ε. Using (3.15) in (3.9) we obtain

$$\begin{aligned}&\operatorname{div}\left\{\sum_{k=0}^{n-2}\binom{n-2}{k}[\Pi_0^{(1)}]^k\Pi^{(n-k)}\frac{1}{\varepsilon^k}\right\}\\&=-\frac{1}{2^2\cdot 3^{n-2}}\operatorname{div}\left\{\sum_{j=0}^{\infty}\frac{1}{\varepsilon^{n-j-1}}F_j(\varepsilon)\sum_{k=0}^{n-2}\binom{n-2}{k}(n-k)^{j-1}(-1)^k\right\},\end{aligned}\quad (3.16)$$

where we can restrict the j-summation to $0\leqq j\leqq n-2$. We have now a result similar to (2.11)

$$\sum_{k=0}^{n-2}\binom{n-2}{k}(n-k)^{j-1}(-1)^k=0\,,\qquad 1\leqq j\leqq n-2\,,\quad (3.17)$$

and, using (3.9), we get

$$\sum_{i=0}^{n-1}B_i^{(n)}\frac{1}{\varepsilon^{n-i}}=(-1)^n\frac{1}{2^2\cdot 3^{n-2}}\frac{1}{(n-1)n}\operatorname{div}\left\{\frac{1}{\varepsilon^{n-1}}F_0(\varepsilon)\right\}.\quad (3.18)$$

Since $F_0(\varepsilon)$ is independent of q^2/v^2, this is also true for Z_K^{-1} as it must be.

As before, and after a long calculation, we can see that

$$\begin{aligned}&F_0(\varepsilon)=F(q^2/v^2,0,\varepsilon)=A(\varepsilon)B(\varepsilon)\,,\\&A(\varepsilon)=C(\varepsilon)\,,\qquad B(\varepsilon)=(1-\varepsilon)(1+2\varepsilon)D(\varepsilon)\,,\end{aligned}\quad (3.19)$$

where $C(\varepsilon)$ and $D(\varepsilon)$ are defined in (2.13). As in the study of the γ-function we can write

$$F_0(\varepsilon)=\sum_{i=0}^{\infty}F_i\varepsilon^i\,,\qquad F_0=1\,,\quad (3.20)$$

and clearly

$$F_i=G_i+G_{i-1}-2G_{i-2}\,,\qquad G_{-1}\equiv G_{-2}\equiv 0\,.\quad (3.21)$$

Using (3.20) in (3.18) we obtain immediately for $n\geqq 2$

$$\begin{aligned}&B_0^{(n)}=0\,,\\&B_1^{(n)}=\frac{1}{2^2\cdot 3^{n-2}}\cdot\frac{(-1)^n}{(n-1)n}\\&B_i^{(n)}=\frac{1}{2^2\cdot 3^{n-2}}\cdot\frac{(-1)^n}{(n-1)n}F_{i-1}\,,\qquad 2\leqq i\leqq n-1\,,\end{aligned}\quad (3.22)$$

which satisfy the relations (3.5) as it must be. Finally, from (3.4)

$$\beta(K)=\frac{2}{3}K+\frac{1}{N_F}\frac{1}{2}K^2\sum_{n=0}^{\infty}\left(-\frac{K}{3}\right)^n\frac{F_n}{n+1}+O(1/N_F^2)\,,\quad (3.23)$$

which is the desired expression and allows us to compute the coefficients of the β-function expansion as far as we like, since no important cancellations appear.

From this expansion and the corresponding one for $\gamma(K)$ we can easily see that, up to terms of order $1/N_F$, we have

$$K\frac{d\beta(K)}{dK}-\beta(K)=\frac{K}{3}\left(1+\frac{K}{3}\right)\left(1-\frac{2K}{3}\right)\gamma(K). \tag{3.24}$$

An alternative way of expressing the result (3.23) is

$$\begin{aligned}\beta(K)&=\frac{2}{3}K+\frac{1}{N_F}\frac{3K}{2}\int\limits_{-K/3}^{0}dxF_0(x)+O(1/N_F^2)\\&=\frac{2}{3}K+\frac{1}{N_F}\frac{1}{2}K\int\limits_{-K/3}^{0}dx\frac{\Gamma(4+2x)(1+2x)(1+2x/3)(1-x)}{[\Gamma(2+x)]^2\Gamma(3+x)\Gamma(1-x)}+O(1/N_F^2).\end{aligned} \tag{3.25}$$

A study of $F_0(\varepsilon)$ proves that the radius of convergence of the $\beta(K)$ expansion is $K=15/2$, which is the same as the one of the $\gamma(K)$ expansion in Q.E.D. Furthermore, from (3.24) it is clear that $Kd\beta(K)/dK-\beta(K)$ can still be factorized as the product of two series with characteristics analogous to the ones appearing in the case of the γ-function, but this is not true for $\beta(K)$.

4. Numerical Results and Conclusions

From the expression equivalent to (2.16) for Q.E.D. and from (3.23) we can easily obtain the first terms of both expansions

$$\begin{aligned}\gamma(K)=\frac{3K}{2N_F}\Bigg[&1-\frac{5}{2\cdot 3}\left(\frac{K}{3}\right)-\frac{5\cdot 7}{2^2\cdot 3^2}\left(\frac{K}{3}\right)^2-\left(\frac{83}{2^3\cdot 3^2}-2\zeta(3)\right)\left(\frac{K}{3}\right)^3\\&-\left(\frac{5\cdot 13}{2^4\cdot 3}+\frac{5}{3}\zeta(3)-3\zeta(4)\right)\left(\frac{K}{3}\right)^4-\left(\frac{11\cdot 41}{2^5\cdot 3^2}+\frac{5\cdot 7}{2\cdot 3^2}\zeta(3)\right.\\&\left.+\frac{5}{2}\zeta(4)-2\cdot 3\zeta(5)\right)\left(\frac{K}{3}\right)^5-\left(\frac{13\cdot 79}{2^6\cdot 3^2}+\frac{83}{2^2\cdot 3^2}\zeta(3)\right.\\&\left.+\frac{5\cdot 7}{2^2\cdot 3}\zeta(4)+5\zeta(5)-2\cdot 5\zeta(6)-2\zeta^2(3)\right)\left(\frac{K}{3}\right)^6+\ldots\Bigg],\end{aligned} \tag{4.1}$$

$$\begin{aligned}\beta(K)=\frac{2}{3}K+\frac{K^2}{2N_F}\Bigg[&1-\frac{11}{2\cdot 3}\frac{1}{2}\left(\frac{K}{3}\right)-\frac{7\cdot 11}{2^2\cdot 3^2}\frac{1}{3}\left(\frac{K}{3}\right)^2\\&+\left(\frac{107}{2^3\cdot 3^2}+2\zeta(3)\right)\frac{1}{4}\left(\frac{K}{3}\right)^3+\left(\frac{251}{2^4\cdot 3^2}-\frac{11}{3}\zeta(3)+3\zeta(4)\right)\frac{1}{5}\left(\frac{K}{3}\right)^4\\&+\left(\frac{67}{2^5}-\frac{7\cdot 11}{2\cdot 3^2}\zeta(3)-\frac{11}{2}\zeta(4)+2\cdot 3\zeta(5)\right)\frac{1}{6}\left(\frac{K}{3}\right)^5\\&+\left(\frac{5\cdot 7\cdot 41}{2^6\cdot 3^2}+\frac{107}{2^2\cdot 3^2}\zeta(3)-\frac{7\cdot 11}{2^2\cdot 3}\zeta(4)\right.\\&\left.-11\zeta(5)+2\cdot 5\zeta(6)-2\zeta^2(3)\right)\frac{1}{7}\left(\frac{K}{3}\right)^6+\ldots\Bigg].\end{aligned} \tag{4.2}$$

Table 1. Values of the coefficients γ_n and β_n. $(-n) \equiv x10^{-n}$

n	γ_n	β_n
0	+1	+1
1	−2.777777778(−1)	−3.055555556(−1)
2	−1.080246914(−1)	−7.921810700(−2)
3	4.634577884(−2)	3.602060109(−2)
4	−1.365742438(−3)	1.438230317(−3)
5	−1.594842152(−3)	−1.906442773(−3)
6	2.297699045(−4)	1.521260392(−4)
7	9.292851013(−6)	3.588903124(−5)
8	−4.706120144(−6)	−6.540412881(−6)
9	2.834550611(−7)	−2.129162272(−8)
10	2.943839959(−8)	8.915980505(−8)
11	−4.770448170(−9)	−6.464438468(−9)
12	9.185385637(−11)	−3.738356407(−10)
13	2.646883749(−11)	7.542503419(−11)
14	−2.104472229(−12)	−2.089292409(−12)
15	−1.965280100(−14)	−3.250078716(−13)
16	1.065741625(−14)	2.852169678(−14)
17	−4.476261677(−16)	2.039949218(−17)
18	−1.767283367(−17)	−1.177252018(−16)
19	2.288589571(−18)	5.382600792(−18)

The expression for $\gamma(K)$ is the analogue for Q.E.D. of the result given in [5] for Q.C.D., but obtained much more easily. The expression for $\beta(K)$ is a new result. The terms up to order K^3 have been checked since they can be derived from the results given in [8].

In Table 1 we present the numerical values of the coefficients γ_n and β_n appearing in the expansions

$$\begin{aligned} \gamma(K) &= \frac{3K}{2N_F} \sum_{n=0}^{\infty} \gamma_n K^n , \\ \beta(K) &= \frac{2}{3}K + \frac{K^2}{2N_F} \sum_{n=0}^{\infty} \beta_n K^n . \end{aligned} \tag{4.3}$$

We have checked numerically that the $1/N_F$ term has only the zero at $K=0$ and is positive in the convergence region. Furthermore for $N_F=3$ the $1/N_F$ correction to $\beta(K)$ is never larger than 15% of the leading term $2K/3$. The function $\beta(K)$ is an always increasing function of K for $K \leqq 15/2$.

Acknowledgements. We would like to thank Dr. R. Tarrach for a critical reading of the manuscript. This work has been supported by the Comision Asesora de Investigación Científica y Técnica (contract. n. 0435). One of us (A.P.-M.) acknowledges the financial support of the Instituto Nacional de Asistencia y Promoción del Estudiante.

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Communicated by G. Mack

Received March 30, 1983