Commun. Math. Phys. 94, 537–544 (1984)

Communications in
Mathematical
Physics
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Charges in Spacelike Cones

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Abstract. Starting from a conserved current, operators are defined which measure the charge in certain unbounded stringlike regions which are possible localization regions of charged fields in gauge theories.

1. Introduction

A fundamental feature of the structure of particle states is their division into superselection sectors. In the conventional framework of quantum field theory, these are described by unobservable fields connecting the different sectors. The occurrence of superselection sectors can be understood within the framework of algebraic quantum field theory [1]. Starting from the general structure of the algebra of observables, exploiting essentially the locality principle, one obtains the superselection sectors as equivalence classes of representations describing "situations of interest for particle physics" [2–4]. In [2] this is the class of representations satisfying the spectrum condition, in [3] the class of representations which are equivalent to the vacuum representation on the spacelike complement of bounded regions. The latter class does not contain representations describing charged states in gauge theories. Therefore, in [4] the probably larger class of representations which are equivalent to the vacuum on the spacelike complement of spacelike cones is investigated. This class is known to contain, in massive theories, all factorial representations with single particle states and is supposed to contain all sectors in massive gauge theories which correspond to charges measurable in the outside region of any bounded region by some sort of electric flux (gauge charges).

The quantum numbers labeling the set of superselection sectors may be considered as eigenvalues of charge operators Q. These operators are global observables which do not belong to the quasilocal algebra. However, in Lagrangian field theory. one has certain local observables which are supposed to measure approximately the charges of the states. Recently Doplicher [5] and

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Doplicher and Longo [6] derived the existence of local charge operators from general structural properties of the net of algebras of local fields in the case of locally generated superselection sectors. These abstractly given local charge operators behave essentially as the suitably regularized integrals of the 0^{th} component of a conserved local current. The main input needed for the construction is the existence of interpolating type I factors for the local algebras, a property which seems to be related to the particle content of the theory [7–9]. If interpolating type I factors also exist for the von Neumann algebras associated to spacelike cones, one can find operators measuring the charge in these cones, and one may compare these operators to the integrals of local charge densities over a spatial basis of the spacelike cone.

In this paper we investigate the following technical problem. Let j^μ denote a conserved current, and let **S** denote a spatial cone. Is it possible to give a precise meaning to the integral

$$\int_{\mathbf{S}} j^0(0,\mathbf{x})d^3\mathbf{x}? \tag{1.1}$$

That this problem is not trivial may be seen by looking at the problem of defining the global charge as a limit of local charges. The local charge may be defined by

$$Q_{\mathbf{B},f}=\int_{\boldsymbol{B}} d^3\mathbf{x}j^0(f_{\mathbf{x}}), \tag{1.2}$$

with $\mathbf{B}\subset\mathbb{R}^3$ open and bounded, $f\in D(\mathbb{R}^4)$, $\int f=1$ and $f_{\mathbf{x}}(y)=f(y-(0,\mathbf{x}))$. $Q_{\boldsymbol{B},f}$ is localised in $\{0\}\times\mathbf{B}+\operatorname{supp}f$ and has the same commutation properties as the global charge operator with fields which are localised in the spacelike complement of $\{0\}\times(\mathbb{R}^3\backslash\mathbf{B})+\operatorname{supp}f$. $Q_{\mathbf{B},f}$ is a closable densely defined operator, and one may assume that it has local selfadjoint extensions with the same commutation properties. It is well known (see e.g. [10]) that the operators $Q_{\mathbf{B},f}$ approach the total charge operator Q in a very weak sense as **B** tends to $\mathbb{R}^3$, e.g.

$$[Q_{\mathbf{B},f},F]\Omega\to QF\Omega, \tag{1.3}$$

where Ω denotes the vacuum vector and F any local field operator. Requardt [11] has shown that the convergence can be improved if one chooses an adapted smearing function $f_{\mathbf{B}}$ for each **B**, for instance

$$f_{\lambda\mathbf{B}}(x)=\lambda^{-4}f_{\mathbf{B}}(\lambda^{-1}x). \tag{1.4}$$

Then, in a massive theory

$$\|(Q_{\lambda\mathbf{B},f_{\lambda\mathbf{B}}}-Q)\Omega\|\to 0, \qquad \lambda\to\infty, \tag{1.5}$$

which implies strong convergence of $(Q_{\lambda\mathbf{B},f_{\lambda\mathbf{B}}})$ on a core for Q.

For unbounded regions which are not the whole space one has to find a smearing procedure which preserves locality properties. In the case of a spacelike cone S one may proceed as follows.

Let S_1 be a spacelike cone such that $LS\subset S_1$ for all Poincaré transformations L in some neighbourhood of the identity. Candidates for charge operators Q_S which measure the charge in S and are localised in S_1 are the means of the formal charge operators

$$Q_\Sigma^{(\text{formal})}=\int_\Sigma \varepsilon^{\mu\nu\varrho\tau}j_\mu d\sigma_{\nu\varrho\tau}, \tag{1.6}$$

where Σ runs over spacelike surfaces in S_1 whose causal shadow Σ'' contains S.

We shall show that Q_S exists on Ω under a condition on the 2-point function of j_μ which is satisfied in theories with a mass gap as well as for the electric current in quantum electrodynamics, but not in the case of the massless free scalar field. The existence of Q_S on Ω implies the existence of Q_S as a hermitian operator on a dense domain under some natural conditions. Moreover, Q_S is essentially self-adjoint in the case of a free massive scalar field.

The last result can be used to prove the existence of interpolating type I factors for spacelike cones in the case of the neutral free field by the method of [12]. (A direct proof generalizing the argument in [13] is also possible [14].) On the other hand, the non-existence of interpolating type I factors for algebras of spacelike cones in a dilation invariant theory implies the non-existence of a reasonable charge operator Q_S in the theory of a free massless scalar field.

This negative result suggests that our conditions for the existence of charge operators associated to spacelike cones are in a certain sense optimal. On the other hand, the positive result for quantum electrodynamics supports the heuristic picture that charge carrying fields may exist which are localised in spacelike cones, in analogy to the case of massive gauge theories[1].

2. Construction of Charge Operators

Let j_μ denote a conserved Poincaré covariant local current in a Wightman field theory with a unique vacuum Ω. The 2-point function of j_μ is given by

$$(\Omega, j_\mu(x) j_\nu(y)\Omega) = \int d\varrho(m^2) \int \frac{d^3p}{p_0} (p_\mu p_\nu - m^2 g_{\mu\nu}) e^{ip(x-y)} (p_0 = (|\mathbf{p}|^2 + m^2)^{1/2}) \tag{2.1}$$

with a positive tempered measure ϱ. We assume that ϱ is of the form

$$d\varrho(m^2) = m^2 d\varrho'(m^2) \tag{2.2}$$

with a measure ϱ'. This assumption is clearly fulfilled in theories with a mass gap, but also the electric current in quantum electrodynamics satisfies this condition. The property (2.2) of ϱ guarantees that the local charge operators regularized by the method of [11] converge strongly on a dense set to the global charge [11, 15].

We try to find an analogous method for the case of spacelike cones. Let $S \subset S_1$ denote spacelike cones centered around the x_1-axis, with opening angles 2ε and 2δ and apices 0 and $(0, -a, 0, 0)$, respectively, $0 < \varepsilon < \delta < \frac{\pi}{2}$, $a > 0$, i.e. $(x_\perp = (x_2, x_3))$,

$$\begin{aligned} S &= \{x \in \mathbb{R}^4, |x_0| + |x_\perp| \cos\varepsilon < x_1 \sin\varepsilon\}, \\ S_1 &= \{x \in \mathbb{R}^4, |x_0| + |x_\perp| \cos\delta < (x_1 + a) \sin\varepsilon\}. \end{aligned} \tag{2.3}$$

Consider the family of spacelike surfaces $(0 < b < a,\ \varepsilon < \gamma < \delta)$,

$$\Sigma_{y,\lambda} = \{x \in \mathbb{R}^4, x_0 - y_0 = \lambda(x_1 - y_1 + b), |x_\perp - y_\perp| < (x_1 - y_1 + b) \sin\gamma\} \tag{2.4}$$

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1 In the framework of perturbation theory, Steinmann [19] has shown that such fields exist and can be used for a formulation of quantum electrodynamics in terms of gauge invariant quantities

with $y \in \mathbb{R}^4$ and $\lambda \in \mathbb{R}$. It is straightforward to check that for y and λ sufficiently small $\Sigma_{y,\lambda} \subset S_1$ and $\Sigma''_{y,\lambda} \supset S$. Hence for these values of y and λ,

$$\int_{\Sigma_{y,\lambda}} \varepsilon^{\mu\nu\varrho\sigma}[j_\sigma(x), A] dx_\mu dx_\nu dx_\varrho = \begin{cases} \int_{x_0=0} d^3x [j_0(x), A], & \text{if } A \text{ is localised in } S \\ 0, & \text{if } A \text{ is localised in } S_1' \end{cases} \tag{2.5}$$

Here A denotes a polynomial of Wightman fields smeared with test functions, and A is said to be localised in a region $\mathscr{R}$ if the support of all these test functions is contained in $\mathscr{R}$.

Equation (2.5) suggests the definition of a charge operator Q_S which is localised in S_1 by the formula

$$Q_S = \int d^4y \int d\lambda h(y, \lambda) \int_{\Sigma_{y,\lambda}} \varepsilon^{\mu\nu\varrho\sigma} dx_\mu dx_\nu dx_\varrho j_\sigma(x), \tag{2.6}$$

where $h \in \mathscr{D}(\mathbb{R}^4 \times \mathbb{R})$, $\int d^4y \int d\lambda h(y, \lambda) = 1$, and

$$\operatorname{supp} h \subset \{(y, \lambda) \in \mathbb{R}^4 \times \mathbb{R} | \Sigma_{y,\lambda} \subset S_1, \Sigma''_{y,\lambda} \supset S\}.$$

The right-hand side of Eq. (2.6) is not well defined since it corresponds to the smearing of the Wightman field j_μ with the function h_S^μ,

$$h_S^\mu(y) = \int d\lambda \int_{\Sigma_{0,\lambda}} \varepsilon^{\mu\nu\varrho\sigma} dx_\nu dx_\varrho dx_\sigma h(x-y, \lambda), \tag{2.7}$$

which is not strongly decreasing. Therefore we consider the cutoff functions $h_{S,R}^\mu \in \mathscr{D}(\mathbb{R}^4)$, $R > 0$,

$$h_{S,R}^\mu(y) = \int d\lambda \int_{\substack{\Sigma_{0,\lambda} \\ x_1 < R}} \varepsilon^{\mu\nu\varrho\sigma} dx_\nu dx_\varrho dx_\sigma h(x-y, \lambda). \tag{2.8}$$

For each local operator A one can choose R large enough such that the commutator $[Q_{S,R}, A]$ does not depend on R, $Q_{S,R} = j(h_{S,R})$.

Proposition 2.1. *The sequence $(Q_{S,R}\Omega)$ converges strongly as R tends to infinity, provided the spectral measure ϱ fulfills condition* (2.2).

Proof. Let P_ε denote the projection on spectral values of the squared mass operator M^2 in the interval $[0, \varepsilon]$. We show

(i) $\|P_\varepsilon Q_{S,R}\Omega\| \to 0$ as $\varepsilon \to 0$, uniformly in R.

(ii) $((1-P_\varepsilon)Q_{S,R}\Omega)$ is a Cauchy sequence for each $\varepsilon > 0$.

The Fourier transforms of the functions $h_{S,R}^\mu$, are

$$\tilde{h}_{S,R}^0(p) = \int_{-b}^{R} dx_1 \int_{|x_\perp| < (x_1+b)\tan\gamma} d^2x_\perp e^{ip_1x_1 + ip_\perp x_\perp} \tilde{h}(p, p_0 x_1),$$

$$\tilde{h}_{S,R}^1(p) = \int_{-b}^{R} dx_1 \int_{|x_\perp| < (x_1+b)\tan\gamma} d^2x e^{ip_1x_1 + ip_\perp x_\perp} i\partial_5 \tilde{h}(p, p_0 x_1),$$

$$\tilde{h}_{S,R}^i \equiv 0, \quad i = 2, 3 \left(x_\perp = (x_2, x_3), \partial_5 \tilde{h}(p, \beta) = \frac{\partial}{\partial \beta} \tilde{h}(p, \beta) \right).$$

For $p_0 \geqq \varepsilon > 0$ the functions $\tilde{h}_{S,R}^\mu$ converge in each Schwartz space norm for $R \to \infty$. So the temperedness of ϱ implies (ii). To prove (i) we use the estimates

$$|\tilde{h}^0_{S,R}(p)| \leqq p_0^{-3} \int_{-bp_0}^{Rp_0} dx_1 \int_{|x_\perp| < (x_1+b)\tan\gamma} d^2x_\perp |\tilde{h}(p, x_1)|$$
$$\leqq p_0^{-3} \sup_{q \in \mathbb{R}^4} \int dx_1 \pi |x_1 + bq_0|^2 |\tilde{h}(q, x_1)|,$$
$$|\tilde{h}^1_{S,R}(p)| \leqq p_0^{-3} \sup_{q \in \mathbb{R}^4} \int dx_1 \pi |x_1 + bq_0|^2 |\partial_5 \tilde{h}(q, x)|.$$

Hence

$$\|P_\varepsilon Q_{S,R}\Omega\|^2 = \int_0^\varepsilon d\varrho(m^2) \int \frac{d^3p}{p_0}(p_\mu p_\nu - m^2 g_{\mu\nu})|\tilde{h}^\mu_{S,R}(p)\tilde{h}^\nu_{S,R}(p)|,$$
$$(p_0 = (|\mathbf{p}|^2 + m^2)^{1/2})$$
$$\leqq \int_0^\varepsilon d\varrho(m^2) \int_m^\infty dp_0 p_0^{-3} \,\mathrm{const} \leqq 2\,\mathrm{const} \int_0^\varepsilon \frac{d\varrho(m^2)}{m^2},$$

where the constant does neither depend on ε nor on R, so (ii) follows from (2.2). q.e.d.

The convergence of the sequence $(Q_{S,R})$ on the vacuum implies the convergence on a dense set if there are sufficiently many bounded operators which map the invariant domain of the field operators in itself and commute with sufficiently far spacelike separated fields. Without an assumption on the existence of local bounded operators, in the general framework of Wightman field theory, we do not know whether $(Q_{S,R})$ converges always on a dense set. But in a theory with a mass gap one finds the following result.

Proposition 2.2. *Let A denote a local polynomial in Wightman fields. Then the sequence $(Q_{S,R}A\Omega)$ converges strongly.*

Proof. Let $R' > R$, R sufficiently large. Then by locality and the cluster theorem [9],

$$\|(Q_{S,R'} - Q_{S,R})A\Omega\|^2 = (\Omega, A^*A(Q_{S,R'} - Q_{S,R})^2\Omega)$$
$$\leqq (\Omega, A^*A\Omega)(\Omega, (Q_{S,R'} - Q_{S,R})^2\Omega) + f(R)\|A^*A\Omega\| \, \|(Q_{S,R'} - Q_{S,R})^2\Omega\|, \quad (*)$$

where f is a function which depends on the mass gap and on the localisation region of A and decreases faster than any power of R as R tends to infinity. Now the squares of the norms $\|(Q_{S,R'} - Q_{S,R})^2\Omega\|$ arise from the 4-point function of the current j_μ by smearing with the test functions

$$H^{\mu_1 \dots \mu_4}_{R',R}(x_1, \dots, x_4) = (h^{\mu_1}_{S,R'} - h^{\mu_1}_{S,R})(x_1) \dots (h^{\mu_4}_{S,R'} - h^{\mu_4}_{S,R})(x_4),$$

whose Schwartz space norms are bounded by polynomials in R. Since the 4-point function is a tempered distribution, $\|(Q_{R'} - Q_R)^2\Omega\|$ is also bounded by a polynomial in R. Hence the second term in the second line of relation $(*)$ vanishes in the limit $R \to \infty$, whereas the first term vanishes because of Proposition 2.1. q.e.d.

For a physical interpretation, however, Q_S should be not only densely defined but self-adjoint. We study this problem in the case of the free charged scalar massive field.

Let $\Phi(x) = 2^{-1/2}(\Phi_1(x) + i\Phi_2(x))$, $x \in \mathbb{R}^4$, denote the free charged field, where Φ_j, $j = 1, 2$, denote independent neutral fields. A conserved current is defined by

$$j_\mu(x) = \Phi_1(x)\partial_\mu\Phi_2(x) - \Phi_2(x)\partial_\mu\Phi_1(x). \quad (2.9)$$

In terms of annihilation and creation operators $a_j(\mathbf{p})$, $a_j^*(\mathbf{p})$, $j=1,2$, normalized such that

$$[a_j(\mathbf{p}), a_j^*(\mathbf{q})]=\delta^3(\mathbf{p}-\mathbf{q})\,; \tag{2.10}$$

Q_S can be written in the form

$$Q_S=\int d^3\mathbf{p}\int d^3\mathbf{q}\{a_1(\mathbf{p})a_2(\mathbf{q})W(\mathbf{p},\mathbf{q})+a_1(\mathbf{p})a_2^*(\mathbf{q})V(\mathbf{p},\mathbf{q})+\text{h.c.}\}\,, \tag{2.11}$$

where W and V denote the distributions

$$W(\mathbf{p},\mathbf{q})=(2\pi)^{-3}(4\mu(\mathbf{p})\mu(\mathbf{q}))^{-1/2}\int_{-b}^{\infty}dx_1\int_{|x_\perp|<(x_1+b)\tan\gamma}d^2x_\perp e^{i(\mathbf{p}+\mathbf{q})\mathbf{x}}$$
$$\cdot[\mu(\mathbf{p})-\mu(\mathbf{q})-i(p_1-q_1)\partial_5]\tilde{h}(\mu(\mathbf{p})+\mu(\mathbf{q}),\mathbf{p}+\mathbf{q},(\mu(\mathbf{p})+\mu(\mathbf{q}))(x_1+b))\,,$$
$$V(\mathbf{p},\mathbf{q})=(2\pi)^{-3}(4\mu(\mathbf{p})\mu(\mathbf{q}))^{-1/2}\int d\lambda\int_{-b}^{\infty}dx_1\int_{|x_\perp|<(x_1+b)\tan\gamma}d^2x_\perp e^{i(\mathbf{p}-\mathbf{q})\mathbf{x}}$$
$$\cdot[\mu(\mathbf{p})+\mu(\mathbf{q})-\lambda(p_1+q_1)]\hat{h}(\mu(\mathbf{p})-\mu(\mathbf{q}),\mathbf{p}-\mathbf{q},\lambda)e^{-i(\mu(\mathbf{p})-\mu(\mathbf{q}))\lambda(x_1+b)}\,,$$
$$\Bigg(\mu(\mathbf{p})=(|\mathbf{p}|^2+m^2)^{1/2},\ \hat{h}(p,\lambda)=\int d^4y e^{-ipy}h(y,\lambda),\ \tilde{h}(p,\beta)$$
$$=\int d\lambda e^{-i\beta\lambda}\hat{h}(p,\lambda),\ \partial_5\tilde{h}(p,\beta)=\frac{\partial}{\partial\beta}\tilde{h}(p,\beta)\Bigg)\,. \tag{2.12}$$

Equation (2.11) has to be understood as a relation between sesquilinear forms on the domain $\mathscr{D}$ of vectors with finite particle number and wave functions from the Schwartz class.

Proposition 2.3. *There exists a unique essentially self-adjoint operator Q_S on $D(N)$ (the domain of the particle number operator) which fulfills Eq.* (2.11) *in the sense of sesquilinear forms on $\mathscr{D}$.*

Proof. It is sufficient to show that $W\in\mathscr{L}^2(\mathbb{R}^3\times\mathbb{R}^3)$ and that V is the kernel of a bounded operator in $\mathscr{L}^2(\mathbb{R}^3)$ (see e.g. [16]). Since $\|W\|_2=\|Q_S\Omega\|<\infty$, according to Proposition 2.1, we only need to investigate V. This investigation turns out to be rather delicate since there is no damping factor in the integration over x_1 in (2.12). The idea of the proof is to relate the kernel V by suitable operations to the kernels

$$A_{c,\lambda}(\mathbf{p},\mathbf{q})=(2\pi)^{-3}\int_{-c}^{\infty}dx_1\int_{|x_\perp|<\eta(x_1+c)}d^2xe^{i(\mathbf{p}-\mathbf{q})}\equiv\tilde{\chi}_{c,\eta}(\mathbf{p}-\mathbf{q})\,,$$

$\chi_{c,\eta}$ denoting the characteristic function of a certain spatial cone $\mathbf{S}_{c,\eta}$, which clearly are kernels of operators with norm 1.

The first step consists in smearing $A_{c,\eta}$ with a space-time function $\varphi\in S(\mathbb{R}^4)$. Let $\mathbf{P}=(P_1,P_2,P_3)$ denote the momentum operator in $\mathscr{L}^2(\mathbb{R}^3)$, $(P_if)(\mathbf{p})=P_if(\mathbf{p})$, $i=1,2,3$, and let $\bar{P}=(\mu(\mathbf{P}),\mathbf{P})$. The operator

$$B_{c,\eta}^{(\varphi)}=\int d^4x\varphi(x)e^{-i\bar{P}x}A_{c,\eta}e^{i\bar{P}x}$$

(the integral is understood in the weak sense) has a norm which is bounded by $\int d^4x|\varphi(x)|$ and the kernel

$$B_{c,\eta}^{(\varphi)}(\mathbf{p},\mathbf{q})=A_{c,\eta}(\mathbf{p},\mathbf{q})\tilde{\varphi}(\mu(\mathbf{p})-\mu(\mathbf{q}),\mathbf{p}-\mathbf{q})\,.$$

Using $i[\mu(\mathbf{P}), B^{(\varphi)}_{c,\eta}] = B^{(\partial_0\varphi)}_{c,\eta}$ and $\|\mu(\mathbf{P})^{-1}\| = m^{-1}$, one finds

$$\|\mu(\mathbf{P})B^{(\varphi)}_{c,\eta}\mu(P)^{-1}\|\,, \qquad \|\mu(\mathbf{P})^{-1}B^{(\varphi)}_{c,\eta}\mu(\mathbf{P})\| \leqq \int d^4x\{|\varphi(x)| + m^{-1}|\partial_0\varphi(x)|\}\,.$$

Furthermore, by complex interpolation [17] the same estimate holds for

$$C^{(\varphi)}_{c,\eta} = \tfrac{1}{2}\{\mu(\mathbf{P})^{1/2}B^{(\varphi)}_{c,\eta}\mu(\mathbf{P})^{-1/2} + \mu(\mathbf{P})^{-1/2}B^{(\varphi)}_{c,\eta}\mu(\mathbf{P})^{1/2}\}\,.$$

The final step now consists in a suitable smearing over Lorentz boosts in the x_1-direction. Let $\Lambda \to U(\Lambda)$ denote the unitary representation of $L^{\uparrow}_{+}$ in $\mathscr{L}^2(\mathbb{R}^3)$ given by

$$(U(\Lambda)f)(\mathbf{p}) = (\mu(\mathbf{\Lambda}^{-1}\mathbf{p})/\mu(\mathbf{p}))^{1/2}f(\mathbf{\Lambda}^{-1}\mathbf{p})\,.$$

($\mathbf{\Lambda}^{-1}\mathbf{p}$ is the spatial part of $\Lambda^{-1}(\mu(\mathbf{p}), \mathbf{p}$.) Kernels of operators transform according to

$$[U(\Lambda)C^{(\varphi)}_{c,\eta}U(\Lambda)^{-1}](\mathbf{p},\mathbf{q}) = (\mu(\mathbf{\Lambda}^{-1}\mathbf{p})/\mu(\mathbf{p}))^{1/2}(\mu(\mathbf{\Lambda}^{-1}\mathbf{q})/\mu(\mathbf{q}))^{1/2} \cdot C^{(\varphi)}_{c,\eta}(\mathbf{\Lambda}^{-1}\mathbf{p}, \mathbf{\Lambda}^{-1}\mathbf{q})\,.$$

Let the boosts in the x_1-direction be parametrized by

$$\Lambda(\lambda) = \begin{pmatrix} (1-\lambda^2)^{-1/2} & \lambda(1-\lambda^2)^{-1/2} & & 0 \\ \lambda(1-\lambda^2)^{-1/2} & (1-\lambda^2)^{-1/2} & & \\ & 0 & 1 & \\ & & & 1 \end{pmatrix}, \qquad -1 < \lambda < 1\,.$$

Then

$$\begin{aligned}[U(\Lambda(\lambda))C^{(\varphi)}_{c,\eta}U(\Lambda(\lambda))^{-1}](\mathbf{p},q) &= \varphi(\Lambda(\lambda)^{-1}(\mu(\mathbf{p}) - \mu(\mathbf{q}), \mathbf{p}-\mathbf{q})) \\ &\cdot (2\pi)^{-3}(4\mu(\mathbf{p})\mu(\mathbf{q}))^{-1/2}(\mu(\mathbf{p}) + \mu(\mathbf{q}) - \lambda(p_1+q_1)) \\ &\cdot \int_{-c(1-\lambda^2)^{-1/2}}^{\infty} dx_1 \int_{|x_\perp| < \eta(1-\lambda^2)^{1/2}(x_1 + c(1-\lambda^2)^{-1/2})} d^2x_\perp e^{-i(\mu(\mathbf{p})-\mu(\mathbf{q}))\lambda x_1} e^{-i(\mathbf{p}-\mathbf{q})\mathbf{x}}\,.\end{aligned}$$

Comparison with V gives

$$V(\mathbf{p},\mathbf{q}) = \int d\lambda [U(\Lambda(\lambda))C^{(\varphi_\lambda)}_{c_\lambda,\eta_\lambda}U(\Lambda(\lambda))^{-1}](\mathbf{p},\mathbf{q})\,,$$

with $\varphi_\lambda(x) = h(\Lambda(\lambda)x, \lambda)$, $c_\lambda = b(1-\lambda^2)^{1/2}$, $\eta_\lambda = (1-\lambda^2)^{-1/2}\tan\gamma$. Thus

$$\|V\| \leqq \int d\lambda \int d^4x\{|h(x,\lambda)| + m^{-1}(1-\lambda^2)^{-1/2}|(\partial_0 + \lambda\partial_1)h(x,\lambda)|\} < \infty\,. \quad \text{q.e.d.}$$

The explicit dependence of some of the estimates used in the proof of Proposition 2.3 on the mass m raises doubts whether a similar result may hold in the massless case. Actually, one can show that it is impossible, in the massless theory, to find a self-adjoint operator Q_S affiliated to $\mathfrak{A}(S_1)$[2] such that

$$e^{i\alpha Q_S}\Phi(f)e^{-i\alpha Q_S} = e^{i\alpha}\Phi(f) \tag{2.13}$$

for $\alpha \in \mathbb{R}$ and $\operatorname{supp} f \subset S$. The reasoning is as follows.

Suppose that an operator Q_S with the above properties exists. Then, by the method of [12], one can prove the existence of a type I factor $\mathscr{N}$ in the theory of the free neutral field with $\mathfrak{A}_0(S) \subset \mathscr{N} \subset \mathfrak{A}_0(S_1)$[2]. But from an argument of Driessler

2 $\mathfrak{A}(\mathscr{R})$ and $\mathfrak{A}_0(\mathscr{R})$, for a region $\mathscr{R} \subset \mathbb{R}^4$, denote the von Neuman algebras generated by all bounded functions of the field Φ and the neutral field Φ_0, respectively, smeared with test functions with support in $\mathscr{R}$

[18] one knows that such a type I factor $\mathcal{N}$ cannot exist in a dilation invariant theory.

By the same reasoning, one can also exclude the possibility, in the massive free field, that Q_S can be defined such that it is affiliated with $\mathfrak{A}(S_1)$, where S_1 has the same opening angle as S, i.e. $S_1 \subset S + x$ for some x.

Acknowledgements. We thank Sergio Doplicher for helpful discussions and the Universities of Freiburg, L'Aquila and Roma as well as the C.N.R.S., Luminy for the hospitality extended to us at various stages of this work.

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Communicated by R. Haag

Received March 10, 1984