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A Simplified SO(6,2) Model of SU(3)

A. J. Bracken*

Department of Physics, University of Colorado, Boulder, CO 80309, USA

Abstract. A new realization is obtained of the representation of so(6,2) which has been shown recently by Flath and Biedenharn, and also by Bracken and MacGibbon, to define a model of SU(3). In contrast to the realization in terms of six pairs of boson operators used previously, which involved cubic expressions, the new realization involves only quadratic expressions in eight pairs of boson operators, and is manifestly hermitian. Properties of this new "oscillator realization", and in particular its advantages over the old realization, are discussed briefly. It is deduced that the representation of so(6,2) is integrable to a unitary group representation.

Introduction

Following Bernšteĭn, Gel'fand and Gel'fand [1], a model of a compact group G is defined as a realization of a representation of G which contains in direct sum exactly one representative from each and every equivalence class of irreducible representations (irreps) of G.

Recently a remarkable model of SU(3) has been discovered by Flath and Biedenharn [2–5] and also by Bracken and MacGibbon [6]. There exists a realization in terms of boson operators of an irrep of the Lie algebra so(6, 2), which contains every distinct hermitian irrep of the Lie subalgebra su(3) exactly once and so defines a model of SU(3). Furthermore, basis elements of the so(6,2) algebra are represented in this case by Wigner tensor operators (that is, tensor shift-operators) for SU(3).

Flath and Biedenharn have emphasized that this model provides the framework for an elegant description of the algebra of SU(3) tensor operators, including a complete resolution of the multiplicity problem for such operators. On the other hand, Bracken and MacGibbon have emphasized that this realization of so(6,2) defines "creation and annihilation" operators which, when applied to a

* On leave from Department of Mathematics, University of Queensland, Australia

"vacuum vector," generate the whole representation space and permit the construction of a basis for each SU(3) irrep, in much the same way as the basis vectors are constructed in Schwinger's model [7] of SU(2).

In both approaches the realization of the so(6,2) representation is the same: in the notation of Bracken and MacGibbon, it involves six boson creation operators $\bar{\alpha}^r$, $\bar{\beta}_r (r=1,2,3)$ and corresponding annihilation operators α_r, β^r. Thus

$$[\alpha_r, \bar{\alpha}^s] = \delta^s_r = [\beta^s, \bar{\beta}_r], \tag{1}$$

with all other commutators vanishing. (Here and in what follows, algebraic relations are defined on the space obtained by applying to a vacuum vector, finite polynomials in boson creation operators; and numerical multiples of the unit operator are represented by the corresponding numbers.) The so(6,2) operators are then, for $r, s = 1, 2, 3$,

$$\begin{aligned} \bar{A}^r &= (N+1)\,\bar{\alpha}^r - (\bar{\alpha}^s\bar{\beta}_s)\beta^r, \qquad A_r = \alpha_r, \\ \bar{B}_r &= (N+1)\,\bar{\beta}_r - (\bar{\alpha}^s\bar{\beta}_s)\alpha_r, \qquad B^r = \beta^r, \\ T_{rs} &= \alpha_r\bar{\beta}_s - \alpha_s\bar{\beta}_r, \qquad T^{rs} = \beta^r\bar{\alpha}^s - \beta^s\bar{\alpha}^r, \\ A^r_s &= \bar{\alpha}^r\alpha_s - \bar{\beta}_s\beta^r, \qquad M = N+2, \end{aligned} \tag{2}$$

where

$$N = \bar{\alpha}^r\alpha_r + \bar{\beta}_r\beta^r. \tag{3}$$

All indices here are contravariant (upper) or covariant (lower) with respect to the U(3) subalgebra spanned by the A^r_s, and the su(3) subalgebra of interest is spanned by the operators

$$T^r_s = A^r_s - \tfrac{1}{3}\delta^r_s A^t_t. \tag{4}$$

The $\bar{A}^r$ and $\bar{B}_r$ are the new creation operators, and A_r, B^r the new annihilation operators, discussed by Bracken and MacGibbon. They generate under commutation the whole so(6,2) algebra.

In the space generated by the (finite) action of the boson operators (1) on a vacuum vector Ψ_0, for which

$$\alpha_r\Psi_0 = \beta^r\Psi_0 = 0, \tag{5}$$

the operators (2) define a representation of so(6,2) which is in fact reducible. The desired irrep is obtained in the subspace generated by the action of the operators (2) on Ψ_0. This irreducible subspace is characterized by the so(6,2) – invariant condition

$$\alpha_r\beta^r\Psi = 0. \tag{6}$$

If an alternative set of so(6,2) basis operators $J_{AB}(= -J_{BA}, A, B = 1, 2, \ldots, 8)$ is defined [6] by

$$\begin{aligned} J_{2r-1,2s-1} &= -\tfrac{1}{2}i(T_{rs} + A^r_s - A^s_r + T^{rs}), \\ J_{2r-1,2s} &= -\tfrac{1}{2}(T_{rs} + A^r_s + A^s_r - T^{rs}), \\ J_{2r,2s} &= \tfrac{1}{2}i(T_{rs} - A^r_s + A^s_r + T^{rs}), \end{aligned}$$

$$
\begin{aligned}
J_{7,2r-1} &= -\tfrac{1}{2}(A_r + \bar{A}^r - \bar{B}_r - B^r)\,, \\
J_{7,2r} &= \tfrac{1}{2}i(A_r - \bar{A}^r - \bar{B}_r + B^r)\,, \\
J_{8,2r-1} &= \tfrac{1}{2}i(A_r - \bar{A}^r + \bar{B}_r - B^r)\,, \\
J_{8,2r} &= \tfrac{1}{2}(A_r + \bar{A}^r + \bar{B}_r + B^r)\,, \\
J_{78} &= M\,,
\end{aligned} \tag{7}
$$

for $r, s = 1, 2, 3$, then the so(6,2) commutation relations take the familiar form

$$[J_{AB}, J_{CD}] = i(g_{AC}J_{BD} + g_{BD}J_{AC} - g_{BC}J_{AD} - g_{AD}J_{BC})\,, \tag{8}$$

with the metric tensor $g_{AB} = diag(1, 1, 1, 1, 1, 1, -1, -1)$.

Note that some of the operators (2) are linear, some quadratic and some cubic in the boson operators. This makes algebraic manipulation rather complicated. Note also that if α_r and β^r are hermitian conjugate to $\bar{\alpha}^r$ and $\bar{\beta}_r$, then A_r and B^r are not hermitian conjugate to $\bar{A}^r$ and $\bar{B}_r$, so that the operators J_{7i} and J_{8i} $(i = 1, 2, \ldots 6)$ are not hermitian, and the representation of so(6,2) associated with Eqs. (2) or (7) is not hermitian. Nevertheless, the subrepresentation defined by adding the condition (6) has been shown to be equivalent to an hermitian representation, [5, 6] although this hermiticity could only be made manifest [6] by more seriously complicating the expressions for the so(6,2) basis operators.

It is the purpose of this note to show that these two blemishes upon this "beautiful algebraic structure" [4] can be removed. Another realization of this so(6,2) irrep is found, in which every basis operator is quadratic in boson operators, confirming a conjecture made by Bracken and MacGibbon [6]. Furthermore, this new realization is manifestly hermitian, and indeed it can now be deduced that this irrep of so(6,2) is integrable to a unitary irrep of (a double covering group of) SO(6,2).

2. New Realization of the so(6,2) Representation

Let $j_{AB} (= -j_{BA}, A, B = 1, 2, \ldots, 8)$ be the generators of one of the two fundamental (eight-dimensional) spinor representations of SO(6,2), chosen in such a way that they satisfy relations of the form (8), and also

$$\beta j_{AB}{}^{\dagger} = j_{AB}\beta\,, \tag{9}$$

where $j_{AB}{}^{\dagger}$ denotes the hermitian conjugate of j_{AB}, and β is the matrix $diag(1, 1, 1, 1, -1, -1, -1, -1)$; it can also be arranged that $j_{78} = \frac{1}{2}\beta$. [For definiteness, choose the representation with highest weight $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ for the generators $(j_{21}, j_{43}, j_{65}, j_{78})$, which represent a basis for a maximal compact Cartan subalgebra of so(6,2). Use of the other spinor representation in what follows would lead to an inequivalent realization of so(6,2) in which the J_{8A} obtained below would be replaced by their negatives. Since either choice leads to the same su(3) substructure, only the first possibility will be considered.]

Now form an "oscillator realization" of so(6,2) by the technique familiar [8] from the case of so(4,2). To this end, introduce eight boson creation operators $\bar{\theta}, \bar{\theta}^r$,

$\bar{\phi}$ and $\bar{\phi}^r$ ($r=1,2,3$), and corresponding annihilation operators θ, θ_r, ϕ and ϕ_r, with

$$\begin{aligned}&[\theta,\bar{\theta}]=1=[\phi,\bar{\phi}]\,,\\&[\theta_r,\bar{\theta}^s]=\delta_r^s=[\phi_r,\bar{\phi}^s]\,,\end{aligned}\tag{10}$$

and all other commutators vanishing. Define

$$\eta=[\theta\theta_1\theta_2\theta_3\bar{\phi}^1\bar{\phi}^2\bar{\phi}^3\bar{\phi}]^T=\;[\eta_\mu]^T,\;\mu=1,\,2,\,\ldots 8\,,\tag{11}$$

so that

$$[\eta_\mu,\,\eta_\nu{}^\dagger]=\beta_{\mu\nu}\,,\tag{12}$$

where $\{\beta_{\mu\nu}\}$ are the matrix elements of β. Finally, define

$$J_{AB}=\eta_\mu{}^\dagger(\beta j_{AB})_{\mu\nu}\eta_\nu\,.\tag{13}$$

It is not difficult to check with the use of Eqs. (12) that these J_{AB}, which are all quadratic in boson operators, satisfy the commutation relations (8). Furthermore if $\bar{\theta}$, $\bar{\theta}^r$ etc. are hermitian conjugate to θ, θ_r etc. then the J_{AB} are hermitian as a consequence of Eqs. (9).

The Eqs. (7) can be inverted [6] to express $\bar{A}^r$, A_r etc. in terms of the J_{AB}. When the operators (13) are substituted into these expressions, the so(6,2) basis operators of the oscillator realization are obtained in the form

$$\begin{aligned}&\bar{A}^r=\bar{\phi}^r\bar{\theta}+\bar{\theta}^r\bar{\phi}\,,\qquad A_r=\phi_r\theta+\theta_r\phi\,,\\&\bar{B}_r=\varepsilon_{rst}\bar{\phi}^s\bar{\theta}^t\,,\qquad B^r=\varepsilon^{rst}\phi_s\theta_t\,,\\&T_{rs}=\varepsilon_{rst}(\phi\bar{\phi}^t-\theta\bar{\theta}^t)\,,\qquad T^{rs}=\varepsilon^{rst}(\bar{\theta}\theta_t-\bar{\phi}\phi_t)\,,\\&A_s^r=\bar{\theta}^r\theta_s+\bar{\phi}^r\phi_s+\tfrac{1}{2}\delta_s^r(\bar{\theta}\theta-\bar{\theta}^t\theta_t+\bar{\phi}\phi-\bar{\phi}^t\phi_t)\,,\\&M=\tfrac{1}{2}(\bar{\theta}\theta+\bar{\theta}^r\theta_r+\bar{\phi}\phi+\bar{\phi}^r\phi_r)+2\,,\end{aligned}\tag{14}$$

where ε_{rst} and ε^{rst} are alternating tensors, with $\varepsilon_{123}=\varepsilon^{123}=1$. These expressions may be contrasted with those of Eqs. (2).

This new realization of so(6,2) is reducible, as all the operators (14) evidently commute with

$$S=\bar{\theta}\theta+\bar{\theta}^r\theta_r-\bar{\phi}\phi-\bar{\phi}^r\phi_r\,.\tag{15}$$

In the old realization, determined by Eqs. (2) and (6), the operator M has 2 as an eigenvalue, on the vacuum vector of the α_r and β^r. The only vector on which M as in Eqs. (14) has eigenvalue 2 is the vacuum vector Φ_0 of the operators θ, θ_r, ϕ, and ϕ_r.

Consider therefore, in the Fock space of these boson operators and their conjugates, the subspace $\mathfrak{A}$ obtained from Φ_0 by the (finite) action of the $\bar{A}^r$ and $\bar{B}_r$ of Eqs. (14). It can be seen [6] that such a subspace is invariant under the action of all the so(6,2) operators (14), and has no invariant proper subspaces. Any vector in $\mathfrak{A}$ satisfies

$$S\Phi=0\,.\tag{16}$$

Suppose that the operators J_{21}, J_{43}, J_{65}, and J_{78} ($=A_1^1$, A_2^2, A_3^3, and M) take eigenvalues λ_1, λ_2, λ_3, and m, respectively, on some common eigenvector in $\mathfrak{A}$.

Then $(\lambda_1, \lambda_2, \lambda_3, m)$ is a weight for the irrep on $\mathfrak{A}$. If $(\lambda'_1, \lambda'_2, \lambda'_3, m')$ is another such weight, the two can be ordered by considering firstly the ordering of m and m', then, if $m=m'$, the ordering of λ_3 and λ'_3, and so on. It is now seen that (0, 0, 0, 2) is the lowest weight for this irrep, and corresponds to the vacuum vector Φ_0 as eigenvector. In a similar way, it can be seen that (0, 0, 0, 2) is also the lowest weight for the irrep of so(6,2) associated with Eqs. (2) and (6), in this case corresponding to the vacuum vector Ψ_0 as eigenvector. Since each of these irreps is also hermitian with respect to a corresponding scalar product which is (up to normalization) unique in each case, the argument which has been used by Mack [9, Sect. 4] in the study of hermitian representations of so(4,2) which admit lowest weights, may now be invoked to deduce that the two irreps of so(6,2) at hand are indeed unitarily equivalent.

The task of constructing a manifestly hermitian realization of the relevant irrep *of so(6,2), quadratic in boson operators, has therefore been accomplished.*

For the purpose of further characterizing this irrep, the following observation may be noted. On $\mathfrak{A}$, the operators J_{AB} of (13) satisfy the identity

$$J_{AB}J^B{}_C + 3iJ_{AC} = 2g_{AC}, \tag{17}$$

where $J^B{}_C = g^{BD}J_{DC}$ and $g^{BD} = g_{BD}$. Then contraction on the left in turn with g^{CA}, then J^{CA}, then $J^{CD}J_D{}^A$ and so on, enables the values of all so(6,2) scalar invariants to be determined. (There is no non-trivial pseudo-scalar invariant in this case.) The existence of the quadratic identity (17) generalizes a result known [10] in the case of the ladder representations of so(4,2), and reflects the degeneracy of the irrep of so(6,2) involved here.

3. Concluding Remarks

Eight boson pairs have been used in place of the six used in the old realization, but this may be regarded as a small price to pay for the advantages of the new realization. Apart from the increased ease of manipulation of the operators because of their quadratic form, note that the expressions (14) for the operators $\bar{A}^r$ and $\bar{B}_r$ only involve boson creation operators, in contrast with the expressions (2). This considerably simplifies the calculation of the lengths of vectors of the form

$$\Phi^{rs\ldots t}_{kl\ldots m} = \bar{A}^r\bar{A}^s\ldots\bar{A}^t\bar{B}_k\bar{B}_l\ldots\bar{B}_m\Phi_0\,. \tag{18}$$

The expressions obtained when the forms (14) for the $\bar{A}$- and $\bar{B}$-operators are substituted in Eq. (18) are similar in structure to ones obtained, for example by Holman and Biedenharn [11], in earlier treatments of SU(3), where the calculation of such lengths has been discussed in some detail. As shown by Bracken and MacGibbon [6], the vectors of the form (18), in which p of the $\bar{A}$-operators and q of the $\bar{B}$-operators appear, span a space carrying the irrep $(p+q, q)$ of SU(3); and as p *and q run independently over the non-negative integers,* a basis for $\mathfrak{A}$ is obtained in which the appearance of each irrep of SU(3) exactly once is evident. The "number operators" whose eigenvalues are p and q are given by

$$P = \bar{\theta}\theta + \bar{\phi}\phi\,, \qquad Q = \tfrac{1}{2}[\bar{\theta}^t\theta_t + \bar{\phi}^t\phi_t - \bar{\theta}\theta - \bar{\phi}\phi]\,. \tag{19}$$

In the new realization, the relations

$$[\bar{A}^r, \bar{A}^s] = [\bar{B}_r, \bar{B}_s] = [\bar{A}^r, \bar{B}_s] = 0, \qquad \bar{A}^r \bar{B}_r = 0, \tag{20}$$

are obvious, whereas they are difficult to verify in the old realization. These relations are crucial to the model, and ensure in particular that $\Phi^{rs\ldots t}_{kl\ldots m}$ as in Eq. (18) is separately symmetric in its upper and lower indices, and satisfies the zero-trace condition

$$\Phi^{rs\ldots t}_{rl\ldots m} = 0, \tag{21}$$

so that it transforms as an irreducible SU(3) tensor.

Although the operators $\bar{\theta}$ and $\bar{\phi}$ and their conjugates are apparently necessary for the construction of a model of SU(3) in which all so(6,2) basis operators are quadratic in boson operators, they play something of a background role in the construction of SU(3) basis vectors from the vacuum vector Φ_0, since they are SU(3) scalars and so change no SU(3) "state labels," unlike the $\bar{\theta}^r$ and $\bar{\phi}^r$.

The role of "background bosons" like θ and ϕ can be seen more clearly in an analogous SU(2) case. It was remarked by Bracken and MacGibbon [6] that an SU(2,1) model of SU(2) can be defined. The structure of the revised SO(6,2) model of SU(3) also suggests a simpler realization of that model. Take as su(2,1) basis operators, in terms of boson operators θ, $\theta_r (r = 1, 2)$ and their conjugates,

$$\begin{aligned} &\bar{A}^r = \bar{\theta}\bar{\theta}^r, \qquad A_r = \theta\theta_r, \\ &A^r_s = \bar{\theta}^r\theta_s + \tfrac{1}{3}\delta^r_s(\bar{\theta}\theta - \bar{\theta}^t\theta_t + 1), \end{aligned} \tag{22}$$

acting on the space generated by these operators (22) from a normalized vacuum vector χ_0. The SU(2) generators are then $A^r_s - \frac{1}{2}\delta^r_s A^t_t$, and a complete set of SU(2) basis vectors $\chi_{jm}(j = 0, \frac{1}{2}, 1\ldots;\ m = j, j-1, \ldots -j)$ can be constructed as

$$\chi_{jm} = c_{jm}(\bar{A}^1)^{j+m}(\bar{A}^2)^{j-m}\chi_0 = c_{jm}(\bar{\theta})^j(\bar{\theta}^1)^{j+m}(\bar{\theta}^2)^{j-m}\chi_0, \tag{23}$$

where the normalization constant evidently must satisfy

$$|c_{jm}|^{-2} = j!\ (j+m)!\ (j-m)\ . \tag{24}$$

The powers of $\bar{\theta}^1$ and $\bar{\theta}^2$ determine the essential SU(2) character of the vector (23), as do the powers of the boson creation operators in Schwinger's model [7] of SU(2). The SU(2) scalars θ and $\bar{\theta}$ apparently have to be added to Schwinger's bosons if his non-semisimple model is to be converted to a simple [SU(2,1)] model.

As Flath [2] has indicated, the problem of constructing a generalization of the so(6,2) model of SU(3) to the case of SU(4) seems to be "significantly more complex." If the operators (14) in the SU(3) case are regarded as generalizations of (22) in the SU(2) case, the problem seems to be, in part at least: Find the next Lie algebra in the sequence

$$\mathrm{su}(2,1), \mathrm{so}(6,2), \ldots. \tag{25}$$

This seems a most unlikely progression. Perhaps the fact that su(2,1) and so(6,2) are both subalgebras of the Lie algebras of pseudo-unitary groups provides a clue for an approach to SU(n), $n > 3$.

The irrep of so(6,2) defined by Eqs. (14) on the subspace $\mathfrak{A}$ is in fact a subrepresentation of the "oscillator representation" of u(4,4), whose hermitian basis operators have the form (13), except that the twenty-eight j_{AB} are replaced by all sixty-four of the eight-by-eight complex matrices A satisfying

$$\beta A^{\dagger} = A\beta . \tag{26}$$

This representation of u(4,4) has been shown by Anderson et al. [12] to be integrable to a unitary representation of U(4,4). It follows that the irrep of so(6,2) is integrable to a unitary irrep of SO(6,2), or more accurately, of a double-covering of that group.

These oscillator representations can be realized in Bargmann spaces [12, 13], and it may well be that the most elegant treatment of the representations and tensor operators of SU(3) will be obtained when the SO(6,2) model is formulated in one of these beautiful reproducing-kernel Hilbert spaces of entire functions.

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