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Borel Summability of the Unequal Double Well

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Abstract. Unlike the $\varepsilon=0$ case, the perturbation series of the unequal double well $p^2+x^2+2gx^3+g^2(1+\varepsilon)x^4$ are Borel summable to the eigenvalues for any $\varepsilon>0$.

The best known example (see e.g. [13, Sect. XII.4]) of a non-Borel summable perturbation series is represented by the Rayleigh-Schrödinger perturbation expansion (hereafter RSPE) of the standard double well oscillator $H(g)=p^2+x^2+x^2(1+gx)^2$ in $L^2(\mathbb{R})$, $g\in\mathbb{R}$. This fact is of course due to the instability of the eigenvalues as $g\to 0$, i.e. to their asymptotic degeneracy as $g\to 0$. However there are examples, such as the Herbst and Simon [5] one, $K(g)=p^2+x^2(1+gx)^2-2gx-1$, in which there is stability but no Borel summability to the eigenvalues. Hence, also on account of recent investigations on Borel summability in four dimensional field theories [6, 7], it could be interesting to relate the lack of summability to some other more subtle physical mechanism of well defined meaning also in a more general context. To this end, T. Spencer has suggested considering the following "unequal" double well oscillator

$$H(g,\varepsilon)=p^2+x^2(1+gx)^2+\varepsilon g^2x^4\,, \tag{1}$$

which in the limit $g\to 0$ has an infinite action instanton for any $\varepsilon\geqq 0$. (A standard reference for the notion of instanton in problems of this type is Coleman [1]; additional discussion can be found in [2, 11].) This model could in addition have some interest in itself: as a matter of fact, in some sense it represents the slightest modification of the non-summable example, and it is natural to ask to what extent the non-summability as "accidental," i.e. how sensitive is its dependence on the choice of the parameters in $H(g)$? Furthermore it can be easily proved through the Hunziker-Vock technique [8] that any eigenvalue E of $H(0,\varepsilon)\equiv H(0)=p^2+x^2$ is stable for $g\in\mathbb{R}$ small as an eigenvalue of $H(g,\varepsilon)$, $\varepsilon>0$, because the second minimum of $V(g,\varepsilon)\equiv x^2(1+gx)^2+\varepsilon g^2x^4$ tends to $+\infty$ as $g\to 0$, $\varepsilon>0$.

In this note we prove that, for $\varepsilon>0$, any eigenvalue E is actually stable as an eigenvalue $E(g,\varepsilon)$ of $H(g,\varepsilon)$ for g complex, $|g|$ suitably small, $|\arg g| \leqq \frac{\pi}{4}$, and that the RSPE near E is Borel summable to $E(g,\varepsilon)$ for g positive and small. To this end, let us first collect some well known results on the operator families $H(g,\varepsilon)$ acting in $L^2(\mathbb{R})$ under the form of a proposition whose proof can be easily traced out of [13, Sect. XII.3, 4].

Proposition 1. *Let $\varepsilon>0$ be fixed and $g\in\mathbb{C}$, $g=|g|e^{i\theta}$, $|g|\geqq 0$, $|\theta|<\frac{\pi}{2}$. Let the operator family $H(g,\varepsilon)$ in $L^2(\mathbb{R})$ be defined as the action of $p^2+V(g,\varepsilon)$ on the domain $D(H(g,\varepsilon))=D(p^2)\cap D(x^4)$, $g\neq 0$, and $H(0)$ as the action of p^2+x^2 on $D(0)=D(p^2)\cap D(x^2)$ for $g=0$. Then for any fixed g, $H(g,\varepsilon)$ has compact resolvent, and any eigenvalue $E(g,\varepsilon)$ is a locally holomorphic function of g in the complex sector*

$$S\equiv\left\{g\in\mathbb{C}:|g|>0,\ |\theta|<\frac{\pi}{2}\right\}.$$

Our result can be stated as follows.

Proposition 2. *Let $\varepsilon>0$ be fixed. Then:*

(i) There is $B(E)>0$ such that any eigenvalue E of $H(0)$ is stable (in the sense of Kato [9, Sect. VIII.1.4]) as an eigenvalue $E(g,\varepsilon)$ of $H(g,\varepsilon)$ for $|g|<B(E)$, $|\theta|\leqq\frac{\pi}{4}$.

(ii) $E(g,\varepsilon)$ is a holomorphic function of g at least in the sector

$$Q\equiv\left\{0<|g|<B(E);\ |\theta|\leqq\frac{\pi}{4}\right\},$$

and is continuous as $|g|\to 0$, $|\theta|\leqq\frac{\pi}{4}$.

(iii) Let $\sum_{n=0}^{\infty}A_n(\varepsilon)g^n\sim E(g,\varepsilon)$ be the RSPE *of $E(g,\varepsilon)$ near $E, R_N(g,\varepsilon)=E(g,\varepsilon)-\sum_{n=0}^{N-1}A_n(\varepsilon)g^n$ its N^{th} order remainder. Then $A_{2n+1}(\varepsilon)\equiv 0$ $\forall n$, and there is $D>0$ independent of g such that*

$$|R_{2N}(g,\varepsilon)|\leqq DN!|g^2|^N,\qquad N=1,2,\ldots \tag{2}$$

as long as $|g|\leqq B(E)$, $|\theta|\leqq\frac{\pi}{4}$.

Remarks. (i) Statement (iii) implies the Borel summability of $\sum_{n=0}^{\infty}A_n(\varepsilon)g^n$ to $E(g,\varepsilon)$ for $0\leqq g\leqq B(E)$. For $A_{2n+1}\equiv 0$ $\forall n$ implies that $E(g,\varepsilon)$ is a function of g^2, which by (ii) is holomorphic for $0<|g|^2<B(E)^2$, $|\arg g^2|\leqq\frac{\pi}{2}$, and continuous as $|g|\to 0$, $|\arg g^2|\leqq\frac{\pi}{2}$. Then (2) holds with g replaced by g^2 in the left-hand side so that by the Watson-Nevanlinna theorem (for details see Sokal [14]) the summability takes place for g as above.

(ii) Proposition 1 allows us to apply the standard complex scaling argument (see e.g. Simon [12] for details). Hence the operators $H(g.\varepsilon)$ and $e^{-2i\phi}H(g,\varepsilon,\phi)$,

$H(g,\varepsilon,\phi)$ defined as the action of $p^2+e^{4i\phi}V(ge^{i\phi},\varepsilon)$ on $D(H(g,\varepsilon))$ have the same eigenvalues as long as $|\theta+3\phi|<\frac{\pi}{2}$.

Let us now proceed to prove Proposition 2. By Remark (ii) we can consider $H(g,\varepsilon,\phi)$ instead of $H(g,\varepsilon)$. Given δ, $0<\delta<\frac{\pi}{4}$, we take $\phi=-\theta\equiv-\arg g$ for $|\theta|\leqq\frac{\pi}{4}-\delta$, i.e. we consider $H(|g|,\varepsilon,-\theta)$ as long as $|\theta|\leqq\frac{\pi}{4}-\delta$. For $\theta=\frac{\pi}{4}-\theta'$, $0\leqq\theta'\leqq\delta_1$, $\delta<\delta_1<\frac{\pi}{4}$, we take $\phi=-\frac{\pi}{4}+\eta$, $\delta_1<\eta<\frac{\pi}{4}$, $\eta<\operatorname{arctg}\sqrt{\varepsilon}$, i.e. we consider $H(|g|,\varepsilon,\chi)\equiv p^2+e^{-i(\pi-4\eta)}V(|g|e^{i\chi},\varepsilon)$ with $\chi=\eta-\theta'$, $\delta_1-\delta\leqq\chi\leqq\eta$. For $\theta=-\frac{\pi}{4}+\theta'$ we obviously consider $H(|g|,\varepsilon,-\chi)$. The condition $\eta<\operatorname{arctg}\sqrt{\varepsilon}$ ensures that the zeros $|g|x=(1+\varepsilon)^{-1}e^{-i\chi}(-1\pm i\sqrt{\varepsilon})$ of $V(|g|e^{i\chi},\varepsilon)$ have non-vanishing imaginary part.

It is clearly enough to prove Proposition 2 for $H(|g|,\varepsilon,-\theta)$ and $H(|g|,\varepsilon,\chi)$ separately. We proceed by means of ODE techniques of WKB type because, while $H(|g|,\varepsilon,-\theta)$ can be analyzed by means of the Hunziker and Vock [8] stability theorem, this is not the case for $H(|g|,\varepsilon,\chi)$, because the union over $|g|>0$ of the numerical ranges is the whole of $\mathbb{C}$.

Lemma 3. *Let $\varepsilon>0$, $|g|\geqq 0$, $0<\chi\leqq\eta$. Then the* ODE $H(|g|,\varepsilon,\chi)\psi=0$, *i.e.*

$$-\psi''+e^{-i(\pi-4\eta)}x^2[(1+|g|e^{i\chi}x)^2+\varepsilon|g|^2e^{2i\chi}x^2]\psi=0 \tag{3}$$

admits a unique solution $\psi_-(x,|g|,\varepsilon,\chi)$ (respectively $\psi_+(x,|g|,\varepsilon,\chi)$) which is L^2 at $-\infty$ (respectively at $+\infty$), and such that

$$\lim_{|g|\to 0}\psi_\pm(x,|g|,\varepsilon,\chi)=\psi_\pm(x,0,\varepsilon,\chi)\equiv\psi_\pm(x,0,\chi)$$

uniformly with respect to $(x,\chi)\in[-a,a]\times[\bar{\chi},\eta]$, $0<a<+\infty$, $0<\bar{\chi}<\eta$. An analogous statement holds for the solutions $\psi_\pm(x,|g|,\varepsilon,\theta)$ of the ODE $H(|g|,\varepsilon,-\theta)\psi=0$, *$|\theta|\leqq\delta$, and for the solutions $\psi_\pm(x,|g|,\varepsilon,-\chi)$ of the* ODE $H(|g|,\varepsilon,-\chi)\psi=0$, *$-\eta\leqq\chi\leqq 0$.*

Proof. We limit ourselves to consider the case of $\varepsilon>0$ suitably small, because this is clearly the most interesting and delicate situation. The general case requires only lengthier computations. For any $\varepsilon>0$ the function $V(|g|e^{i\chi},\varepsilon)$ vanishes only at $x=0$ if $x\in\mathbb{R}$. Therefore we can define:

$$f_\pm(x,|g|,\varepsilon,\chi)=e^{i(\pi/4-\eta)}V(|g|e^{i\chi},\varepsilon)^{-1/4}\exp\left(\pm e^{-i\pi/2+2i\eta}\int_0^x V(|g|e^{i\chi},\varepsilon)^{1/2}dt\right). \tag{4}$$

It is known (see e.g. Sibuya [10, Lemma 13.1]) that $\psi_\pm(\cdot)$ exist, with $\psi_\pm(x,|g|,\varepsilon,\chi)=(1+o(1))f_\pm(x,|g|,\varepsilon,\psi)$ as $x\to\pm\infty$, uniformly with respect to $(|g|,\chi)$, but this does not necessarily imply $f_-(x,|g|,\cdot)\to f_-(x,0,\cdot)$. Consider now $f_-(\cdot)$. For $x<0$, setting $R(u,\varepsilon)=(1-u)^2+\varepsilon u^2$ we have:

$$\int_0^x V(|g|e^{i\chi},\varepsilon)^{1/2}dt=|g|^{-2}\int_0^{|gx|}tR(te^{i\chi},\varepsilon)^{1/2}dt=|g|^{-2}F(|gx|,\chi,\varepsilon),$$

where

$$F(|gx|,\chi,\varepsilon)=\tfrac{1}{3}(1+\varepsilon)^{-1}e^{-2i\chi}R(|gx|e^{i\chi},\varepsilon)^{3/2}$$
$$+\tfrac{1}{2}(1+\varepsilon)^{-2}e^{-2i\chi}((1+\varepsilon)e^{i\chi}|gx|-1)R(|gx|e^{i\chi},\varepsilon)^{1/2}$$
$$+\tfrac{1}{2}\varepsilon(1+\varepsilon)^{-3/2}e^{-2i\chi}\ln[2(1+\varepsilon)^{1/2}e^{i\chi}R(|gx|e^{i\chi},\varepsilon)^{1/2}+2|gx|e^{2i\chi}-2e^{i\chi}]$$
$$-\tfrac{1}{3}(1+\varepsilon)^{-1}e^{-2i\chi}+\tfrac{1}{2}e^{-2i\chi}(1+\varepsilon)^{-2}-\tfrac{1}{2}\varepsilon(1+\varepsilon)^{-3/2}e^{-2i\chi}\ln(2(1+\varepsilon)^{1/2}e^{i\chi}-2e^{i\chi}).$$

Set now $|gx|=1+y$, $-1\leqq y\leqq+\infty$. For ε and $\eta<\varepsilon$ suitably small, we can replace $e^{-i\pi/2+2i\eta}|g|^{-2}F(1+y,\chi,\varepsilon)$ through its first order Taylor expansion up to a relative error of order ε^2, which is uniform with respect to y: namely, on account also of $\eta\geqq\chi$, we have:

$$\operatorname{Im}e^{2i\eta}|g|^{-2}F(1+y,\chi,\varepsilon)\geqq\operatorname{Im}e^{2i\chi}|g|^{-2}F(1+y,\chi,\varepsilon)$$
$$=\chi|g|^{-2}\frac{y^2(1+y)^2+\varepsilon(3y^4+3y+1+(1+y)(8y^2+1/2)}{(1+2\varepsilon)(y^2+\varepsilon(1+y)^2)^{1/2}}(1+a(y;\varepsilon,\chi)\varepsilon^2)$$

for some $a(y;\varepsilon,\chi)$ bounded independently of $(y;\varepsilon,\chi)$. Hence

$$\operatorname{Im}\left(e^{2i\eta}\int_0^x V(|g|e^{i\chi},\varepsilon)^{1/2}dt\right)>0$$

strictly, independently of x and $|g|$, as long as $\eta>0$, $\chi>0$, i.e. $\frac{\pi}{4}\geqq\theta\geqq\frac{\pi}{4}-\eta$. Therefore given $\bar{\varepsilon}>0$, there is $M(\bar{\varepsilon})<0$ independent of $|g|$ such that $|f_-(\cdot)|<\bar{\varepsilon}$ for $x<M(\bar{\varepsilon})$, and hence $|\psi_-(x,|g|,\chi,\varepsilon)|<\bar{\varepsilon}$ for $x<M(\bar{\varepsilon})$, uniformly with respect to $(|g|,\chi)$. This implies $\lim_{|g|\to 0}\psi_-(x,|g|,\chi,\varepsilon)=\psi_-(x,0,\chi)$, with the stated uniformities by the theorem of the continuous dependence on the parameters applied to the ODE $H(|g|,\varepsilon,\chi)\psi=0$. For $\psi_+(x,|g|,\chi,\varepsilon)$ the statement is obvious because the real part of the integrand never undergoes a cancellation. An even simpler argument applies to $\psi_\pm(x,|g|,\theta,\varepsilon)$: in this case indeed one has to consider the real part of

$$\pm|g|^{-2}e^{-2i\theta}\int_0^{|g|x}tR(-|g|t,\varepsilon)^{1/2}dt,$$

which for $|\theta|<\frac{\pi}{4}$ is trivially uniformly positive as $x\to\pm\infty$, respectively. This proves Lemma 3, and explains why a different scaling is needed for $\theta=\pm\pi/4$.

Remark. The above statement is not true for $\varepsilon=0$. Taking indeed $\theta=0$, the double zero of $V(g,0)=x^2(1+gx)^2$ at $x=-\frac{1}{g}$ forces the exponent of the WKB solution to switch sign near $-\infty$ as $g\to 0$.

Lemma 4. *Let* $R(|g|,\varepsilon,\chi;z)=(H(|g|,\varepsilon,\chi)-z)^{-1}$, $R(|g|,\varepsilon,\theta,z)=(H(|g|,\varepsilon,-\theta)-z)^{-1}$ *denote the resolvent, of* $H(|g|,\varepsilon,\chi)$ *and* $H(|g|,\varepsilon,-\theta)$, *respectively, which are compact operators in* L^2 *by Proposition 1 for* $z\notin\sigma(H(\cdot))$. *Let* $R(0,\theta,z)=(p^2+e^{-4i\theta}x^2-z)^{-1}$, $R(0,\eta,z)=(p^2+e^{-i\pi+4i\eta}x^2-z)^{-1}$ *denote the unperturbed scaled resolvents,* $z\neq e^{-2i\theta}(2n+1)$, $z\neq e^{-i\pi/2+2i\eta}(2n+1)$, $n=0,1,\dots$, *respectively. Then, as* $|g|\to 0$, *and* $z\neq e^{-2i\theta}(2n+1)$, $z\neq e^{-i\pi/2+2i\eta}(2n+1)$, *respectively,* $\|R(|g|,\varepsilon,\theta,z)-R(0,\theta,z)\|\to 0$,

$\|R(|g|,\varepsilon,\chi,z)-R(0,\eta,z)\|\to 0$ *the convergence being uniform with respect to* $\theta, |\theta|\leqq\frac{\pi}{4}-\delta$, *and* $\chi, 0<\chi\leqq\eta$, *respectively.*

Proof. Denote by $W_\theta(|g|,\varepsilon)$, $W_\chi(|g|,\varepsilon)$ the Wronskians of $(\psi_-(x,|g|,\varepsilon,\theta)$. $\psi_+(x,|g|,\varepsilon,\theta))$ and $(\psi_-(x,|g|,\varepsilon,\chi)$, $\psi_+(x,|g|,\varepsilon,\chi))$, respectively, and by $W_\theta(0)$, $W_\eta(0)$ the Wronskians of $(\psi_-(x,0,\theta)$, $\psi_+(x,0,\theta))$, $(\psi_-(x,0,\eta)$, $\psi_+(x,0,\eta))$, respectively. Since $W_\theta(0)\neq 0$, $W_\eta(0)\neq 0$, by Lemma 3 there is $\bar{g}>0$ independent of θ, χ, respectively, such that $W_\theta(|g|,\varepsilon)\neq 0$, $W_\chi(|g|,\varepsilon)\neq 0$ for $|g|\leqq\bar{g}$. Then, through standard ODE arguments (see e.g. Hellwig [4]), one can easily check the following Green's function representations, valid for any $u\in L^2$

$$(R(|g|,\varepsilon,\theta;0)u)(x)=\int_{\mathbb{R}} G(x,y;|g|,\varepsilon,\theta)u(y)dy\,,$$

$$(R(|g|,\varepsilon,\chi;0)u)(x)=\int_{\mathbb{R}} G(x,y;|g|,\varepsilon,\chi)u(y)dy\,,$$

$$(R(0,\theta;0)u)(x)=\int G(x,y;0,\theta)u(y)dy\,,$$

$$(R(0,\eta;0)u)(x)=\int_{\mathbb{R}} G(x,y;0,\eta)u(y)dy\,,$$

where

$$G(x,y;|g|,\varepsilon,\theta)=W_\theta(|g|,\varepsilon)^{-1}\begin{cases}\psi_-(y,\cdot)\psi_+(x,\cdot), & y\leqq x\\ \psi_+(y,\cdot)\psi_-(x,\cdot), & y\geqq x,\end{cases}$$

$$G(x,y;|g|,\varepsilon,\chi)=W_\chi(|g|,\varepsilon)^{-1}\begin{cases}\psi_-(y,\cdot)\psi_+(x,\cdot), & y\leqq x\\ \psi_+(y,\cdot)\psi_-(x,\cdot), & y\geqq x,\end{cases}$$

and analogous definitions for $G(x,y;0,\theta)$, $G(x,y;0,\eta)$, with $W_\theta(|g|,\varepsilon)$, $W_\chi(|g|,\varepsilon)$ replaced by $W_\theta(0)$, $W_\eta(0)$, respectively. Starting from the asymptotic behaviours of $\psi_\pm(\cdot)$, it is not difficult to check that $G(x,y;|g|,\varepsilon,\chi)$ and $G(x,y;|g|,\varepsilon,\theta)$ are Hilbert-Schmidt integral kernels, as well as $G(\cdot,0,\eta)$ and $G(\cdot,0,\theta)$. Proceeding as in Lemma 3, one easily proves that, given $\bar{\varepsilon}>0$, there is $M(\bar{\varepsilon})>0$ independent of $|g|$ and θ, χ (respectively) such that

$$\iint_{x^2+y^2\geqq M(\bar{\varepsilon})} |G(x,y;|g|,\varepsilon,\theta)|^2dxdy<\bar{\varepsilon}\,,$$

$$\iint_{x^2+y^2\geqq M(\bar{\varepsilon})} |G(x,y;|g|,\varepsilon,\chi)|^2dxdy<\bar{\varepsilon}\,.$$

Therefore, by the continuity of $\psi_\pm(x,|g|,\varepsilon,\theta)$, $\psi_\pm(x,|g|,\varepsilon,\chi)$ as $|g|\to 0$, uniform with respect to θ and χ, respectively, and with respect to x in the compacts of $\mathbb{R}$, and by the uniform convergence of $W_\theta(|g|,\varepsilon)$, $W_\chi(|g|,\varepsilon)$ towards $W_\theta(0)$ and $W_\eta(0)$, respectively, we have for $|g|\to 0$,

$$\int_{\mathbb{R}^2} |G(x,y;|g|,\varepsilon,\theta)-G(x,y,0,\theta)|^2dxdy\to 0\,,$$

$$\int_{\mathbb{R}^2} |G(x,y;|g|,\varepsilon,\chi)-G(x,y,0,\eta)|^2dxdy\to 0\,.$$

This is enough to prove the assertion because the Hilbert-Schmidt norm majorizes the operator norm, and the norm resolvent convergence for $z=0$ implies the norm resolvent convergence for all z as above. This proves Lemma 4.

Proof of Proposition 2. Let $E=(2j+1); j=0,1,\ldots$ be an unperturbed eigenvalue, which is an eigenvalue also of $e^{-2i\phi}H_0(\phi)=e^{-2i\phi}(p^2+e^{+4i\phi}x^2)$, $|\phi|<\frac{\pi}{4}$. By Lemma 4 and standard arguments of perturbation theory (see e.g. Reed and Simon [13, Sect. XII.3), E is stable both as an eigenvalue of $e^{2i\theta}H(|g|,\varepsilon,-\theta)$, $|\theta|\leqq\frac{\pi}{4}-\delta$, and as an eigenvalue of $e^{-i(\pi/2-2\eta)}H(|g|,\varepsilon,\pm\chi)$, $0\leqq\chi\leqq\eta$. This implies that given the circle $\Gamma_\nu:\{z:|z-E|=\nu\}$ there is $B(E)>0$ such that $e^{2i\theta}H(|g|,\varepsilon,-\theta)$ and $e^{-i(\pi/2-2\eta)}H(|g|,\varepsilon,\pm\chi)$ have one and only one eigenvalue, denoted by $E(|g|,\theta)$ and $E(|g|;\pm\chi)$, respectively, inside Γ_ν for all g such that $|g|\leqq B(E)$, with

$$\lim_{|g|\to 0} E(|g|,\theta)=\lim_{|g|\to 0} E(|g|,\chi)=E.$$

By rescaling the phase of g we can thus conclude that $H(g,\varepsilon)$ has one and only one eigenvalue $E(g,\varepsilon)$ inside Γ_ν as long as $|g|<B(E)$, $|\theta|=|\arg g|\leqq\frac{\pi}{4}$, with $E(g,\varepsilon)\to E$ as $g\to 0$. In addition $E(g,\varepsilon)$ is a holomorphic function of g at least for $0<|g|<B(E)$, $|\arg g|\leqq\frac{\pi}{4}$. This proves assertions (i) and (ii). To see (iii), by the scaling invariance of the RSPE and well known arguments of asymptotic perturbation theory (see e.g. Reed and Simon [13, Sect. XII.4]) it is enough to check that both $R(|g|,\varepsilon,\chi,z)$ and $R(|g|,\varepsilon,\theta,z)$ are uniformly bounded with respect to $(|g|,z)\in(0,B(E))\times\Gamma_{1/2}$. This is once more a consequence of the norm resolvent convergence.

Remark. The limit $\varepsilon\to 0$ of $H(g,\varepsilon)$ is of course highly irregular. Examined in the light of the stability of the boundary conditions, for g real it would clearly correspond to a fixed choice of sign in $\sqrt{(1+gx)^2+\varepsilon g^2x^2}$ at the limit $\varepsilon=0$ also for $x=-\frac{1}{g}$ while the square root is forced to switch sign at the double zero $x=-\frac{1}{g}$ by analyticity. Not surprisingly, the Borel sum of the RSPE of $H(g)$ does not represent an eigenvalue of the problem and has been identified [3] as a complex eigenvalue of a non-self-adjoint problem described by the same equation with L^2 conditions at infinity imposed along complex directions.

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