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Large Solutions for Harmonic Maps in Two Dimensions*

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Abstract. We seek critical points of the functional $E(u) = \int_\Omega |\nabla u|^2$, where Ω is the unit disk in $\mathbb{R}^2$ and $u: \Omega \to S^2$ satisfies the boundary condition $u = \gamma$ on $\partial\Omega$. We prove that if γ is not a constant, then E has a local minimum which is different from the absolute minimum. We discuss in more details the case where $\gamma(x, y) = (Rx, Ry, \sqrt{1 - R^2})$ and $R < 1$.

Introduction

Let $\Omega = \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 < 1\}$ and $S^2 = \{(x, y, z) \in \mathbb{R}^2;\ x^2 + y^2 + z^2 = 1\}$. Let $\gamma: \partial\Omega \to S^2$ be given and assume that γ is the restriction to $\partial\Omega$ of some function in $H^1(\Omega; S^2)^1$. We set

$$E(u) = \int_\Omega |\nabla u|^2 \quad \text{for} \quad u \in H^1(\Omega; \mathbb{R}^3)$$

and

$$\mathscr{E} = \{u \in H^1(\Omega; S^2); u = \gamma \quad \text{on} \quad \partial\Omega\}.$$

We seek critical points of E on $\mathscr{E}$. It is obvious that there exists some $\underline{u} \in \mathscr{E}$ such that

$$E(\underline{u}) = \operatorname*{Inf}_{\mathscr{E}} E.$$

Our first result is the following:

Theorem 1. *If γ is not a constant, there exists a critical point of E on $\mathscr{E}$ which is different from $\underline{u}$.*

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1 We use the standard notation for Sobolev spaces:
$H^1(\Omega; \mathbb{R}^3) = \{u \in L^2(\Omega; \mathbb{R}^3);\quad u_x, u_y \in L^2(\Omega; \mathbb{R}^3)\}$ and
$H^1(\Omega; S^2) = \{u \in H^1(\Omega; \mathbb{R}^3);\quad u(x, y) \in S^2$ a.e. on $\Omega\}$

In order to prove Theorem 1, we introduce

$$Q(u)=\frac{1}{4\pi}\int_{\Omega} u\cdot u_x \wedge u_y$$

for $u\in L^{\infty}(\Omega;\mathbb{R}^3)$ with $u_x, u_y\in L^2(\Omega;\mathbb{R}^3)$ and we observe (see Lemma 1) that

$$Q(u_1)-Q(u_2)\in\mathbb{Z} \quad \forall u_1,u_2\in\mathscr{E}.$$

For every $k\in\mathbb{Z}$ we define the class $\mathscr{E}_k=\{u\in\mathscr{E};\ Q(u)-Q(\underline{u})=k\}$. Each class $\mathscr{E}_k$ is (non-empty) closed and open in $\mathscr{E}$ for the topology induced by the norm of $H^1(\Omega;\mathbb{R}^3)$.

In order to find other critical points of E on $\mathscr{E}$ it is tempting to consider

$$\operatorname*{Inf}_{\mathscr{E}_k} E \quad \text{for} \quad k\neq 0.$$

When trying to prove that $\operatorname*{Inf}_{\mathscr{E}_k} E$ is achieved one encounters a major difficulty due to the fact that $\mathscr{E}_k$ is not closed under weak H^1 convergence. Nevertheless we shall prove that at least one of the two infima $\operatorname*{Inf}_{\mathscr{E}_1} E$ or $\operatorname*{Inf}_{\mathscr{E}_{-1}} E$ is achieved. The argument involves some ideas used by the authors in [2]; related difficulties also occur in [1, 3, 7, 8, 11]. Notice that the assumption "γ is not a constant" in Theorem 1 is essential. Indeed when $\gamma=C$ is a constant, Lemaire [6] has proved that $u\equiv C$ is the only critical point of E on $\mathscr{E}$.

For simplicity we consider only maps with values into S^2. The same result holds if S^2 is replaced by a Riemannian surface homeomorphic to S^2 (see Remark 2).

The paper is organized as follows: In Sect. 1 we present some technical lemmas. In Sect. 2 we prove Theorem 1. In Sect. 3 we discuss a simple example, namely

$$\gamma(x,y)=\begin{pmatrix} Rx \\ Ry \\ \sqrt{1-R^2}\end{pmatrix} \quad \text{with } R<1.$$

We prove (see Theorem 2) that $\operatorname*{Inf}_{\mathscr{E}_k} E$ is *not* achieved, except when $k=0$ and $k=-1$. We have collected in the Appendix various useful facts and in particular an important density result due to R. Schoen and K. Uhlenbeck [10].

After our work was completed we learned that J. Jost [5] has obtained independently a result similar to our Theorem 1.

1. Some Technical Lemmas

We start with

Lemma 1. *Assume* $u_1,u_2\in\mathscr{E}$. *then* $Q(u_1)-Q(u_2)\in\mathbb{Z}$.

Proof. We consider $w\colon\mathbb{R}^2\to S^2$ defined as follows:

$$\begin{cases} w(x,y)=u_1(x,y) & \text{if} \quad x^2+y^2<1 \\ w(x,y)=u_2\left(\dfrac{x}{x^2+y^2},\dfrac{y}{x^2+y^2}\right) & \text{if} \quad x^2+y^2>1.\end{cases}$$

It is easy to check that $w \in L^\infty(\mathbb{R}^2; S^2)$, $w_x, w_y \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ and

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} w \cdot w_x \wedge w_y = Q(u_1) - Q(u_2).$$

On the other hand if $\phi \in C^\infty(\mathbb{R}^2; S^2)$ and ϕ is constant far out then

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} \phi \cdot \phi_x \wedge \phi_y \in \mathbb{Z}.$$

In fact this integer is the degree of the map $\phi \circ \pi : S^2 \to S^2$, where $\pi : S^2 \to \mathbb{R}^2$ is a stereographic projection (see for example the analytic expression of the degree given in [9]). It follows by density (see Lemma A.1) that

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} \phi \cdot \phi_x \wedge \phi_y \in \mathbb{Z} \quad \forall \phi \in L^\infty(\mathbb{R}^2, S^2) \text{ with } \phi_x, \phi_y \in L^2(\mathbb{R}^2, \mathbb{R}^3), \tag{1}$$

and thus we obtain the conclusion of Lemma 1.

Our next lemma plays a crucial role in the proof of Theorem 1. We assume now that γ is not a constant and we fix some $\underline{u} \in \mathscr{E}$ such that $E(\underline{u}) = \operatorname{Inf}_{\mathscr{E}} E$.

Lemma 2. *There is some* $v \in \mathscr{E}$ *such that*

$$|Q(v) - Q(\underline{u})| = 1 \tag{2}$$

and

$$E(v) < E(\underline{u}) + 8\pi. \tag{3}$$

Proof. By Morrey's regularity theory we know that $\underline{u} \in C^\infty(\Omega; \mathbb{R}^3)$. Since γ is not a constant it follows that $\nabla \underline{u}(x_0, y_0) \neq 0$ for some point $(x_0, y_0) \in \Omega$. Rotating coordinates in the (x, y) plane we may always assume that

$$\underline{u}_x(x_0, y_0) \cdot \underline{u}_y(x_0, y_0) = 0.$$

[Indeed, if we set

$$x' = (\cos\theta)x + (\sin\theta)y, \quad y' = (-\sin\theta)x + (\cos\theta)y,$$

we find

$$u_{x'} \cdot u_{y'} = -(|u_x|^2 - |u_y|^2)\sin\theta\cos\theta + u_x u_y(\cos^2\theta - \sin^2\theta)].$$

In addition we have

$$\underline{u} \cdot \underline{u}_x = \underline{u} \cdot \underline{u}_y = 0 \quad \text{on} \quad \Omega,$$

since $|\underline{u}|^2 = 1$ on Ω.

Therefore we may choose an orthonormal basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ in $\mathbb{R}^3$ such that (in the basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$)

$$\underline{u}(x_0, y_0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \underline{u}_x(x_0, y_0) = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \quad \underline{u}_y(x_0, y_0) = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$$

with $a \geqq 0$, $b \geqq 0$ and $a + b \neq 0$. (Notice that the basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ could possibly have a different orientation than the canonical basis of $\mathbb{R}^3$ which was used to define Q.) Let $\varepsilon > 0$ be small enough. We define a function $u^\varepsilon: \Omega \to \mathbb{R}^3$ in the following way:

Let $r = [x - x_0)^2 + (y - y_0)^2]^{1/2}$ and θ such that $x - x_0 = r\cos\theta$, $y - y_0 = r\sin\theta$. We set[1]

a) If $r > 2\varepsilon$,

$$u^\varepsilon(x, y) = \underline{u}(x, y).$$

b) If $r < \varepsilon$, we set (in the basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$)

$$u^\varepsilon(x, y) = \frac{2\lambda}{\lambda^2 + r^2}\begin{pmatrix} x - x_0 \\ y - y_0 \\ -\lambda \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

where $\lambda = c\varepsilon^2$ and c is a constant to be fixed later.

c) If $\varepsilon \leqq r \leqq 2\varepsilon$ we set (in the basis (i, j, k))

$$u^\varepsilon(x, y) = \begin{pmatrix} A_1 r + B_1 \\ A_2 r + B_2 \\ \sqrt{1 - (A_1 r + B_1)^2 - (A_2 r + B_2)^2} \end{pmatrix},$$

where A_1, A_2, B_1, B_2 depend only on θ, ε are determined in such a way as to make u^ε continuous on Ω; more precisely

$$\begin{cases} 2\varepsilon A_i + B_i = \underline{u}^i(x_0 + 2\varepsilon\cos\theta, y_0 + 2\varepsilon\sin\theta) \quad i = 1, 2 \\ \varepsilon A_1 + B_1 = \dfrac{2\lambda\varepsilon}{\lambda^2 + \varepsilon^2}\cos\theta \\ \varepsilon A_2 + B_2 = \dfrac{2\lambda\varepsilon}{\lambda^2 + \varepsilon^2}\sin\theta. \end{cases}$$

Clearly we have, as $\varepsilon \to 0$,

$$\int_{[r > 2\varepsilon]} |\nabla u^\varepsilon|^2 = \int_{[r > 2\varepsilon]} |\nabla \underline{u}|^2 = \int_\Omega |\nabla \underline{u}|^2 - 4\pi(a^2 + b^2)\varepsilon^2 + o(\varepsilon^2). \tag{4}$$

We claim that

$$\int_{[\varepsilon < r < 2\varepsilon]} |\nabla u^\varepsilon|^2 = 4\pi\varepsilon^2[a^2 + b^2 - 2c^2 + (a^2 + b^2 + 8c^2 - 4ac - 4bc)\log 2] + o(\varepsilon^2), \tag{5}$$

and

$$\int_{[r < \varepsilon]} |\nabla u^\varepsilon|^2 = 8\pi - 8\pi\varepsilon^2 c^2 + o(\varepsilon^2). \tag{6}$$

We postpone for a moment the verification of (5) and (6). Combining (4), (5), and (6) we find

$$\int_\Omega |\nabla u^\varepsilon|^2 = \int_\Omega |\nabla u|^2 + 8\pi - 4\pi\varepsilon^2[4c^2 - (8c^2 + a^2 + b^2 - 4ac - 4bc)\log 2] + o(\varepsilon^2).$$

We choose c in such a way that

$$4c^2 - (8c^2 + a^2 + b^2 - 4ac - 4bc)\log 2 > 0$$

1 A related construction appears in [12]

for example $c = \mathrm{Max}\{a/2, b/2\}$). Therefore $v = u^\varepsilon$ satisfies (3) provided ε is small enough.

Verification of (5). We have

$$\begin{aligned} A_1 &= 2(a-c)\cos\theta + o(1),\\ B_1 &= 2\varepsilon(2c-a)\cos\theta + o(\varepsilon),\\ A_2 &= 2(b-c)\sin\theta + o(1),\\ B_2 &= 2\varepsilon(2c-b)\sin\theta + o(\varepsilon), \end{aligned}$$

and similar expressions for the θ derivatives. Thus we obtain (5) since

$$\int_{[\varepsilon<r<2\varepsilon]} |\nabla u^\varepsilon|^2 = \int_{[\varepsilon<r<2\varepsilon]} \sum_{i=1}^{2} \left[|A_i|^2 + \left(A_{i\theta} + \frac{B_{i\theta}}{r} \right)^2 \right] r\,dr\,d\theta.$$

Verification of (6). We have $|\nabla u^\varepsilon|^2 = 8\lambda^2/(\lambda^2+r^2)^2$ and therefore

$$\int_{[r<\varepsilon]} |\nabla u^\varepsilon|^2 = 16\pi\lambda^2 \int_0^\varepsilon \frac{r dr}{(\lambda^2+r^2)^2},$$

which leads to (6).

We turn now to property (2). We claim that

$$Q(u^\varepsilon) = Q(\underline{u}) - 1 + 0(\varepsilon^2). \quad \text{if} \quad \mathbf{i}\cdot\mathbf{j}\wedge\mathbf{k} = +1, \tag{7}$$

and

$$Q(u^\varepsilon) = Q(\underline{u}) + 1 + 0(\varepsilon^2) \quad \text{if} \quad \mathbf{i}\cdot\mathbf{j}\wedge\mathbf{k} = -1. \tag{7'}$$

We shall verify only (7) (the proof of (7′) is identical). We write

$$\begin{aligned} Q(u^\varepsilon) &= \frac{1}{4\pi}\int_\Omega u^\varepsilon\cdot u_x^\varepsilon \wedge u_y^\varepsilon = \frac{1}{4\pi}\int_{[r>2\varepsilon]} + \frac{1}{4\pi}\int_{[\varepsilon<r<2\varepsilon]} + \frac{1}{4\pi}\int_{[r<\varepsilon]}\\ &\equiv I + II + III. \end{aligned}$$

We have

$$I = Q(\underline{u}) + 0(\varepsilon^2), \tag{8}$$

and

$$|II| \leqq \frac{1}{8\pi}\int_{[\varepsilon<r<2\varepsilon]} |\nabla u^\varepsilon|^2 = 0(\varepsilon^2) \quad \text{by (5)} \tag{9}$$

In order to evaluate III we note that in the region $[r<\varepsilon]$ we have

$$u^\varepsilon\cdot u_x^\varepsilon \wedge u_y^\varepsilon = -\frac{8\lambda^4}{(\lambda^2+r^2)^3} + \mathbf{k}\cdot u_x^\varepsilon \wedge u_y^\varepsilon,$$

and thus by (5),

$$III = -4\int_0^\varepsilon \frac{\lambda^4 r\,dr}{(\lambda^2+r^2)^3} + 0(\varepsilon^2) = -1 + 0(\varepsilon^2). \tag{10}$$

Combining (8), (9), and (10) we obtain (7).

Remark. 1. The conclusion of Lemma 2 asserts that there is some $v \in \mathscr{E}$ such that $Q(v) - Q(\underline{u}) = \pm 1$, and $E(v) < E(\underline{u}) + 8\pi$.
In general one can *not* find two v's $v_1, v_2 \in \mathscr{E}$ such that

$$\begin{cases} Q(v_1) - Q(\underline{u}) = +1 \\ Q(v_2) - Q(\underline{u}) = -1 \\ E(v_i) < E(\underline{u}) + 8\pi, \quad i = 1,2. \end{cases} \tag{11}$$

When (11) happens to be true one can prove that both $\operatorname{Inf}_{\mathscr{E}_{+1}} E$ and $\operatorname{Inf}_{\mathscr{E}_{-1}} E$ are achieved (see the proof of Theorem 1). [However there are simple examples where only one of these two infima is achieved (see Sect. 3).] Notice that (11) holds in the following cases:

a) There is some point $(x_0, y_0) \in \Omega$ such that

$$\nabla \underline{u} \neq 0 \quad \text{at} \quad (x_0, y_0),$$
$$\underline{u} \cdot \underline{u}_x \wedge \underline{u}_y = 0 \quad \text{at} \quad (x_0, y_0).$$

b) There are two points $(x_0, y_0) \in \Omega$, $(x_1, y_1) \in \Omega$ such that

$$\underline{u} \cdot \underline{u}_x \wedge \underline{u}_y > 0 \quad \text{at} \quad (x_0, y_0),$$
$$\underline{u} \cdot \underline{u}_x \wedge \underline{u}_y < 0 \quad \text{at} \quad (x_1, y_1).$$

[This is a direct consequence of the argument we have used in the proof of Lemma 2.]

2. Proof of Theorem 1.

Let $v \in \mathscr{E}$ be given by Lemma 2. We shall establish that if $v \in \mathscr{E}_1$ (respectively $v \in \mathscr{E}_{-1}$) then $\operatorname{Inf}_{\mathscr{E}_1} E$ (respectively $\operatorname{Inf}_{\mathscr{E}_{-1}} E$) is achieved. We consider just the case where $v \in \mathscr{E}_1$ (the other case is similar). Let (u^n) be a minimizing sequence, i.e. $u^n \in \mathscr{E}_1$ and $E(u^n) = \operatorname{Inf}_{\mathscr{E}_1} E + o(1)$ (as $n \to \infty$). We may extract a subsequence still denoted by u^n such that $u^n \rightharpoonup \bar{u}$ weakly in $H^1(\Omega; \mathbb{R}^3)$. Clearly $\bar{u} \in \mathscr{E}$ and $E(\bar{u}) \leqq \operatorname{Inf}_{\mathscr{E}_1} E$. It remains to prove that $\bar{u} \in \mathscr{E}_1$. Suppose by contradiction that $\bar{u} \notin \mathscr{E}_1$. It follows that

$$|Q(u^n) - Q(\bar{u})| \geqq 1. \tag{12}$$

Assume for example that

$$Q(u^n) \geqq Q(\bar{u}) + 1. \tag{13}$$

Set

$$F(v) = E(v) - 8\pi Q(v) = \int_\Omega |\nabla v|^2 - 2 \int_\Omega v \cdot (v_x \wedge v_y). \tag{14}$$

Using the same argument as in [2] (see the proof of Lemma 1) one obtains

$$F(\bar{u}) \leqq \liminf F(u^n). \tag{15}$$

Combining (13), (14) and (15) we find

$$E(\bar{u}) - 8\pi Q(\bar{u}) \leqq \liminf \{E(u^n) - 8\pi Q(\bar{u}) - 8\pi\}.$$

Hence

$$E(\bar{u}) \leqq \operatorname*{Inf}_{\mathscr{E}_1} E - 8\pi. \tag{16}$$

On the other hand, by Lemma 2, there is some $v \in \mathscr{E}_1$ such that

$$E(v) < E(\underline{u}) + 8\pi,$$

a contradiction with (16).

Remark. 2. The conclusion of Theorem 1 still holds if we replace S^2 by a Riemannian surface M homeomorphic to S^2. Using a conformal diffeomorphism between M and S^2, this amounts to establish Theorem 1 for $E'(u) = \int_\Omega g(u)|\nabla u|^2$ instead of E, where $g \in C^1(S^2; (0, \infty))$ and $u: \Omega \to S^2$. We replace Q by

$$Q'(u) = \frac{1}{\Sigma} \int_\Omega g(u) u \cdot u_x \wedge u_y,$$

where $\Sigma = \int_{S^2} g \, d\sigma$ and $u: \Omega \to S^2$. Instead of Lemma 1 and 2 we have now, with nearly the same proofs:

Lemma 1'. *Assume* $u_1, u_2 \in \mathscr{E}$, *then* $Q'(u_1) - Q'(u_2) \in \mathbb{Z}$.

Let $u' \in \mathscr{E}$ *be such that* $E'(u') = \operatorname*{Inf}_{\mathscr{E}} E'$.

Lemma 2'. *There is some* $v \in \mathscr{E}$ *such that* $|Q'(v) - Q'(u')| = 1$ *and* $E'(v) < E'(u') + 2^{\varepsilon}$.

Then we proceed as in the proof of Theorem 1.

3. A Simple Example

We consider now the case where $\gamma(x, y) = (Rx, Ry, \sqrt{1 - R^2})$ for $(x, y) \in \partial\Omega$ with $0 < R < 1$. In that case we shall give a complete description of the solution of the problems $\operatorname*{Inf}_{\mathscr{E}} E$ and $\operatorname*{Inf}_{\mathscr{E}_k} E$. For this purpose, we set

$$\lambda = \frac{1}{R} + \sqrt{\frac{1}{R^2} - 1}, \quad \mu = \frac{1}{R} - \sqrt{\frac{1}{R^2} - 1},$$

$$\underline{u}(x, y) = \frac{2\lambda}{\lambda^2 + r^2} \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \bar{u}(x, y) = \frac{2\mu}{\mu^2 + r^2} \begin{pmatrix} x \\ y \\ -\mu \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

with $(x, y) \in \Omega$ and $r^2 = x^2 + y^2$.

Theorem 2. *We have*

(A) $\underline{u} \in \mathscr{E}$ *and* $E(\underline{u}) = \operatorname*{Inf}_{\mathscr{E}} E$;

moreover $\underline{u}$ *is the unique element which minimizes E on* $\mathscr{E}$.

(B) $\bar{u} \in \mathscr{E}_{-1}$ *and* $E(\bar{u}) = \operatorname*{Inf}_{\mathscr{E}_{-1}} E$;

moreover $\bar{u}$ *is the unique element which minimizes E on* $\mathscr{E}_{-1}$.

(C) $\operatorname*{Inf}_{\mathscr{E}_k} E$ *is not achieved if* $k \notin \{0, -1\}$.

Proof. Part A. Let $\tilde{u}\in\mathscr{E}$ be such that

$$E(\tilde{u}) \leqq E(v) \quad \forall v\in\mathscr{E}. \tag{17}$$

First we claim that

$$Q(\tilde{u}) = Q(\underline{u}). \tag{18}$$

Assume by contradiction that $|Q(\tilde{u}) - Q(\underline{u})| \geqq 1$ (see Lemma 1). We introduce $w\colon \mathbb{R}^2 \to S^2$ defined as follows:

$$\begin{cases} w(x, y) = \tilde{u}(x, y) & \text{for } r < 1 \\ w(x, y) = \underline{u}\left(\dfrac{x}{r^2}, \dfrac{y}{r^2}\right) & \text{for } r > 1, \end{cases}$$

so that $w\in L^\infty(\mathbb{R}^2, S^2)$ and $w_x, w_y \in L^2(\mathbb{R}^2; \mathbb{R}^3)$. By the proof of Lemma 1 we have

$$\frac{1}{4\pi}\int_{\mathbb{R}^2} w\cdot w_x \wedge w_y = Q(\tilde{u}) - Q(\underline{u}),$$

and thus

$$\frac{1}{4\pi}\left|\int_{\mathbb{R}^2} w\cdot w_x \wedge w_y\right| \geqq 1.$$

Therefore we obtain

$$\int_{\mathbb{R}^2} |\nabla w|^2 \geqq 8\pi. \tag{19}$$

Obviously we have

$$\int_{\mathbb{R}^2} |\nabla w|^2 = E(\tilde{u}) + E(\underline{u}), \tag{20}$$

and a direct computation shows that

$$E(\underline{u}) = 4\pi(1 - \sqrt{1 - R^2}). \tag{21}$$

Combining (19), (20) and (21) we obtain

$$E(\tilde{u}) \geqq 4\pi(1 + \sqrt{1 - R^2}) > E(\underline{u})$$

—a contradiction with (17). Hence we have proved (18).

Next we consider the function $\bar{w}\colon \mathbb{R}^2 \to S^2$ defined as follows:

$$\begin{cases} \bar{w}(x, y) = \tilde{u}(x, y) & \text{for } r < 1; \\ \bar{w}(x, y) = \bar{u}\left(\dfrac{x}{r^2}, \dfrac{y}{r^2}\right) & \text{for } r > 1. \end{cases}$$

We have

$$\int_{\mathbb{R}^2} |\nabla \bar{w}|^2 = E(\tilde{u}) + E(\bar{u}) \leqq E(\underline{u}) + E(\bar{u}), \tag{22}$$

and a direct computation shows that

$$E(\bar{u}) = 4\pi(1 + \sqrt{1 - R^2}). \tag{23}$$

Combining (21), (22), and (23) we see that

$$\int_{\mathbb{R}^2} |\nabla \bar{w}|^2 \leqq 8\pi. \tag{24}$$

Moreover, we have (using (18)),

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} \bar{w} \cdot \bar{w}_x \wedge \bar{w}_y = Q(\tilde{u}) - Q(\bar{u}) = Q(\underline{u}) - Q(\bar{u}) = 1$$

(the last equality follows from a direct computation). Thus, $\int_{\mathbb{R}^2} |\nabla \bar{w}|^2 \geqq 8\pi$ and in fact (by (24)), $\int_{\mathbb{R}^2} |\nabla \bar{w}|^2 = 8\pi$. The conclusion of Lemma A.2 asserts that $\bar{w}$ is analytic on $\mathbb{R}^2$. Finally we consider $\bar{\bar{w}}: \mathbb{R}^2 \to S^2$ defined as follows:

$$\begin{cases} \bar{\bar{w}}(x, y) = \underline{u}(x, y) & \text{for } r < 1 \\ \bar{\bar{w}}(x, y) = \bar{u}\left(\dfrac{x}{r^2}, \dfrac{y}{r^2}\right) & \text{for } r > 1. \end{cases}$$

It is readily seen (by direct inspection) that $\bar{w}$ is analytic in $\mathbb{R}^2$. On the other hand we have $\bar{\bar{w}} = \bar{w}$ for $r > 1$, and therefore $\bar{\bar{w}} = \bar{w}$, i.e. $\underline{u} = \tilde{u}$.

Part B. Let $v \in \mathscr{E}_{-1}$; we shall first check that

$$E(\bar{u}) \leqq E(v). \tag{25}$$

Let $w: \mathbb{R}^2 \to S^2$ be defined as follows:

$$\begin{cases} w(x, y) = v(x, y) & \text{for } r < 1 \\ w(x, y) = \underline{u}\left(\dfrac{x}{r^2}, \dfrac{y}{r^2}\right) & \text{for } r > 1. \end{cases}$$

We have

$$\int_{\mathbb{R}^2} |\nabla w|^2 = E(v) + E(\underline{u}), \tag{26}$$

and moreover

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} w \cdot w_x \wedge w_y = Q(v) - Q(\underline{u}) = -1. \tag{27}$$

Thus

$$\int |\nabla w|^2 \geqq 8\pi. \tag{28}$$

Combining (26), (28), (21), and (23) we obtain (25).

Finally we assume in addition that

$$E(v) = E(\bar{u}) \quad \text{with } v \in \mathscr{E}_{-1}. \tag{29}$$

We deduce from (26), (29), (21), and (23) that

$$\int_{\mathbb{R}^2} |\nabla w|^2 = 8\pi.$$

Again, by Lemma A.2, w is analytic on $\mathbb{R}^2$ and we conclude as in Part A that $v = \bar{u}$.

Part C. We assume for example that $k > 0$ (the argument is similar for $k \leqq -2$). Suppose, by contradiction, that there is some $v \in \mathscr{E}_k$ such that

$$E(v) = \operatorname*{Inf}_{\mathscr{E}_k} E. \tag{30}$$

It is a well known fact that

$$\operatorname*{Inf}_{\mathscr{E}_k} E \leqq E(\underline{u}) + 8k\pi. \tag{31}$$

[The technique is similar to the one used in the proof of Lemma 2, except that it is much simpler since we don't require a strict inequality. Given $\varepsilon > 0$, one considers, for example, $v^\varepsilon : \Omega \to S^2$ such that

a) If $r > 2\varepsilon$, $v^\varepsilon(x, y) = \underline{u}(x, y)$.

b) If $r < \varepsilon$ we set

$$v^\varepsilon(x, y) = \frac{2\varepsilon^{k+1}}{\varepsilon^{2k+2} + r^{2k}} \begin{pmatrix} r^k \cos k\theta \\ -r^k \sin k\theta \\ -\varepsilon^{k+1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

c) If $\varepsilon < r < 2\varepsilon$ we proceed as in the proof of Lemma 2.

One checks that $v^\varepsilon \in \mathscr{E}_k$ and $E(v^\varepsilon) = E(\underline{u}) + 8k\pi + o(1)$.]

Finally we consider the function $w : \mathbb{R}^2 \to S^2$ defined as follows:

$$\begin{cases} w(x, y) = v(x, y) & \text{for } r < 1 \\ w(x, y) = \bar{u}\left(\dfrac{x}{r^2}, \dfrac{y}{r^2}\right) & \text{for } r > 1, \end{cases}$$

so that

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} w \cdot w_x \wedge w_y = Q(v) - Q(\bar{u}) = k + 1. \tag{32}$$

We deduce from (30) and (31) that

$$\int_{\mathbb{R}^2} |\nabla w|^2 = E(v) + E(\bar{u}) \leqq E(\underline{u}) + E(\bar{u}) + 8k\pi = 8(k+1)\pi. \tag{33}$$

Once more it follows from Lemma A.2 that w is analytic and thus (as in Part A), $v = \underline{u}$—a contradiction since $v \in \mathscr{E}_k$ $(k \neq 0)$.

Appendix

We start with a useful density result due to Schoen–Uhlenbeck [10]. For the convenience of the reader we sketch its proof.

Lemma A.1. *Given $u \in L^\infty(\mathbb{R}^2; S^2)$ with $\nabla u \in L^2(\mathbb{R}^2; \mathbb{R}^6)$ there exists a sequence (u_n) such that*

$$\begin{cases} u_n \in C^\infty(\mathbb{R}^2; S^2), \\ \textit{each } u_n \textit{ is constant far out}, \\ u_n \to u \text{ a.e.}, \\ \nabla u_n \to \nabla u \textit{ in } L^2(\mathbb{R}^2; \mathbb{R}^6). \end{cases}$$

Proof. We denote by $\pi: S^2 \to \mathbb{R}^2$ the stereographic projection which maps the south pole into 0. We set $v(p) = u(\pi(p))$ for $p \in S^2$. It is well known that $v \in H^1(S^2; S^2)$. Let $v_n(p)$ denote the average of v over $B_{1/n}(p) = \{q \in S^2; |q-p| < 1/n\}$ and thus we have

$$v_n \in C(S^2; \mathbb{R}^2) \cap H^1(S^2; \mathbb{R}^2),$$

and $v_n \to v$ in $H^1(S^2; \mathbb{R}^3)$. Note that v_n does not take its values into S^2. However Poincaré's inequality shows that

$$\int_{B_{1/n}(p)} |v(q) - v_n(p)| \, dq \leqq \frac{C}{n^2} \left(\int_{B_{1/n}(p)} |\nabla v|^2 \right)^{1/2},$$

and therefore

$$\operatorname{dist}(v_n(p), S^2) \underset{n \to \infty}{\to} 0 \text{ uniformly in } p \in S^2.$$

By a small modification of v_n we may as well assume that

$$\begin{cases} v_n \in C^\infty(S^2; \mathbb{R}^2), \\ \text{each } v_n \text{ is constant near the north pole,} \\ v_n \to v \text{ in } H^1(S^2; \mathbb{R}^3), \\ \operatorname{dist}(v_n(p), S^2) \to 0 \text{ uniformly in } p \in S^2. \end{cases}$$

Projecting $v_n(p)$ on S^2 we may further assume that $v_n(p) \in S^2$ $\forall n$, $\forall p \in S^2$. The sequence $u_n(x, y) = v_n(\pi^{-1}(x, y))$ satisfies all the required properties.

In our next lemma we extend to Sobolev classes a property which is well known for smooth maps.

Lemma A.2. *We have*

$$\int_{\mathbb{R}^2} |\nabla \phi|^2 \geqq 2 \left| \int_{\mathbb{R}^2} \phi \cdot \phi_x \wedge \phi_y \right| \quad \forall \phi \in L^\infty(\mathbb{R}^2; S^2) \quad \text{with } \phi_x, \phi_y \in L^2(\mathbb{R}^2; \mathbb{R}^3) \tag{$*$}$$

and if equality holds in (*), then ϕ is analytic.

Proof. Inequality (*) is trivial since $|\phi| = 1$. Suppose now that we have some $\phi \in L^\infty(\mathbb{R}^2; S^2)$ with $\phi_x, \phi_y \in L^2(\mathbb{R}^2; \mathbb{R}^3)$, and such that

$$\int_{\mathbb{R}^2} |\nabla \phi|^2 = 2 \int_{\mathbb{R}^2} \phi \cdot \phi_x \wedge \phi_y. \tag{A.1}$$

We shall prove that $\phi \in C^\infty(\mathbb{R}^2; S^2)$. This will imply that ϕ is analytic. Indeed if equality in (A.1) holds then ϕ is a harmonic map and thus ϕ is analytic.

We now prove that ϕ is C^∞ for example near 0. We fix $\rho > 0$ such that

$$\int_D |\nabla \phi|^2 \leqq 2\pi, \tag{A.2}$$

where $D = \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 < \rho^2\}$ and we let $\gamma = \phi|_{\partial D}$. We claim that

$$\int_D |\nabla \phi|^2 \leqq \int_D |\nabla \psi|^2, \quad \forall \psi \in H^1(D; S^2), \psi = \gamma \quad \text{on } \partial D \tag{A.3}$$

—which in turn implies that $\phi \in C^\infty(D; S^2)$ by Morrey's regularity theory.

In order to establish (A.3) we assume by contradiction that there is some $\psi \in H^1(D; S^2)$ with $\psi = \gamma$ on ∂D and

$$\int_D |\nabla \psi|^2 < \int_D |\nabla \phi|^2. \tag{A.4}$$

We have

$$\int_D \psi \cdot \psi_x \wedge \psi_y \neq \int_D \phi \cdot \phi_x \wedge \phi_y. \tag{A.5}$$

Indeed if we had

$$\int_D \psi \cdot \psi_x \wedge \psi_y = \int_D \phi \cdot \phi_x \wedge \phi_y, \tag{A.5$'$}$$

we could introduce the map $\tilde{\phi}: \mathbb{R}^2 \to S^2$ defined as follows:

$$\begin{cases} \tilde{\phi} = \psi & \text{on } D \\ \tilde{\phi} = \phi & \text{on } \mathbb{R}^2 \backslash D, \end{cases}$$

and we would find

$$\int_{\mathbb{R}^2} \tilde{\phi} \cdot \tilde{\phi}_x \wedge \tilde{\phi}_y = \int_{\mathbb{R}^2} \phi \cdot \phi_x \wedge \phi_y,$$

and

$$\int_{\mathbb{R}^2} |\nabla \tilde{\phi}|^2 = \int_D |\nabla \psi|^2 + \int_{\mathbb{R}^2 \backslash D} |\nabla \phi|^2 < \int_{\mathbb{R}^2} |\nabla \phi|^2 \quad \text{(by (A.4))}. \tag{A.6}$$

Applying (*) to $\tilde{\phi}$ and combining the resulting inequality with (A.1) and (A.6) we would obtain a contradiction. Thus we have established (A.5).

Finally we consider the map $h: \mathbb{R}^2 \to S^2$ defined as follows:

$$\begin{cases} h = \psi & \text{in } D \\ h(x, y) = \phi\left(\dfrac{\rho^2 x}{x^2 + y^2}, \dfrac{\rho^2 y}{x^2 + y^2}\right) & \text{for } (x, y) \in \mathbb{R}^2 \backslash D, \end{cases}$$

so that $h \in L^\infty(\mathbb{R}^2; S^2)$, $h_x, h_y \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ and

$$\int_{\mathbb{R}^2} h \cdot h_x \wedge h_y = \int_D \psi \cdot \psi_x \wedge \psi_y - \int_D \phi \cdot \phi_x \wedge \phi_y.$$

We deduce from (A.5) that

$$\left| \int_{\mathbb{R}^2} h \cdot h_x \wedge h_y \right| \geqq 4\pi,$$

and thus $\int_{\mathbb{R}^2} |\nabla h|^2 \geqq 8\pi$. But

$$\int_{\mathbb{R}^2} |\nabla h|^2 = \int_D |\nabla \psi|^2 + \int_D |\nabla \phi|^2 \leqq 2 \int_D |\nabla \phi|^2$$

(by (A.4)). Hence $\int_D |\nabla \phi|^2 \geqq 4\pi$—a contradiction with (A.2).

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