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SU(2) Monopoles of Charge 2

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Abstract. Using the methods of Hitchin, the moduli space of SU(2) monopoles of charge two is computed.

A. Introduction and Notation

The purpose of this paper is to compute the moduli space of SU(2) monopoles of charge two; we find it to be $\mathbb{R}^3 \times T(\mathbb{P}_2(\mathbb{R}))/\pm$, where $T(\mathbb{P}_2(\mathbb{R}))/\pm$ is the manifold obtained from the tangent bundle of $\mathbb{P}_2(\mathbb{R})$ by identifying all tangent vectors v with their inverses $-v$.

The first SU(2) monopoles of charge two were produced by Ward [8]; they possess an axis of symmetry and correspond in the moduli space to ($\mathbb{R}^3 \times$ the zero section of $T(\mathbb{P}_2(\mathbb{R}))$). Later, Ward produced a more general solution [9], but was only able to assert non-singularity in the case of solutions sufficiently close to the axisymmetric ones. It is to be noted that the more general statement of non-singularity needed here follows from the recent work of Nahm [7] and Hitchin [4].

We start by giving a brief summary of the theory; this will also serve the purpose of fixing notation. Details can be found in [3] and [4].

Let P be a principal SU(2)-bundle over $\mathbb{R}^3$, $P_\mathfrak{g}$ its associated su(2)-bundle, ϕ a section of $P_\mathfrak{g}$,∇ a connection on P, with F its associated curvature. The couple (∇, Φ) is an SU(2) monopole if the following conditions are satisfied:

1) $*F = \nabla\Phi$, where $*$ is the Hodge star operator on 2-forms over $\mathbb{R}^3$. (Bogomolny equations; see [1])

2) $\int |F|^2 < \infty$ (finite action) and $|\Phi| = 1 - k/2r + O(r^{-2})$ as $r \to \infty$ (boundary conditions). The su(2) norm is given by $-\operatorname{tr}(x^2)/2$: this is chosen to conform with Hitchin [3]. The k in $k/2r$ is an integer, and is called the *charge* of the monopole.

Monopoles are susceptible to treatment via complex geometry. To do this, one uses the space $\tilde{T}$ of oriented lines in $\mathbb{R}^3$; $\tilde{T}$ has a holomorphic structure determined by the cross product in $\mathbb{R}^3$, and $\tilde{T} \cong T(\mathbb{P}_1(\mathbb{C}))$, the holomorphic tangent bundle of $\mathbb{P}_1(\mathbb{C})$. $\tilde{T}$ has a natural real structure τ, with no fixed points, given by reversal of orientation of the lines. Also, fixing a point p, one obtains a section $s_p\colon \mathbb{P}_1(\mathbb{C}) \to \tilde{T}$, given by all the lines through p; these sections are τ-invariant, and so are called real. Finally, let x be an inhomogeneous coordinate on $\mathbb{P}_1(\mathbb{C})$; $(w, z) \to w\,\partial/\partial x|_{x=z}$ gives local coordinates on $\tilde{T}$, in which $\tau(w, z) = (-\bar{w}/\bar{z}^2 \cdot -1/\bar{z})$.

Let E be the rank 2 complex vector bundle associated to the principal bundle P; associating to each oriented line x the space of sections s over x such that $(\nabla_u - i\Phi)s = 0$ along x, u a positive unit vector, determines a bundle $\tilde{E}$ on $\tilde{T}$. If (∇, Φ) is a solution to the Bogomolny equations, $\tilde{E}$ has a natural holomorphic structure. Let L be the holomorphic line bundle on $\tilde{T}$ defined by the transition function $\exp(-w/z)$ from $\{z \neq \infty\}$ to $\{z \neq 0\}$; let $\mathcal{O}(n)$ be the lift to $\tilde{T}$ of the bundle $\mathcal{O}(n)$ on $\mathbb{P}_1(\mathbb{C})$; set $L(n) = L \otimes \mathcal{O}(n)$; then, if (∇, Φ) also satisfy the boundary conditions, $\tilde{E}$ can be written in two ways as a holomorphic extension:

$$0 \to L(-k) \to \tilde{E} \to L^*(k) \to 0,$$
$$0 \to L^*(-k) \to \tilde{E} \to L(k) \to 0.$$

These are permuted by the real structure. Let S be the curve over which $L(-k)$, $L^*(-k)$ coincide; S is a curve in the linear system $|\mathcal{O}(2k)|$, and is called the *spectral curve* of the monopole. One has:

Theorem 1 [3, 4]. i) *S is compact.*

ii) *S is preserved by τ.*

iii) *L^2 is holomorphically trivial on S.*

iv) *As $\tau^* L^2 = L^{*2}$, the natural pairing of sections of L^2 on S: $\langle s, s' \rangle = \tau^* s(s')$ is $(-1)^{k+1}$ definite.*

v) *$H^0(S, L^t(k-2)) = 0$ for $t \in (0, 2)$.*

Note that i) implies that S is of the form $0 = w^k + a_1(z)w^{k-1} + \cdots + a_k(z)$, with a_i polynomial, of degree $2i$. iv) is equivalent to asking that $L(k-1)$ have a real structure on S. v) is akin to the instantion vanishing theorem, and is the condition that ensures non-singularity.

Furthermore, from a curve S satisfying the spectral curve conditions i) to v), it is possible to recreate an $\tilde{E}$, and hence a monopole, and so one can parametrise the monopoles of a given charge by the space of corresponding spectral curves.

B. The Case of Charge Two

In the case of charge two, the curves S are in the linear system $|\mathcal{O}(4)|$; they are either smooth and elliptic, or pairs of sections $\mathbb{P}_1(\mathbb{C}) \to \tilde{T}$ (curves in $|\mathcal{O}(2)|$); the reason is that if S has singular points (if it does not, it is elliptic), the real structure forces them to come in pairs. As $(\text{section} \cap S) = 4$ points, a section through two singular points of S and another point (such sections exist) is then a component of S, which must then be the union of two sections of $|\mathcal{O}(2)|$.

1. Reduction and Symmetries

The Euclidean group on $\mathbb{R}^3$ acts on the space of divisors $|\mathcal{O}(4)|$. We start by factoring out this action, on the subspace $W \simeq \mathbb{R}^8$ of real compact curves.

The general compact, real curve of $|\mathcal{O}(4)|$ is of the form

$$0 = w'^2 + (c'_{10} + r'_1 z' - \bar{c}'_{10} z'^2) w' + (c'_{20} + c'_{21} z' + r'_2 z'^2 - \bar{c}'_{21} z'^3 + \bar{c}'_{20} z'^4), \tag{1}$$

with $c'_{ij} \in \mathbb{C}$, $r'_i \in \mathbb{R}$, and w', z' our standard coordinates on $\tilde{T}$.

a) One first factors out the translation action, by choosing an origin in $\mathbb{R}^3$; one does this by eliminating the w-term. One gets:

$$0 = w''^2 + (c''_{20} + c''_{21} z'' + r''_{22} z''^2 - \bar{c}''_{21} z''^3 + \bar{c}''_{20} z''^4). \tag{2}$$

Call the subspace of these centred divisors W_c.

b) Now, one factors out the remaining SO(3) action: the z''-term above has four roots a, b, $-1/\bar{a}$, $1/\bar{b}$; rotate so that one of these is sent to zero (another goes to infinity); then rotate around the axis in $\mathbb{R}^3$ corresponding to the points $\{0, \infty\}$ in $\mathbb{P}_1(\mathbb{C}) = \{w'' = 0\}$, so that one gets the reduced form (which is well defined)

$$w^2 = r_1 z^3 - r_2 z^2 - r_1 z, r_i \in \mathbb{R}, r_1 \geqq 0. \tag{3}$$

Call the space $[0, \infty) x \mathbb{R}$ of such reduced divisors W_{red}; one therefore has the projection map $P: W \longrightarrow W_{\text{red}}$, factoring as $P = P_c \circ Q$, with $Q: W \longrightarrow W_c$, $P_c: W_c \longrightarrow W_{\text{red}}$.

The isotropy group of this reduced form is $\mathbb{Z}_2 \times \mathbb{Z}_2$ when $r_1 > 0$ and $S^1 \times \mathbb{Z}_2$ when $r_1 = 0$ (axisymmetric case); the $\mathbb{Z}_2$ factors correspond in $\mathbb{R}^3$ to the rotations by π that permute the roots 0, ∞, a, $-1/a$ of $r_1 z^3 - r_2 z^2 - r_1 z$ in $\mathbb{P}_1(\mathbb{C}) \cong S^2$.

In passing, note that any symmetry of the spectral curve determines a symmetry of the monopole: therefore,

Proposition 2. *The symmetry subgroup (of the group of proper Euclidean motions) of a monopole of charge two is:*

a) $\mathbb{Z}_2 \times \mathbb{Z}_2$ in the non axisymmetric case, and corresponds to rotations by π around 3 orthogonal axes intersecting at a point.

b) $S^1 \times \mathbb{Z}_2$ in the axisymmetric case, and corresponds to rotations about a point mapping an axis through that point to itself.

2. Link with the Standard From of an Elliptic Curve

Consider a curve $S: w^2 = r_1 z^3 - r_2 z^2 - r_1 z$ corresponding to (r_1, r_2) in W_{red}. If $r_1 = 0$, S is the union of two sections; if $r_1 > 0$, the curve is non-singular, elliptic. Setting

$$w = \tilde{w} \cdot (r_1/4)^{1/2} \overset{\text{def}}{=} \tilde{w} \cdot k_1,$$

$$z = \tilde{z} + (r_2/3r_1) \overset{\text{def}}{=} \tilde{z} + k_2,$$

we obtain the normal form of an elliptic curve:

$$\tilde{w}^2 = 4\tilde{z}^3 - g_2\tilde{z} - g_3, \text{ with}$$
$$g_2 = 12k_2{}^2 + 4, \quad g_3 = 8k_2{}^3 + 4k_2,$$
$$27g_3{}^2 = (g_2 - 4)(g_2 + 2)^2, \quad k_2 = 3g_3/(2g_2 + 4). \tag{4}$$

Then S is the embedding in $\tilde{T}$ via the Weierstrass $\wp$-function and its derivative ($\tilde{w} = \wp'$, $\tilde{z} = \wp$) of ($\mathbb{C}$ modulo a lattice $\mathscr{L}$). As S is real, $\mathscr{L}$ is rhombic or rectangular; as the real structure has no fixed points, $\mathscr{L}$ is rectangular, with positive real and positive imaginary generators l_r, l_i. (For a detailed treatment, see [2].)

If we consider the modular function $I(g_2, g_3) = 27g_3{}^2/g_2{}^3$ of the lattice, we get I as a function of $k_2{}^2$; this is a diffeomorphism for $k_2{}^2 \geqq 0$, with $k_2 = 0$ giving $I = 0$ (square lattice); furthermore, for $I \neq 0$, there are, up to scale, two rectangular lattices with real and imaginary generators giving the same I, one horizontal ($l_r > l_i$), one vertical ($l_i > l_r$); the ratio l_i/l_r is smoothly parametrised by $I^{1/2}$, and so by k_2. Therefore, if $I(r_1, r_2)$ = modulus of S, $H(r_1, r_2) = l_i/l_r(S)$, there are smooth diffeomorphisms $\tilde{I}$, $\tilde{H}$: $\mathbb{R} \to \mathbb{R}$ such that

$$I(r_1, r_2) = \tilde{I}((r_2/r_1)^2), \quad H(r_1, r_2) = \tilde{H}(r_2/r_1).$$

3. Spectral Curve Conditions on S in W_{red}

a) $r_1 \neq 0$.
Consider again $S: w^2 = r_1 z^3 - r_2 z^2 - r_1 z$. One has:

Proposition 3.

i) *L^2 is trivial on the curve $S \Leftrightarrow 4k_1 \in$ the lattice $\mathscr{L}$.*

ii) *$H^0(S, L^t) = 0$, $t \in (0, 2)$, for S a curve with L^2 trivial $\Leftrightarrow 4k_1$ is a real generator of the lattice $\mathscr{L}$.*

iii) *When $4k_1$ is a real generator of $\mathscr{L}$, $\langle \cdot, \cdot \rangle$ is negative definite.*

Proof. i) L^2 is trivial on S iff there are holomorphic functions f_1, f_2 on $S \cap \{z \neq \infty\}$, $S \cap \{z \neq 0\}$ respectively with $f_2 = \exp(-2w/z) f_1$ on the overlap. Taking dlog, this is equivalent to

$$\operatorname{dlog}(f_2) = d(-2w/z) + \operatorname{dlog}(f_1), \tag{5}$$

where $d(-2w/z)$ is a one-form on S with double poles and no residues at $z = 0$, $z = \infty$; pulling back to $\mathbb{C}/\mathscr{L}$, $d(-2w/z) = -2k_1 d(\wp'/(\wp + k_2))$; letting u be the standard coordinate on $\mathbb{C}$, $d(-2w/z) = (-4k_1/u^2 + O(1))du$ near 0. In turn, $d\log(f_1)$, must be of the form $(4k_1/u^2 + O(1))du$; as this is its only pole on S,

$$\operatorname{dlog}(f_1) = (4k_1\wp + c)\, du, e \in \mathbb{C}.$$

Moreover, $\operatorname{dlog}(f_1)$ has periods $2\pi n i$, $n \in \mathbb{Z}$; for any $l \in \mathscr{L}$,

$$4k_1\eta(l) + cl \overset{\text{def}}{=} \int_{u_0}^{u_0 + l} (4k_1\wp + c)\, du = 2\pi i n(l), n(l) \in \mathbb{Z}. \tag{6}$$

However, one has the Legendre relation, for l_r, l_i [2]:

$$\eta(l_r)l_i - \eta(l_i)l_r = 2\pi i. \tag{7}$$

Comparing (6) and (7) gives

$$n(l_r)l_i - n(l_i)l_r = 4k_1, \tag{8}$$

and so $4k_1 \in \mathscr{L}$.

Conversely, if $4k_1 \in \mathscr{L}$, one has (8) for some $n(l_r)$, $n(l_1)$; one can choose c so that (6) holds for l_r; (7) then implies that (6) holds also for l_i, hence for all l; integrating and taking the exponential gives a section.

ii) As for i), the condition that $H^0(S, L^t) = 0$ is just that $2tk_1 \notin \mathscr{L}$; as $4k_1 \in \mathscr{L}$, $4k_1$ must be a generator.

iii) $\langle \cdot,\cdot \rangle$ is negative definite $\Leftrightarrow f_1(x) f_2(\tau(x)) = r$, $r < 0$, $r \in \mathbb{R}$. If Γ is a path on S from x to $\tau(x)$, this is equivalent to asking that $\int_{\Gamma + \tau(\Gamma)} (4k_1 \wp + c)du = 2\pi n i$, n odd. (This property is independent of the Γ chosen). The real structure on $\mathbb{C}/\mathscr{L}$ must be a lattice preserving map of the form $u \to a\bar{u} + b$, with $|a| = 1, b$ a half-period ([2]); τ acts continuously on a family of real elliptic curves in $\tilde{T}$; considering the rectangular lattices in the family gives $a = \pm 1$; considering the square lattice of the family ($k_2 = 0$) and the action of τ on $\tilde{T}$ gives that $\wp(u)$ real, $\wp'(u)$ imaginary positive $\Rightarrow \wp'(\tau(u))$ imaginary positive, which, referring to [2], yields $a = -1, b = (l_r + l_i)/2$. Taking Γ as the segment $[0, (l_r + l_i)/2]$ then gives $l_i \cong \Gamma + \tau(\Gamma)$; if $4k_1$ is a real generator of the lattice, (8) gives $n(l_i) = -1$, and so

$$\int_{\Gamma + \tau(\Gamma)} (4k_1 \wp + c)\, du = -2\pi i.$$

b) $r_1 = 0$.

Proposition 4. *The curve $S: w^2 = -(\pi z/2)^2$ is the only curve with $r_1 = 0$ in W_{red} satisfying the spectral curve conditions.*

Proof: The curves in W_{red} with L^2 trivial, $r_1 = 0$ are given by $r_2 = [(2n-1)\pi/2]^2$, $n \in N$; see [3]. The condition $H^0(S, L^t) = 0$ gives, by the same type of argument, $n = 1$. Finally, for $n = 1$, we can take $f_1(z) = 1$, $f_2(z) = -1$, and so $\langle \cdot,\cdot \rangle$ is negative definite.

We now fit Propositions 3 and 4 together, and obtain:

Proposition 5. i) *The set C of spectral curves in W_{red} is a smooth curve intersecting $r_1 = 0$ transversely at $r_2 = \pi^2/4$.*

ii) *In a neighbourhood of this point, C can be described by $f(r_1, r_2) = 0$, with f a smooth function, even in r_1, with $\partial f/\partial r_2 \neq 0$.*

Proof. For $r_1 > 0$, we show that C is the graph in polar coordinates (r, θ) of a smooth function $r = g(\theta)$. On $r_1 > 0$, C is the set of elliptic curves with $4k_1$ generating $\mathscr{L}$. Now for a curve in C, $\tilde{H}(r_2/r_1)$, $g_2(r_2/r_1)$ are smooth functions of θ; however, for a fixed $\mathscr{L}$, $g_2(m\mathscr{L}) = m^{-4} g_2(\mathscr{L})$, and so the real generator $4k_1$ of $\mathscr{L}$ is a smooth well defined function of r_1/r_2, for $\mathscr{L}$ corresponding to a curve in C; but $4k_1{}^2 = r_1$, and so C is as claimed.

Consider now how C behaves as $r_2/r_1 \to +\infty$; from (4),

$$\frac{r_2}{3r_1} = k_2 = \frac{3(4k_1)^{-6}}{2(4k_1)^{-4}} \cdot \frac{G_3(k_2)}{(G_2(k_2) + 2(4k_1)^4)}$$

where G_2, G_3 are g_2, g_3 of our lattice, normalised so that the real generator is 1. This yields

$$r_2 = \frac{9}{8} \frac{G_3(k_2)}{(G_2(k_2) + 32r_1{}^2)},$$

$k_2 \to \infty$ implies $G_3(k_2) \to 8\pi^6/27$, $G_2(k_2) \to 4\pi^4/3$ [5]. Also, $k_2 \to \infty$ means $g_2 \to \infty$: the scale of the lattice (which is vertical, for $k_2 > 0$) tends to zero, and so $r_1{}^2 = (2k_1)^4 \to 0$; thus

$$\lim_{k_2 \to \infty} = \frac{\pi^2}{4},$$

i.e., the curve C tends to the axisymmetric case. Also, expanding in series [5], one has

$$G_2(s) = \frac{(2\pi)^4}{12}\left(1 + 240 \sum_{n=1} \sigma_3(n) q^n\right),$$

$$G_3(s) = \frac{(2\pi)^6}{216}\left(1 - 504 \sum_{n=1} \sigma_5(n) q^n\right),$$

$$(1 - I(s)) = \frac{G_2{}^3 - 27G_3{}^2}{G_2{}^3} = 1728\left(q + \sum_{n>2} a_n q^n\right),$$

where $\sigma_k(n) = \sum_{d|n} d^k$, and $G_2(s)$, $G_3(s)$, $1 - I(s)$ are computed for the lattice generated by $(1, s)$, $\operatorname{Im}(s) > 0$, and $q = \exp(2\pi i s)$. Inverting near $q = 0$, one can write G_2, G_3 as $G_2(1 - I)$, $G_3(1 - I)$; but at $1 - I = 0$, I is a smooth function of $(r_1/r_2)^2$, and so, near $(r_1, r_2) = (0, \pi^2/4)$, C is defined by $F(r_1, r_2) = 0$, with

$$F(r_1, r_2) = r_2 - \frac{9}{8} \frac{G_3((r_1/r_2)^2)}{(G_2((r_1/r_2)^2) + 32r_1{}^2)},$$

which proves ii).

Note. One can show that $r_2 \to -\infty$, $r_1 \to 0$ as $k_2 \to -\infty$. One uses the case $k_2 \to +\infty$, plus the fact that $\pm k_2$ determine identical lattices, but with one horizontal, one vertical. The spectral curve thus tends to the union of two well separated real sections; as these are the spectral curves of monopoles of charge one, one sees that this limiting case conforms with Taubes' construction of n-monopoles by "glueing" well separated monopoles of charge 1 [6].

4. $P^{-1}(C)$, *C a Curve in* W_{red}

We have computed the curve C of spectral curves (r_1, r_2) in W_{red} satisfying our general conditions for determining a bundle E that generates a monopole. The space M of all such curves is just $P^{-1}(C)$, where $P:W\to W_{\text{red}}$ was our map defined in 1). We now prove a smoothness criterion for $P^{-1}(C)$.

Proposition 6. *Let C be a smooth curve in W_{red}, intersecting $\{0\}\times\mathbb{R}$ transversally in a discrete set, with $(0,0)\notin C$. $P^{-1}(C)$ is smooth* iff *the following condition (*) holds: (*) for each point p in $C\cap\{0\}\times\mathbb{R}$, the curve can be expressed locally as $f(r_1,r_2)=0$, with f a smooth function, even in r_1, with $\partial f/\partial r_2\neq 0$ at p.*

Proof. $\Leftarrow$): As P factors as $P_c\circ Q$, and Q is just the factoring out of the translation action, it suffices to show $P_c^{-1}(C)$ smooth. Away from $\{0\}\times\mathbb{R}$ in W_{red}, P_c is a submersion; the only points of C for which we have to check smoothness are those in $\{0\}\times\mathbb{R}$. Let $(r_1,r_2)=(0,x)\in C$; $(c_{20},c_{21},r_2)=(0,0,x)$ is in $P_c^{-1}(C)$; we show that $P_c^{-1}(C)$ is smooth at this point; because of the group action, this is sufficient.

i) P_c restricted to $W_i=\{(0,c_{21},r_2)\in W_c\}$ is just $(0,c_{21},r_2)\to(|c_{21}|,r_2)$, and so at $(0,0,x)$, $P_c^{-1}(C)\cap W_i$ is smooth iff (*) holds.

ii) $P_c^{-1}(C)$ smooth iff $P_c^{-1}(C)\cap W_i$ smooth at $(0,0,x)$: In a neighbourhood of $(0,0,x)$, $x\neq 0$, $W_c\cong S^2(U)\times I$, where $S^2(U)$ is the symmetric square of a neighbourhood U of zero in $\mathbb{C}$, and I is an interval in $\mathbb{R}$; one has an unordered pair of roots, a, b defining the intersection $z=a,\ b,\ -1/\bar{a},\ -1/\bar{b}$ of the divisor with $w=0$, and an SO(3)-invariant scale factor r; along $(0,0,y)$, $y\in\mathbb{R}$ in W_c the scale factor can be taken as y. Similarly, $W_i\cong U\times I$.

Let Sym: $U\times U\times I\to S^2(U)\times I$ be the natural projection; we consider $(P_c\circ\text{Sym})^{-1}(C)$. Define $T_a(z)$: $U\to\mathbb{P}_1(\mathbb{C})$ by $z\to(z-a)/(\bar{a}z+1)$; note that if a and b are two points in U, there is a real constant c such that $T_a(b)=\exp(2\pi ic)T_b(a)$. Let $F(a,r)$: $U\times I\to\mathbb{R}$ denote an axially symmetric (i.e., $F(a,r)=F(ca,r)$, for $|c|=1$) smooth function defining $P_c^{-1}(C)\cap W_i$ at $(0,0,x)$, with $\partial F/\partial r\neq 0$ at $(0,0,x)$. We define the smooth composition

$$\tilde{F}(a,b,r)=F(T_a(b),r);$$

this defines $(P_c\circ\text{Sym})^{-1}(C)$ locally, and $\tilde{F}(a,b,r)=\tilde{F}(b,a,r)$; $\tilde{F}$ then factors to a smooth function on $S^2(U)\times I$ defining $P_c^{-1}(C)$ locally, with $\partial\tilde{F}/\partial r\neq 0$ at $(0,0,x)$.
$\Rightarrow$): W_{red} embeds in W_c naturally; at $(0,x)\in C$, $\partial/\partial r_2$ is transversal to C in W_{red}, by hypothesis; but $\partial/\partial r_2$ is also transversal to the group action at $(0,0,x)$ and so $\partial/\partial r_2$ is transversal to $P_c^{-1}(C)$ in W_c; the two-plane containing W_{red} thus intersects $P_c^{-1}(C)$ transversally, and so the condition (*) is realised.

5. *The moduli space*

Let M be $P^{-1}(C)$; $M\cong$ moduli space of monopoles of charge two.

Theorem 7. i) *M is a 7-dimensional manifold, smoothly embedded in $W=\mathbb{R}^8$.*

ii) *M is diffeomorphic to $\mathbb{R}^3\times T(\mathbb{P}_2(\mathbb{R}))/\pm$, where $T(\mathbb{P}_2(\mathbb{R}))/\pm$ is the tangent bundle of the real projective plane, with the vectors v, $-v$ identified for all v.*

Proof. i) is a consequence of Propositions 5 and 6.

ii) $P^{-1}(C)\cong\mathbb{R}^3\times T(\mathbb{P}_2(\mathbb{R}))/\pm$ iff $P_c^{-1}(C)\cong T(\mathbb{P}_2(\mathbb{R}))/\pm$.

W_c is a 5 dimensional irreducible representation of SO(3), and W_{red} is the quotient space. Think of W_c as the space of symmetric, tracefree real 3×3 matrices, and of W_{red} as the space of diagonalised matrices in W_c, with decreasing values along the diagonal; let $(0,a)$ in W_{red} correspond to the diagonal matrices with eigenvalues $a/\sqrt{6}$, $a/\sqrt{6}$, $-2a/\sqrt{6}$,. C is diffeomorphic to the curve of unit vectors in W_{red} minus the point $(0,-1)$: see the proof of Proposition 6. The inverse image of $(0,-1)$ is the set of matrices of norm 1 in W_c with two equal, negative eigenvalues. This is just $\mathbb{P}_2(\mathbb{R})$, as such a matrix is determined by its positive eigenspace. $P_c^{-1}(C)$ is therefore $S^4 - \mathbb{P}_2(\mathbb{R})$; to see that this is $T(\mathbb{P}_2(\mathbb{R}))/\pm$, note that an element of $P_c^{-1}(C)$ corresponds to a matrix which is determined by three orthogonal eigenspaces, with eigenvalues $a_1 \geqq a_2 > a_3$, $\Sigma a_i = 0$, $\Sigma a_i^2 = 1$. The eigenspace corresponding to a_3 determines a point x in $\mathbb{P}_2(\mathbb{R})$; the eigenspace corresponding to a_2, when $a_2 > a_1$, determines a direction in the tangent space T_x of x; $(1-(a_2/a_1))/(1+2(a_2/a_1))$ can then determine a norm; thus the matrix corresponds to a couple v, $-v$ in the tangent bundle; the case $a_2 = a_1$ corresponds to the zero section of the tangent bundle.

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