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Exponential Lower Bounds to Solutions of the Schrödinger Equation: Lower Bounds for the Spherical Average

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Abstract. For a large class of generalized N-body-Schrödinger operators, H, we show that if $E<\Sigma=\inf\sigma_{\text{ess}}(H)$ and ψ is an eigenfunction of H with eigenvalue E, then

$$\lim_{R\to\infty} R^{-1}\ln\left(\int_{S^{n-1}}|\psi(R\omega)|^2 d\omega\right)^{1/2}=-\alpha_0,$$

with α_0^2+E a threshold. Similar results are given for $E\geqq\Sigma$.

I. Introduction

In this paper we will be concerned with operators of the form

$$H=-\Delta+V(x) \tag{1.1}$$

in $L^2(\mathbb{R}^n)$, where

$$V(x)=\sum_{i=1}^{M} v_i(\pi_i x). \tag{1.2}$$

In (1.2) π_i is the orthogonal projection onto a subspace X_i of $\mathbb{R}^n$ and v_i is a real valued function on X_i satisfying

$$v_i(-\Delta_i+1)^{-1} \quad \text{is compact}, \tag{1.3}$$

$$(-\Delta_i+1)^{-1}y\cdot\nabla v_i(y)(-\Delta_i+1)^{-1} \quad \text{extends to a compact operator}. \tag{1.4}$$

Here Δ_i is the Laplacian in $L^2(X_i)$. By (1.4) we mean the following: Let $\mathcal{S}(X_i)$ be the Schwartz space of test functions on X_i and T_i the tempered distribution given by $y\cdot\nabla v_i(y)$. Define the sesquilinear form

$$q(f,g)=T_i((-\Delta_i+1)^{-1}\bar{f}(-\Delta_i+1)^{-1}g)$$

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on $\mathscr{S}(X_i)\times\mathscr{S}(X_i)$. Then by definition (1.4) is satisfied if $q(f,g)$ extends to the form of a compact operator on $L^2(X_i)$.

The set $\mathscr{T}(H)$ of thresholds of the self-adjoint operator H is defined as follows: $\lambda\in\mathscr{T}(H)$ if and only if either $\lambda=0$ or λ is an eigenvalue of some subsystem operator, H_X, where

$$H_X=-\Delta_X+\sum_{\{i:X_i\subset X\}} v_i(\pi_i x).$$

Here X is a *proper* subspace of $\mathbb{R}^n$ and Δ_X is the usual Laplace operator in $L^2(X)$.

In [6] the following result was proved:

Theorem 1.1. *Suppose H is given by* (1.1) *with v_i satisfying* (1.3) *and* (1.4). *Suppose $H\psi=E\psi$. Define*

$$\alpha_0=\sup\{\alpha:\exp(\alpha|x|)\psi\in L^2(\mathbb{R}^n)\}.$$

Then either $\alpha_0=+\infty$ or $\alpha_0^2+E\in\mathscr{T}(H)$.

Under certain regularity assumptions on the potential, the possibility that $\alpha_0=+\infty$ can be eliminated. This was shown in [6, 8]:

Theorem 1.2. *Suppose H is as in Theorem 1.1. Let $v_i=\dim X_i$ and $p_i=\mathrm{Max}(v_i-1,2)$. Suppose either*

(a) *for some b_1 and b_2 with $b_1<2$*

$$x\cdot\nabla V\leqq -b_1\Delta+b_2,$$

or

(b) *each $v_i\in L^{p_i}(X_i)+L^\infty(X_i)$ and $v_i=v_i^{(1)}+v_i^{(2)}$, with $(1+|y|)v_i^{(1)}\in L^{p_i}(X_i)+L^\infty(X_i)$ and for each $\varepsilon>0$,*

$$y\cdot\nabla v_i^{(2)}\leqq -\varepsilon\Delta_i+b_\varepsilon \quad \textit{for some } b_\varepsilon.$$

Suppose $H\psi=E\psi$ with $\psi\neq 0$, and let α_0 be as in Theorem 1.1. Then $\alpha_0<\infty$. In addition $\mathscr{T}(H)\cup\sigma_{\mathrm{p.p.}}(H)\subset(-\infty,0]$.

These theorems contain upper L^2 exponential bounds of the form $\|\exp(\alpha|x|)\psi\|<\infty$; $\alpha<\alpha_0$. As is well known [1,2], in the presence of some additional regularity of V this information can be converted into a pointwise bound for the eigenfunction:

$$|\psi(x)|\leqq c_\alpha\exp(\alpha|x|)\,; \qquad \alpha<\alpha_0.$$

For more information about upper bounds we refer the reader to [1]. Our main concern in this paper is the conversion of the crude L^2 exponential lower bound given by Theorem 1.1, $\|\exp(\alpha|x|)\psi\|=\infty$; $\alpha>\alpha_0$, into something closer to a pointwise lower bound.

For the ground state of $-\Delta+V$, which can be taken everywhere positive, pointwise lower bounds of the form $\psi(x)\geqq c_\gamma\exp(-\gamma\varrho_E(x))$; $\gamma>1$ are known [4]. Here $c_\gamma>0$ and $\varrho_E(x)$ is the distance to the origin given by the "Agmon metric" [1] (which depends on the eigenvalue E). For other eigenfunctions, the nodal surface, $\{x:\psi(x)=0\}$, is in general unbounded and quite complicated so that a pointwise lower bound for $|\psi(x)|$ of any consequence could not be simple.

Let $B(x;r)$ be the ball of radius $r>0$, centered at x, and $\|\psi\|_B^2 = \int_B |\psi|^2 d^n x$. One might expect lower bounds of the form $\|\psi\|_{B(x;r)} \geqq c_\gamma \exp(-\gamma \varrho_E(x))$; $\gamma > 1$, where again $\varrho_E(x)$ is the distance to the origin given by the Agmon metric appropriate to the eigenvalue E. However, this is not the case in general [8]. In particular it is well known that even if each v_i is in $C_0^\infty(X_i)$, eigenvalues of H can be embedded in the continuous spectrum of H with corresponding eigenfunctions decaying exponentially in all directions. In this case $\varrho_E(x)$ is not even defined. The general situation is complicated as can be seen by an analysis of examples similar to Example 4.3 in [8]. One approach to the problem is to solve the Schrödinger equation explicitly for large $|x|$. This line of attack was used by Mercuriev [11] in the three-body problem. To illustrate the nature of the problem as we see it, we introduce some notation and formulate a conjecture:

Let $\mathscr{L}$ be the family of subspaces of $\mathbb{R}^n$ which contains $\{0\}$ and all subspaces of the form $X_{i_1} + \ldots + X_{i_l}$; $1 \leqq i_j \leqq M$. For each $\omega \in S^{n-1}$ let $X(\omega)$ be the largest subspace in $\mathscr{L}$ which is perpendicular to ω. For each X in $\mathscr{L}$ define

$$H(X) = -\Delta + \sum_{\{i : X_i \subset X\}} v_i(\pi_i x),$$

and

$$S(X) = \{\omega \in S^{n-1} : X(\omega) = X\}.$$

It is not difficult to see that $S(X)$ is a non-empty relatively open subset of $S^{n-1} \cap X^\perp$. It is thus a union of connected components, $S(X, \beta)$, $\beta = 1, 2, \ldots, n(X)$ where as is easily seen $n(X)$ is finite. Clearly if $X \neq Y$, $S(X) \cap S(Y) = \emptyset$, and in addition $\bigcup_{X \in \mathscr{L}} S(X) = S^{n-1}$. Given $E \in \mathbb{R}$, a function $t : S^{n-1} \to \mathscr{T}(H)$ is said to be E-admissable if

(1) $t(\omega)$ is a threshold of $H(X)$ for each $\omega \in S(X)$,
(2) t is constant on $S(X, \beta)$, and,
(3) $t(\omega) \geqq E$.

Conjecture. *Suppose H is as in Theorem 1.2 and $H\psi = E\psi$ with $\psi \neq 0$. Then there exists an E-admissable function, t_ψ, with the following property. Let $\varrho_\psi(x)$ be the distance from x to the origin computed in the metric $(ds)^2 = (t_\psi(x/|x|) - E)|dx|^2$. Then for each $r > 0$*

$$\lim_{R \to \infty} R^{-1}(\ln \|\psi\|_{B(R\omega; r)} + \varrho_\psi(R\omega)) = 0,$$

uniformly for $\omega \in S^{n-1}$.

Unfortunately we are far from proving such a result. We will instead forget about trying to prove direction dependent estimates and concentrate on obtaining lower bounds for the average of $|\psi|^2$ on a sphere. As was remarked in [8], if $E \notin \sigma_{\text{ess}}(H)$, the average of $|\psi|^2$ over a sphere of large enough radius cannot vanish and thus the quantity is a reasonable one to investigate in proving lower bounds. If $E \in \sigma_{\text{ess}}(H)$, we cannot eliminate the possibility that ψ can vanish on a sphere of arbitrarily large radius. Thus in Sect. II we prove lower bounds on the integral of $|\psi|^2$ over a spherical shell, which hold for arbitrary eigenvalues E. The lower bounds for the integral of $|\psi|^2$ over a sphere are then a corollary of the latter estimates in conjunction with estimates proved in Sect. III concerning the Dirichlet problem for $-\Delta+V$ in the region $\Omega(R)=\{x:|x|>R\}$.

Lower bounds for the average of $|\psi|^2$ over a sphere of radius R were proved by Bardos and Merigot [3] for $V\in C^\infty(\Omega(R_0))$ with $\lim_{|x|\to\infty}(|V(x)|+|x\cdot\nabla V(x)|)=0$. Their work motivated ours. In addition our method of analyzing the Dirichlet problem in $\Omega(R)$ (see Sect. III) borrows heavily from their work. However the techniques of Sect. II come directly from [6].

The problems analyzed here are treated in greater depth in the dissertation of Froese. In particular, it turns out that the potential v_{i_0} associated with the projection $\pi_{i_0}=I$ deserves special treatment. Some of the results given in Sect. II are also true under the assumption that $(1+|x|)^{1/2}v_{i_0}(-\Delta+1)^{-1}$ is compact [5] (see also [8, 9]).

II. Lower Bounds

Our first main result involves lower bounds for the L^2 norm of ψ over a spherical shell $\Omega(R_1,R_2)=\{x:R_1<|x|<R_2\}$. We use the notation

$$\|\phi\|_{\Omega(R_1,R_2)}=\left(\int_{\Omega(R_1,R_2)}|\phi(x)|^2d^nx\right)^{1/2}$$

for this norm.

Theorem 2.1. *Suppose $H=-\Delta+V$ in $L^2(\mathbb{R}^n)$ with $V(x)=\sum_{j=1}^{M}v_j(\pi_jx)$, and that the v_j satisfy* (1.3) *and* (1.4). *Suppose $H\psi=E\psi$. Let*

$$\alpha_0=\sup\{\alpha:\exp(\alpha|x|)\psi\in L^2(\mathbb{R}^n)\}. \tag{2.1}$$

(Note that α_0 may be $+\infty$.) Suppose that the positive function $\delta(R)$ does not decrease too rapidly in the sense that

$$\liminf_{R\to\infty}R^{-1}\ln\delta(R)\geqq 0. \tag{2.2}$$

Then

$$\lim_{R\to\infty}R^{-1}\ln\|\psi\|_{\Omega(R,R+\delta(R))}=-\alpha_0. \tag{2.3}$$

Let $d\omega$ be Lebesgue measure on S^{n-1}. Suppose ψ and H are as in Theorem 2.1. It then follows (see Sect. III) that

$$|\psi|_R\equiv\left(\int_{S^{n-1}}|\psi(R\omega)|^2d\omega\right)^{1/2}$$

is equal almost everywhere to a continuous function. It is this continuous function to which we refer in our second main result:

Theorem 2.2. *Suppose that in addition to the hypotheses of Theorem 2.1 we also have $E<\inf\sigma_{\mathrm{ess}}(H)$. Then*

$$\lim_{R\to\infty}R^{-1}\ln|\psi|_R=-\alpha_0. \tag{2.4}$$

Theorem 2.2 is an easy corollary of Theorem 2.1 and the following result proved in Sect. III.

Theorem 2.3. *Suppose* $H = -\Delta + V$, *where* $V(x) = \sum_{i=1}^{M} v_i(\pi_i x)$ *and the* v_j *satisfy* (1.3). *Suppose* $H\psi = E\psi$. *Then for some constant* c_1 *and* $R > 1$,

$$|\psi|_R \leqq c_1 R^{-(n-1)/2} \|\psi\|_{\Omega(R-1)}. \tag{2.5}$$

If in addition $E < \inf \sigma_{\mathrm{ess}}(H)$ *then for some constants* c_2 *and* $R_0 > 0$ *and all* $R \geqq R_0$,

$$\|\psi\|_{\Omega(R)} \leqq c_2 R^{n/2} |\psi|_R. \tag{2.6}$$

We do not give the proof of Theorem 2.2 nor do we prove the simple upper bound inherent in (2.3). Rather, the rest of this section is devoted to showing that if $\alpha_0 < \infty$, then

$$\liminf_{R\to\infty} R^{-1} \ln \|\psi\|_{\Omega(R, R+\delta(R))} \geqq -\alpha_0. \tag{2.7}$$

We begin with a computation analogous to Lemma 2.2 of [6].

Lemma 2.4. *Let* H *and* ψ *be as in Theorem 2.1 and suppose* χ *is a real function in* $C_0^\infty(\mathbb{R}^n \setminus \{0\})$. *Let* $A = x \cdot \nabla + n/2$ *and* $\psi_\alpha = \exp(\alpha r)\psi$, $r = |x|$. *Then* $\chi\psi_\alpha \in D(H)$ *and*

$$(H - E - \alpha^2)\chi\psi_\alpha = -2\alpha r^{-1} A \chi\psi_\alpha + f(\chi), \tag{2.8}$$

$$(\chi\psi_\alpha, (H - E - \alpha^2)\chi\psi_\alpha) = Q_1(\chi), \tag{2.9}$$

$$(\chi\psi_\alpha, [H, A]\chi\psi_\alpha) = -4\alpha \|r^{-1/2} A\chi\psi_\alpha\|^2 + Q_2(\chi), \tag{2.10}$$

where

$$f(\chi) = \alpha r^{-1}\chi\psi_\alpha + 2\alpha r^{-1}(x\cdot\nabla\chi)\psi_\alpha - (\Delta\chi)\psi_\alpha - 2\nabla\chi\cdot\nabla\psi_\alpha, \tag{2.11}$$

$$Q_1(\chi) = (\psi_\alpha, \{\alpha r^{-1} x \cdot \nabla(\chi^2) + (\nabla\chi)^2\}\psi_\alpha), \tag{2.12}$$

$$Q_2(\chi) = 2\,\mathrm{Re}\{(A\chi\psi_\alpha, \nabla\chi\cdot\nabla\psi_\alpha) + (A\chi\psi_\alpha, (\Delta\chi - 2\alpha r^{-1} x\cdot\nabla\chi)\psi_\alpha)\} + \alpha\|r^{-1/2}\chi\psi_\alpha\|^2. \tag{2.13}$$

Note that we do not assume $\psi_\alpha \in L^2(\mathbb{R}^n)$. However, since $\chi \in C_0^\infty(\mathbb{R}^n\setminus\{0\})$, all terms above make sense when properly interpreted. We leave the computation to the reader (see also [5, 6, 8]).

Now suppose (2.7) is false. Clearly we can assume $\delta(R) \leqq 1$. There is thus a $\beta > \alpha_0$ and a sequence $R_i \uparrow \infty$ with $R_{i+1} > 1 + R_i$ such that with $\Omega_i = \Omega(R_i, R_i + \delta(R_i))$,

$$\|\psi\|_{\Omega_i} \leqq e^{-\beta R_i}.$$

Now choose α such that $\alpha_0 < \alpha < \beta$ and $\alpha^2 + E \notin \mathcal{T}(H)$. [This is possible because $\mathcal{T}(H)$ is countable.] Then

$$\|\psi_\alpha\|_{\Omega_i} \leqq c e^{-(\beta-\alpha)R_i}. \tag{2.14}$$

Let $\eta \in C^\infty(\mathbb{R})$ be such that $0 \leqq \eta \leqq 1$ and $\eta(t) = 0$ if $t \leqq 0$, and $\eta(t) = 1$ if $t \geqq 1$. Define

$$\eta_i(x) = \eta((|x| - R_i)/\delta(R_i)),$$

$$\eta_{ij}(x) = \eta_i(x)(1 - \eta_j(x)).$$

Note that since by definition of α_0, $\|\psi_\alpha\| = \infty$, we have for each i

$$\lim_{j\to\infty} \|\eta_{ij}\psi_\alpha\| = \infty\,.$$

Thus we can find $j(i) > i$ so that with $\chi_i = \eta_{ij(i)}$,

$$\lim_{i\to\infty} \|\chi_i\psi_\alpha\| = \infty\,. \tag{2.15}$$

We now prove some simple estimates:

Lemma 2.5. *Let $\chi_i^\lambda = \partial^\lambda \chi_i$, where λ is a multi-index with $|\lambda| \geqq 1$. Then for any $\gamma \geqq 0$,*

$$\lim_{i\to\infty} \|r^\gamma \chi_i^\lambda \psi_\alpha\| = 0\,, \tag{2.16}$$

$$\lim_{i\to\infty} \|r^\gamma \chi_i^\lambda \nabla\psi_\alpha\| = 0\,. \tag{2.17}$$

In addition we have

$$\|(-\Delta + 1)(\chi_i\psi_\alpha)\| \leqq \text{const}\,\|\chi_i\psi_\alpha\|\,. \tag{2.18}$$

Proof. We have

$$\begin{aligned}\|r^\gamma \chi_i^\lambda \psi_\alpha\| \leqq c\{&(R_i+1)^\gamma [\delta(R_i)]^{-|\lambda|} \|\psi_\alpha\|_{\Omega_i} \\ &+ (R_{j(i)}+1)^\gamma [\delta(R_{j(i)})]^{-|\lambda|} \|\psi_\alpha\|_{\Omega_{j(i)}}\}\,,\end{aligned}$$

so that (2.16) follows from (2.2) and (2.14). Let us now use (2.9) with $\chi = r^\gamma \chi_i^\lambda$. Clearly from (2.16), $Q_1(r^\gamma\chi_i^\lambda) \to 0$ as $i\to\infty$. Since $-\Delta \leqq c_1 H + c_2$, we thus have

$$\lim_{i\to\infty} \|\nabla(r^\gamma \chi_i^\lambda \psi_\alpha)\| \leqq (\text{const}) \lim_{i\to\infty} \|r^\gamma \chi_i^\lambda \psi_\alpha\| = 0\,.$$

Thus another application of (2.16) gives (2.17).

From (2.12) and (2.16) we have $Q_1(\chi_i)/\|\chi_i\psi_\alpha\| \to 0$, so that from (2.9)

$$\|\nabla(\chi_i\psi_\alpha)\| \leqq \text{const}\,\|\chi_i\psi_\alpha\|\,. \tag{2.19}$$

Finally note that $\|f(\chi_i)\|/\|\chi_i\psi_\alpha\| \to 0$, so that from (2.8)

$$\lim_{i\to\infty} (\|(H - E - \alpha^2)\chi_i\psi_\alpha\| - 2\alpha\|r^{-1}A\chi_i\psi_\alpha\|)/\|\chi_i\psi_\alpha\| = 0\,. \tag{2.20}$$

Since $\|r^{-1}A\chi_i\psi_\alpha\| \leqq \text{const}\,\|\chi_i\psi_\alpha\|$ by (2.19), (2.18) follows from (2.20).

Now let $\Psi_i = \chi_i\psi_\alpha/\|\chi_i\psi_\alpha\|$. From (2.20) we have

$$\lim_{i\to\infty} (\|(H - E - \alpha^2)\Psi_i\| - 2\alpha\|r^{-1}A\Psi_i\|) = 0\,. \tag{2.21}$$

Now consider (2.10). Using Lemma 2.5 and the Schwarz inequality in (2.13), we find

$$\lim_{i\to\infty} Q_2(\chi_i)/\|\chi_i\psi_\alpha\|^2 = 0\,,$$

so that from (2.10)

$$\lim_{i\to\infty} \{(\Psi_i, [H, A]\Psi_i) + 4\alpha\|r^{-1/2}A\Psi_i\|^2\} = 0\,. \tag{2.22}$$

Since (2.18) implies $\sup_i \|(-\Delta+1)\Psi_i\| < \infty$, and since $(-\Delta+1)^{-1}[H,A] \cdot(-\Delta+1)^{-1}$ is bounded by hypothesis, it follows from (2.22) that

$$\sup_i \| r^{-1/2} A\Psi_i\| < \infty .$$

Hence because $\operatorname{supp}\chi_i \subset \{x : |x| \geqq R_i\}$, we have

$$\lim_{i\to\infty} \|r^{-1} A\Psi_i\| \leqq \lim_{i\to\infty} R_i^{-1/2} \|r^{-1/2} A\Psi_i\| = 0 .$$

Thus the important relation

$$\lim_{i\to\infty} \|(H-E-\alpha^2)\Psi_i\| = 0 \tag{2.23}$$

follows from (2.21).

If J is any open interval containing $E+\alpha^2$, it easily follows from (2.23) that with $E_H(\cdot)$ the spectral projection for H,

$$\lim_{i\to\infty} \|(1-E_H(J))\Psi_i\| = 0 . \tag{2.24}$$

It thus follows from (2.22) that

$$\limsup_{i\to\infty} (\Psi_i, E_H(J)[H,A]E_H(J)\Psi_i) \leqq 0 . \tag{2.25}$$

We now make use of the Mourre estimate [7, 12, 13] which states that given $\lambda \in \mathbb{R}\backslash\mathscr{T}(H)$ there is a $c_0 > 0$, an open interval J containing λ, and a compact operator K_0 such that

$$E_H(J)[H,A]E_H(J) \geqq c_0 E_H(J) + K_0 . \tag{2.26}$$

Taking $\lambda = \alpha^2 + E$ and noting that $\Psi_i \xrightarrow{w} 0$, (2.26) gives

$$\liminf_{i\to\infty} (\Psi_i, E_H(J)[H,A]E_H(J)\Psi_i) \geqq \lim_{i\to\infty} c_0 \|E_H(J)\Psi_i\|^2 = c_0 . \tag{2.27}$$

We obtain a contradiction by comparing (2.25) and (2.27). This completes the proof of (2.7).

III. The Dirichlet Problem for $-\Delta+V$ in $\Omega(R)$

For $f \in C_0^\infty(\mathbb{R}^n)$ and $R>0$, consider the trace, $T_R f(\omega) = f(R\omega)$, of f on the sphere of radius R.

Lemma 3.1. *T_R extends from $C_0^\infty(\mathbb{R}^n)$ to a bounded operator from $D(\Delta) = H^2(\mathbb{R}^n)$ to $L^2(S^{n-1}, d\omega)$. If $f \in H^2(\mathbb{R}^n)$, $T_R f$ (considered as an element of $L^2(S^{n-1}, d\omega)$) is continuously differentiable in the variable R. The following estimate holds*

$$\| T_R \phi \|_{L^2(S^{n-1}, d\omega)} \equiv |\phi|_R \leqq R^{-(n-1)/2} (\|\nabla\phi\|^2_{\Omega(R)} + \|\phi\|^2_{\Omega(R)})^{1/2} . \tag{3.1}$$

We remark that the conclusions of Lemma 3.1 are far from optimal but we will not need optimal results. We prove only (3.1), and refer the reader to [15] from which a proof of the rest is easily constructed.

Suppose $\phi\in C_0^\infty(\mathbb{R}^n)$. Then for $r\geqq R$,

$$\begin{aligned}-\frac{d}{dr}|\phi(r\omega)|^2&=-2\,\mathrm{Re}\overline{\phi(r\omega)}\frac{d}{dr}\phi(r\omega)\\&\leqq|\phi(r\omega)|^2+|\nabla\phi(r\omega)|^2\\&\leqq R^{-(n-1)}(|\phi(r\omega)|^2r^{n-1}+|\nabla\phi(r\omega)|^2r^{n-1}).\end{aligned}$$

Integrating from R to infinity and then over S^{n-1}, (3.1) follows.

Corollary 3.2. *Suppose H is as in Theorem 2.3 and $H\psi=E\psi$. Then for some constant c_1 and $R>1$,*

$$|\psi|_R\leqq c_1R^{-(n-1)/2}\|\psi\|_{\Omega(R-1)}\,. \tag{3.2}$$

Proof. Choose $\tilde\eta\in C^\infty(\mathbb{R}^n)$, $0\leqq\tilde\eta\leqq1$, with $\tilde\eta(t)=0$ if $t\leqq1/2$, and $\tilde\eta(t)=1$ if $t\geqq3/4$. Let $\eta_R(x)=\tilde\eta(|x|-(R-1))$. Then since $T_R\psi=T_R\eta_R\psi$, from (3.1) and the fact that $-\Delta+1\leqq c_1'H+c_2'$, we have

$$|\psi|_R^2\leqq R^{-(n-1)}(\eta_R\psi,(-\Delta+1)\eta_R\psi)\leqq R^{-(n-1)}(\eta_R\psi,(c_1'H+c_2')\eta_R\psi)\,.$$

We now use the identity $\eta_RH\eta_R=(1/2)(H\eta_R^2+\eta_R^2H)+(\nabla\eta_R)^2$ to conclude that

$$|\psi|_R^2\leqq R^{-(n-1)}(\psi,[(c_1'E+c_2')\eta_R^2+(\nabla\eta_R)^2]\psi)\,,$$

which leads to (3.2).

The remainder of this section is devoted to proving (2.6) under the hypotheses given in Theorem 2.3. Again, these hypotheses are far from optimal (some improvements are given in [5]). In order to avoid complications in intermediate stages of the proof we prove (2.6) under the additional hypothesis that $v_j\in C_0^\infty(X_j)$. Inequality (2.6) follows for general v_j by an approximation argument: First approximate [7, 10, 16] v_j by $v_j^{(m)}\in C_0^\infty(X_j)$ so that the corresponding $V^{(m)}$ satisfies $\|(V^{(m)}-V)(-\Delta+1)^{-1}\|\to0$. It is easy to prove that there is an eigenfunction $\psi^{(m)}$ of $H^{(m)}=-\Delta+V^{(m)}$ with eigenvalue $E^{(m)}\to E$ such that

$$\lim_{m\to\infty}\|(-\Delta+1)(\psi^{(m)}-\psi)\|=0\,. \tag{3.3}$$

From the result for $v_j\in C_0^\infty(X_j)$, it follows that

$$\|\psi^{(m)}\|_{\Omega(R)}\leqq c_2R^{n/2}|\psi^{(m)}|_R \tag{3.4}$$

for $R\geqq R_0$. From (3.1)

$$|\psi^{(m)}-\psi|_R\leqq R^{-(n-1)/2}\|(-\Delta+1)^{1/2}(\psi^{(m)}-\psi)\|\,,$$

so that from (3.3), $|\psi^{(m)}|_R\to|\psi|_R$. Thus (2.6) follows from (3.4) as long as c_2 and R_0 can be taken independent of m. This will be evident in what follows.

We will follow a strategy similar to that of Bardos and Merigot [3]. Presumably a Brownian-motion argument like that of [2] would also work.

Given a C^∞ function ϕ on S^{n-1}, ϕ has an expansion $\phi(\omega)=\sum_{k=0}^{\infty}Y_k(\omega)$, where Y_k is a spherical harmonic satisfying $\Delta_SY_k=-k(k+n-2)Y_k$, and where Δ_S is the spherical Laplacian [17]. [For $n=1$ take $S^{n-1}=\{1,-1\}$, $d\omega=\delta(\omega+1)+\delta(\omega-1)$,

$Y_0(\omega)=(\phi(1)+\phi(-1))/2$, $Y_1(\omega)=(\phi(1)-\phi(-1)\omega/2$, and $Y_k=0$ if $k>1$.] The operator $\mathscr{E}$ given by

$$\mathscr{E}\phi(r\omega)=\sum_{k=0}^{\infty} Y_k(\omega)(r/R)^{2-n-k}; \qquad r\geqq R \tag{3.5}$$

defines a function on $\Omega(R)$ where it is harmonic. Clearly $\mathscr{E}\phi(R\omega)=\phi(\omega)$.

Lemma 3.3.

$$\|\mathscr{E}\phi\|_{\Omega(R,3R)}\leqq cR^{n/2}\|\phi\|_{L^2(S^{n-1},d\omega)}. \tag{3.6}$$

Proof.

$$\begin{aligned}\int_R^{3R}(\int|\mathscr{E}\phi(r\omega)|^2d\omega)r^{n-1}dr&=\sum_{k=0}^{\infty}\int_R^{3R}(\int|Y_k|^2d\omega)\left(\frac{r}{R}\right)^{4-2n-2}r^{n-1}dr\\ &=R^n\sum_{k=0}^{\infty}(\int|Y_k|^2d\omega)\left(\int_1^3 t^{3-n-2k}dt\right)\\ &\leqq c^2R^n\sum_{k=0}^{\infty}\int|Y_k|^2d\omega=c^2R^n\|\phi\|^2_{L^2(S^{n-1},d\omega)}.\end{aligned}$$

We are now ready to deal with $H=-\Delta+V$, $V\in C^\infty(\mathbb{R}^n)$. Suppose $H\psi=E\psi$ with $E<\Sigma=\inf\sigma_{\text{ess}}(H)$. It follows from [1] and [14] for example, that there exists an $R_0>0$ so that if $R\geqq R_0$, $(\phi,(H-E)\phi)\geqq\frac{1}{2}(\Sigma-E)(\phi,\phi)$ for all $\phi\in C_0^\infty(\Omega(R))$. The closure of this quadratic form is associated with a positive self-adjoint operator H_R-E with form domain $H_0^1(\Omega(R))$. (H_R is just $-\Delta_D+V$ with Δ_D the Laplacian with Dirichlet boundary conditions on $\partial\Omega(R)$. $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm $(\|\nabla\phi\|^2+\|\phi\|^2)^{1/2}$.) We have

$$\|(H_R-E)^{-1}\|\leqq c=[\tfrac{1}{2}(\Sigma-E)]^{-1}. \tag{3.7}$$

Furthermore the operator $(H_R-E)^{-1}\nabla$ extends from $C_0^\infty(\Omega_R)$ to a bounded operator on $L^2(\Omega(R))$ and

$$\|(H_R-E)^{-1}V\|\leqq K_1, \tag{3.8}$$

$$\|(H_R-E)^{-1}\nabla\|\leqq K_2, \tag{3.9}$$

where K_j is independent of $R\geqq R_0$. (K_j can also be chosen independent of m in $H^{(m)}=-\Delta+V^{(m)}$.)

Now suppose $R>R_0$ and let $u=\mathscr{E}T_R\psi$. From Lemma 3.3

$$\|u\|_{\Omega(R,3R)}\leqq cR^{n/2}|\psi|_R. \tag{3.10}$$

Note that since $V\in C^\infty(\mathbb{R}^n)$, the elliptic regularity theorem tells us that $\psi\in C^\infty(\mathbb{R}^n)$. This means that $T_R\psi\in C^\infty(S^{n-1})$, and thus looking at (3.5), $\|\partial^\alpha u\|_{\Omega(R,3R)}<\infty$ for any α.

Define $\xi_R(x)=\xi(|x|/R)$, where $\xi\in C^\infty(\mathbb{R})$ satisfies $0\leqq\xi\leqq 1$, $\xi(r)=0$ for $r\geqq 3$, and $\xi(r)=1$ for $r\leqq 2$. Define

$$v=(H_R-E)^{-1}(-(V-E)\xi_R u+2\nabla\xi_R\cdot\nabla u+(\Delta\xi_R)u). \tag{3.11}$$

From (3.7) through (3.9) and the uniform boundedness of $\||\nabla\xi_R|+|\Delta\xi_R|\|_{L^\infty(\Omega_R)}$, for $R>R_0$, we see that

$$\|v\|_{\Omega(R)}\leqq c\|u\|_{\Omega(R,3R)}\leqq c'R^{n/2}|\psi|_R.$$

Thus

$$\|\xi_R u+v\|_{\Omega(R)} \leqq c_2 R^{n/2} |\psi|_R .$$

We now show that $\psi=\xi_R u+v$ on $\Omega(R)$. This will complete the proof. Let $f=\psi-\xi_R u-v$, and note that as a distribution $(-\Delta+V-E)f=0$ in $\Omega(R)$. (This computation is straightforward.) From (3.11), $v\in D(H_R)\subset H_0^1(\Omega(R))$. Since $u(R\omega)=\psi(R\omega)$, we have that the function $\psi-\xi_R u$ vanishes on $\partial\Omega(R)$. In addition $\partial^\alpha(\psi-\xi_R u)\in L^2(\Omega(R))$ for all α. It easily follows that $\psi-\xi_R u\in H_0^1(\Omega(R))$, and thus that $f\in H_0^1(\Omega(R))=D((H_R-E)^{1/2})$. This fact and the fact that $(-\Delta+V-E)f=0$ in $\Omega(R)$ as a distribution imply $f\in D(H_R)$ and $(H_R-E)f=0$. Since $H_R-E>0$, $f=0$. This completes the proof of (2.6).

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