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Central Limit Theorem for the Lorentz Process via Perturbation Theory

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Abstract. The Markov partition of the Sinai billiard allows the following heuristic interpretation for the Lorentz process with a $\mathbb{Z}^2$-periodic configuration of scatterers: while executing a (non-Markovian) random walk on $\mathbb{Z}^2$, the particle changes its internal state according to the symbolic dynamics defined by the Markov partition. This picture can be formalized and then the Lorentz process appears as the limit of a sequence of (Markovian!) random walks with a finite but increasing number of internal states and the central limit theorem can be proved for it by perturbational expansions with uniformly bounded – in a sence related to the Perron-Frobenius theorem – coefficients and uniform remainder terms.

1. Introduction

In [K-Sz (1983)] the authors of the present paper proved a local central limit theorem for random walks with internal degrees of freedom (RWwIDF). These generalizations of the classical random walks had been introduced and studied by Sinai [S (1981)] in the hope they would help in understanding the Lorentz process. As a matter of fact, Gyires [Gy (1960)], in his studies on Toeplitz type hypermatrices, proved a local central limit theorem closely related to the theorem of [S (1981)]. His paper refers to a remark of Rényi, who also found a probabilistic interpretation of Gyires' result, namely just in terms of random walks with internal states (cf. also Gyires [Gy (1962)]). Our aim here is to justify Sinai's approach.

In fact, we give a new proof for the central limit theorem (CLT) obtained by Bunimovich and Sinai [B-S (1980)] for the Lorentz process with a periodic configuration of scatterers. At the price of having worked out a sort of uniform – for a family of matrices – perturbation theory, our arguments are simpler and they require less calculations. Moreover, we could immediately obtain more exact results, namely Chebyshev-Edgeworth-Cramér type asymptotic expansions in the

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spirit of [B-R (1976)] but, for brevity, we do not discuss this automatic generalization of our result. Since our proof, too, relies upon the properties of the Markov partition of the Sinai billiard constructed by Bunimovich and Sinai [B-S (1981)] we should also suppose the finiteness of the horizon.

We obtain the CLT for the Lorentz process by approximating it with a sequence of RWwIDF, which have an increasing number of internal states – this is the main technical difficulty!, and by showing that the CLT is valid for them, in a sense, uniformly. We adapt the tools of [K-Sz (1981)]: perturbation theory combined with the Perron-Frobenius theorem, to this situation. Namely, in Sect. 2 we introduce the notion of majorization of a matrix by a matrix with non-negative elements. In Sect. 3 we show that, if the coefficients of the Taylor expansions of a family of matrices are majorized by a family of non-negative matrices, then, under some additional conditions, this remains true for the corresponding families of resolvents and, consequently, for other important characteristics, too. This enables us to uniformly bound the remainder terms in the perturbation series and, in Sect. 4, to prove our main theorem, a CLT for a family of RWwIDF. The application to the Lorentz process is described in Sect. 5. Finally, Sect. 6 gives some comments.

2. Some Facts about Matrices

The set of all eigenvalues of a matrix T will be denoted by spec(T).

Definition 1. The radius $\varrho(T)$ of the set spec(T) is called the spectral radius of the operator T, i.e.

$$\varrho(T)=\max|\lambda|, \qquad \lambda\in\operatorname{spec}(T).$$

Statement 2. $\varrho(T)=\limsup_{n\to\infty}\|T^n\|^{1/n}$, moreover, if $A_1A_2=A_2A_1$, then

$$\varrho(A_1+A_2)\leqq\varrho(A_1)+\varrho(A_2),$$
$$\varrho(A_1A_2)\leqq\varrho(A_1)\varrho(A_2).$$

Definition 3. Let $Q=\{q_{jk}\}$ be a matrix with non-negative elements. The complex matrix $A=\{a_{jk}\}$ is Q-majorized iff $|a_{jk}|\leqq q_{jk}$ for every j and k.

The following lemma is a substatement of Wielandt's lemma, often used in the theory of matrices with non-negative elements.

Lemma 4. *If A is Q-majorized then $\varrho(A)\leqq\varrho(Q)$.*

Corollary 5. *If A is majorized by a positive multiple Q of a stochastic matrix (i.e. Q has the vector $\mathbb{1}=\underbrace{(1,\dots,1)}_{v}$ as an eigenvector) then*

$$\|A\|_{\sup}\overset{\text{def}}{=}\max_j\sum_k|a_{jk}|\leqq\sum_k q_{jk}=\varrho(Q).$$

Remarks. a) If a matrix Q is a positive multiple of a stochastic matrix, then any matrix with non-negative elements, interchangeable with Q has this property.

b) $\|A+B\|_{\sup} \leqq \|A\|_{\sup} + \|B\|_{\sup}$, so $\|\cdot\|_{\sup}$ is a norm.

c) $\|A\| = \sqrt{\sum_{j,k} |a_{jk}|^2} \leqq \sqrt{v}\, \|A\|_{\sup}$.

Lemma 6. *If A_1 is Q_1- and A_2 is Q_2-majorized, respectively, then A_1A_2 is Q_1Q_2-majorized.*

3. Uniform Perturbations of Families of Matrices

First we recall some facts on the perturbation theory of matrices, which can be found in standard textbooks (cf. e.g. [K (1966), Chap. II, Sect. 1]). Since the proof of the CLT for the Lorentz process is based on the local behaviour of families of matrix functions described below, we have to study perturbation theory uniformly, when the size of the matrices increases.

Let us assume that $T(\kappa)$ is an n-times differentiable matrix-function depending on a complex vector-valued parameter $\kappa = (\kappa_1, \ldots, \kappa_d)$ – this means that

$$T(\kappa) = T + A(\kappa) = T + \sum_{\substack{k=1 \\ j_1+\ldots+j_d=k}}^{n} A_{j_1,\ldots,j_d} \kappa_1^{j_1} \ldots \kappa_d^{j_d} + B(\kappa),$$

where $\|B(\kappa)\| = o(\|\kappa\|^n)$.

If $\|B(\kappa)\| = \mathcal{O}(\|\kappa\|^{n+1})$, we shall say that $T(\kappa)$ is n times differentiable in the strong sense.

The main object of perturbation theory is the investigation of the dependence of $\operatorname{spec}(T(\kappa))$ and of the eigen-projections on κ. For our purposes, the most convenient way is to use the Sz.-Nagy-Kato integral formula

Statement 1. If $\Lambda \subset \operatorname{spec}(T)$, then the projection P_Λ onto the maximal invariant subspace of T belonging to Λ (i.e. $P_\Lambda P_\Lambda = P_\Lambda$, $P_\Lambda T = TP_\Lambda$, $\operatorname{spec}(P_\Lambda T) = \Lambda$ and $\dim(P_\Lambda \mathbb{R}^n) \geqq \dim(P\mathbb{R}^n)$, if P satisfies the preceding three conditions) can be represented in the following integral form

$$P_\Lambda = \frac{-1}{2\pi i} \oint_\Gamma (T - \zeta I)^{-1} d\zeta, \tag{3.1}$$

where I is the identity operator, Γ is a closed contour surrounding Λ and containing no eigenvalues from $\operatorname{spec}(T) \setminus \Lambda$.

$R_T(\zeta) = (T - \zeta I)^{-1}$ is called the resolvent operator of T. $R_T(\zeta)$ exists for every $\zeta \notin \operatorname{spec}(T)$.

Lemma 2. *$R_{T(\kappa)}(\zeta)$ is holomorphic in ζ if*

(i) $\zeta \notin \operatorname{spec}(T)$,

(ii) $\varrho(R_T(\zeta)A(\kappa)) < 1$.

Proof.

$$(T(\kappa) - \zeta)^{-1} = (I + R_T(\zeta)A(\kappa))^{-1} R_T(\zeta). \tag{3.2}$$

The inverse operator on the right hand side exists due to (ii).

Definition 3. Suppose $Q^{(\nu)}$ is a family of $\nu \times \nu$ matrices with non-negative elements. The family $\{T^{(\nu)}(\kappa)\}$ of $\nu \times \nu$ matrix functions has uniform Taylor expansion up to the n^{th} order with respect to the family $\{Q^{(\nu)}\}$ iff there exists a $\vartheta > 0$ and a function $\varepsilon_n(\kappa)$ $\left(\varepsilon_n(\kappa) > 0, \lim\limits_{\kappa \to 0} \varepsilon_n(\kappa) = 0\right)$ such that, for every $\|\kappa\| < \vartheta$

$$T^{(\nu)}(\kappa) = T^{(\nu)} + \sum_{\substack{k=1 \\ j_1 + \ldots + j_d = k}}^{n} A^{(\nu)}_{j_1, \ldots, j_d} \kappa_1^{j_1} \ldots \kappa_d^{j_d} + \varepsilon_n(\kappa) \|\kappa\|^n B^{(\nu)}(\kappa),$$

where the matrices $A^{(\nu)}_{j_1, \ldots, j_d}$ and $B^{(\nu)}(\kappa)$ re $Q^{(\nu)}$-majorized.

If the remainder term is $\varepsilon_n(\kappa) \|\kappa\|^{n+1} B^{(\nu)}(\kappa)$, where $\varepsilon_n(\kappa)$ is bounded, then we say that the family $\{T^{(\nu)}(\kappa)\}$ has *a uniform Taylor expansion up to the* n^{th} *order in the strong sense.* The forthcoming statements can be extended to hold in the strong sense, too, but, for brevity, these extensions will not be formulated.

Lemma 4. *Let us assume that the family* $\{T^{(\nu)}\}$ *possesses the following properties*

(i) $T^{(\nu)}$ *are operators with non-negative elements;*

(ii) ζ_0 *is a simple eigenvalue of every* $T^{(\nu)}$*;*

(iii) *there exists a neighbourhood* $|\zeta_0 - \zeta| < \delta$ *of* ζ_0 *such that the operators* $T^{(\nu)}$ *have no other eigenvalues in it.*

If the family $\{T^{(\nu)}(\kappa)\} = \{T^{(\nu)} + A^{(\nu)}(\kappa)\}$ *has a uniform Taylor expansion up to the* 0^{th} *order at the point* $\kappa = \mathbb{0}$ *with respect to the family* $\{T^{(\nu)}\}$, *then the family* $\{R_{T^{(\nu)}(\kappa)}(\zeta)\}$, *for* $\delta > |\zeta - \zeta_0| > \frac{\delta}{2}$, *has a uniform Taylor expansion, too, up to the* 0^{th} *order at* $\kappa = \mathbb{0}$ *with respect to a universal in* ζ *family* $\{Q^{(\nu)}\}$, *where* $Q^{(\nu)}$ *and* $T^{(\nu)}$ *are interchangeable, and there exists a constant D depending only on* δ, *such that* $D\varrho(Q^{(\nu)}) \leqq \varrho(T^{(\nu)})$. *Moreover, the radius* ϑ' *of the Taylor expansion of* $\left\{R_{T^{(\nu)}(\kappa)}(\zeta), \frac{\delta}{2} < |\zeta - \zeta_0| < \delta\right\}$ *depends on that of* $\{T^{(\nu)}(\kappa)\}$ *(i.e. on* ϑ*) and* δ, *only.*

Proof. The statement of the lemma is a straightforward consequence of formula (3.2) and the fact that rational functions of a given operator are interchangeable.

Lemma 5. *If the family* $\{T^{(\nu)}(\kappa)\}$ *satisfies the conditions of Lemma 4 with the modification that it has a uniform Taylor expansion up to the* n^{th} *order, then the family* $\{R_{T^{(\nu)}(\kappa)}(\zeta)\}$ *possesses a uniform Taylor expansion, too, up to* n^{th} *order with the same properties as in Lemma 4.*

Proof. The proof can be carried out by induction using the resolvent formula

$$\begin{aligned}(T^{(\nu)}(\kappa) - \zeta I)^{-1} &- (T^{(\nu)} - \zeta I)^{-1} \\ &= (R_{T^{(\nu)}(\kappa)}(\zeta) - R_{T^{(\nu)}}(\zeta))(T^{(\nu)}(\kappa) - T^{(\nu)}) \\ &\quad - R_{T^{(\nu)}}(\zeta)(T^{(\nu)}(\kappa) - T^{(\nu)}) R_{T^{(\nu)}}(\zeta).\end{aligned}$$

Using formula (3.1) one can obtain the following consequence of the preceding two lemmas:

Corollary 6. *If the family* $\{T^{(\nu)}(\kappa)\}$ *satisfies the conditions of Lemma 5, then there exist two positive numbers* δ_1 *and* ϑ' *depending only on* δ *and* ϑ *such that for* $\|\kappa\|<\vartheta'$,

(i) *in the neighbourhood* $|\zeta_0-\zeta|<\delta_1$, $T^{(\nu)}(\kappa)$ *has only one simple eigenvalue* $\zeta^{(\nu)}(\kappa)$;

(ii) $\zeta^{(\nu)}(\kappa)$ *has a Taylor expansion of the form*

$$\zeta_0+\sum_{\substack{k=1\\ j_1+\ldots+j_d=k}}^{n} a_{j_1,\ldots,j_d}\kappa_1^{j_1}\ldots\kappa_d^{j_d}+\|\kappa\|^n\|Q^{(\nu)}\|_{\sup}\varepsilon_n(\kappa),$$

(iii) *if* $P_{\zeta^{(\nu)}(\kappa)}=\dfrac{-1}{2\pi i}\oint\limits_{|\zeta-\zeta_0|=\delta_1} R_{T^{(\nu)}(\kappa)}(\zeta)d\zeta$, *then*

$$\|P_{\zeta^{(\nu)}(\kappa)}-P_{\zeta^{(\nu)}(0)}\|_{\sup}\leqq \mathrm{const}\,\|Q^{(\nu)}\|_{\sup}\varepsilon_0(\kappa).$$

Remark. Property (ii) comes from the following inequality

$$\left\|T^{(\nu)}(\kappa)P_{\zeta^{(\nu)}(\kappa)}-\left(\zeta_0+\sum_{k=1}^{n} a_{j_1,\ldots,j_d}\kappa_1^{j_1}\ldots\kappa_d^{j_d}\right)P_{\zeta^{(\nu)}(\kappa)}\right\|_{\sup}$$
$$\leqq\|\kappa\|^n\|Q^{(\nu)}\|_{\sup}\varepsilon_n(\kappa). \tag{3.3}$$

4. An Infinite Family of Random Walks with Internal Degrees of Freedom

The objects which we intend to apply the results of the preceding paragraph to were introduced in [K-Sz (1981)]. Here we briefly repeat their definitions.

Definition 1. Let $\xi^{(\nu)}(t)=(\eta^{(\nu)}(t),\varepsilon^{(\nu)}(t))$ $(\eta^{(\nu)}(t)\in\mathbb{Z}^d,\ \varepsilon^{(\nu)}(t)\in\{1,\ldots,\nu\})$ be a Markov chain with the following transition probabilities

$$\mathrm{Prob}\{\xi^{(\nu)}(t)=(x,k)\,|\,\xi^{(\nu)}(t-1)=(y,j)\}=q_{j,k}(x-y),$$

$\xi^{(\nu)}(t)$ is a random walk with ν internal degrees of freedom.

Set

$$\{q_{jk}^{(\nu)}(x)\}_{(j,k=1,\ldots,\nu)}=Q^{(\nu)}(x).$$

For our purposes it will be convenient to work with the $n_\nu{}^{\text{th}}$ power of $Q^{(\nu)}(x)$, where n_ν is a sequence of natural numbers; $n_\nu\to\infty$.

Set $\tilde{Q}^{(\nu)}(x)=(Q^{(\nu)}(x))^{n_\nu}$. The elements of $\tilde{Q}^{(\nu)}(x)$ will be denoted by $\tilde{q}_{j,k}(x)$.

Set

$$M_l^{(\nu)}=\sum_{x\in\mathbb{Z}^d} x_l\tilde{Q}^{(\nu)}(x),$$

$$\Sigma_{l,m}^{(\nu)}=\sum_{x\in\mathbb{Z}^d} x_l x_m\tilde{Q}^{(\nu)}(x),$$

where x_l is the l^{th} co-ordinate of $x\in\mathbb{Z}^d$.

Condition 2 (Finite Horizon). There exists a constant $\mathscr{G}(\geqq 1)$ such that $Q^{(\nu)}(x)=0$ for all $\|x\|>\mathscr{G}$.

Condition 3 (Ergodicity and Doeblin Condition). The stochastic matrix $Q^{(v)} = \sum_{x \in \mathbb{Z}^d} Q^{(v)}(x)$ is irreducible and aperiodic, so there exists a unique stationary vector $\mu^{(v)} : (Q^{(v)})^* \mu^{(v)} = \mu^{(v)}$. Moreover, there exists a positive number δ, such that, for every v, $\tilde{Q}^{(v)} = (Q^{(v)})^{n_v}$ has the only eigenvalue 1 outside the region $|\zeta| < 1 - \delta$.

Condition 4 (Centralization)

$$\sum_{x \in \mathbb{Z}^d} x(Q^{(v)}(x)\mathbb{1}^{(v)}, \mu^{(v)}) = 0,$$

where $\mathbb{1}^{(v)} = (1, \ldots, 1) \in \mathbb{R}^v$

Set $\alpha^{(v)}(\kappa) = \sum_{x \in \mathbb{Z}^d} Q^{(v)}(x) e^{i(x,\kappa)}$, the Fourier-transform matrix of $Q^{(v)}(x)$. Then $\tilde{\alpha}^{(v)}(\kappa) = (\alpha^{(v)}(\kappa))^{n_v}$ is the Fourier transform of $\tilde{Q}^{(v)}(x)$.

Proposition 5. *The family $\{\tilde{\alpha}^{(v)}(\kappa)\}$ satisfies the conditions of Lemma 3.5 and Corollary 3.6 (in the strong sense, too) with the majorizing sequence $\{(n_v \mathcal{G})^n \tilde{Q}^{(v)}\}$.*

Proof. The proposition follows from Conditions 2 and 3 taking the Taylor expansion for the n_vth power of the Fourier-transform matrix $\alpha^{(v)}(\kappa)$. Moreover, Corollary 2.5 provides that the coefficient of $\varepsilon_n(\kappa)$ in Statements (ii) and (iii) of Corollary 3.6 has the order $\mathcal{O}(n_v^3)$.

Proposition 6. *Set $\lambda^{(v)}(\kappa)$ the largest eigenvalue of $\tilde{\alpha}^{(v)}(\kappa)$. The Taylor expansion of $\lambda^{(v)}(\kappa)$ for $\|\kappa\| < \vartheta'$ looks like*

$$\lambda^{(v)}(\kappa) = 1 - \frac{n_v}{2}(\sigma^{(v)}\kappa, \kappa) + n_v^3 \mathcal{O}(\|\kappa\|^3), \tag{4.1}$$

where $\sigma^{(v)}$ is a $d \times d$ matrix and $\mathcal{O}(\|\kappa\|^3)$ is uniform in v.

Proof. By using Proposition 5, formula (4.1) can be deduced from the centralization condition by Schrödinger's implicit method (cf. e.g. [F (1965)]). Here we do not repeat the detailed considerations of [K-Sz (1981)]. The matrix $\sigma^{(v)}$ takes the form

$$\sigma_{l,m}^{(v)} = \frac{1}{n_v}\left\{\left(\mu^{(v)}, \Sigma_{l,m}^{(v)}\mathbb{1}\right) - (\mu^{(v)}, M_l^{(v)}(\tilde{Q}^{(v)} - 1)^{-1} M_m^{(v)}\mathbb{1}) \right.$$
$$\left. - (\mu^{(v)}, M_m^{(v)}(\tilde{Q}^{(v)} - 1)^{-1} M_l^{(v)}\mathbb{1})\right\},$$

which has the probabilistic meaning:

$$\sigma_{l,m}^{(v)} = \lim_{t \to \infty} \frac{1}{t \cdot n_v} E\eta_l^{(v)}(t \cdot n_v) \cdot \eta_m^{(v)}(t \cdot n_v),$$

where $\eta_l^{(v)}(t)$ is the l^{th} co-ordinate in $\mathbb{Z}^d$ of $\eta^{(v)}(t)$ and the expectation is taken with respect to the measure $\delta_{0,x} \otimes \mu^{(v)}$,

$$\left(\delta_{0,x} = \begin{cases} 1 & x = 0 \\ 0 & x \in \mathbb{Z}^d \backslash 0 \end{cases}\right).$$

The uniformness of $O(\|\kappa\|^3)$ is a consequence of Proposition 5.

Now we are in the position to formulate the main result of this paper.

Main Theorem. *If besides Conditions 2–4* $\lim_{v\to\infty} \sigma^{(v)}=\sigma$ *and* $n_v^4=o(t)$ *hold, too, and* $\operatorname{Prob}(\xi^{(v)}(0)=(0,j))=\mu_j^{(v)}$ *(*$\mu_j^{(v)}$ *is the* j^{th} *co-ordinate of the stationary distribution vector* $\mu^{(v)}$*), then the distribution of* $t^{-1/2}\eta^{(v)}(t)$ *tends weakly to the d-dimensional Gaussian distribution* $N(0,\sigma)$.

Proof. The modification to our case of the standard proof of the CLT (cf. proof of Pre-Theorem 2.1 in [K-Sz (1981)]) says that we only need the convergence

$$\lim_{t\to\infty} \int_{\|\kappa\|<\omega} \left|\left(\left(\alpha^{(v)}\left(\frac{\kappa}{\sqrt{t}}\right)\right)^t \mathbb{1}^{(v)}, \mu^{(v)}\right) - e^{-\frac{1}{2}(\sigma\kappa,\kappa)}\right| d\kappa = 0$$

for any $\omega>0$.

In view of $(P(0)\mathbb{1},\mu)=1$ the decomposition

$$\begin{aligned}\left(\alpha^{(v)}\left(\frac{\kappa}{\sqrt{t}}\right)\right)^t - e^{-\frac{1}{2}(\sigma\kappa,\kappa)}P(0) &= \left(\alpha^{(v)}\left(\frac{\kappa}{\sqrt{t}}\right)\right)^t (1-P(0)) + \left(\alpha^{(v)}\left(\frac{\kappa}{\sqrt{t}}\right)\right)^t \left(P(0)-P\left(\frac{\kappa}{\sqrt{t}}\right)\right) \\ &\quad + \left[\left(\alpha^{(v)}\left(\frac{\kappa}{\sqrt{t}}\right)\right)^t P(\kappa) - e^{-\frac{1}{2}(\sigma\kappa,\kappa)}P(\kappa)\right] \\ &\quad + e^{-\frac{1}{2}(\sigma\kappa,\kappa)}(P(\kappa)-P(0)) \\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}\end{aligned}$$

will be suitable to prove this convergence.

Since $|(A\mathbb{1}^{(v)}, \mu^{(v)}| \leqq \|A\|_{\sup}$, Taylor expansions with remainder terms of the type (3.3) can be used.

The equation

$$\left(\alpha^{(v)}\left(\frac{\kappa}{\sqrt{t}}\right)\right)^t = \tilde{\alpha}^{(v)}\left[\frac{\kappa\left(\frac{t}{n_v}\right)^{-1/2}}{\sqrt{n_v}}\right]^{\frac{t}{n_v}}$$

makes it possible to apply Proposition 6. Thus the largest eigenvalue of $\left(\alpha^{(v)}\left(\frac{\kappa}{\sqrt{t}}\right)\right)^t$ has the expansion

$$1-\tfrac{1}{2}(\sigma^{(v)}\kappa,\kappa)+\frac{t}{n_v}\cdot n_v^3\, \mathcal{O}\left(\left\|\frac{\kappa}{\sqrt{t}}\right\|^3\right).$$

The assumption $n_v^4=o(t)$ ensures that the remainder term tends to 0 as $t\to\infty$ uniformly in v and in $\|\kappa\|<\omega$ (ω is arbitrary), thus showing that the integral corresponding to III tends to 0. Moreover, $\left(\alpha^{(v)}\left(\frac{\kappa}{\sqrt{t}}\right)\right)^t$ is Q^t-majorized for every κ and by Lemma 3.5 (see also Corollary 3.6) $P(\kappa)-P(0)$ is (const $\varepsilon_0(\kappa)\cdot Q$)-majorized, if $\|\kappa\|<\vartheta'$. Thus, by Lemma 2.6, $\left(\alpha^{(v)}\left(\frac{\kappa}{\sqrt{t}}\right)\right)^t\left(P(0)-P\left(\frac{\kappa}{\sqrt{t}}\right)\right)$ is (const $\varepsilon_0(\kappa)\cdot Q^{t+1}$)-

majorized. Now Corollary 2.5 implies that

$$\left\|\left(\alpha^{(\nu)}\left(\frac{\kappa}{\sqrt{t}}\right)\right)^t\left(P(0)-P\left(\frac{\kappa}{\sqrt{t}}\right)\right)\right\|_{\sup} \leqq \varrho\left(\text{const } \varepsilon_0\left(\frac{\kappa}{\sqrt{t}}\right)\cdot Q^{t+1}\right) \leqq \text{const } \varepsilon_0\left(\frac{\kappa}{\sqrt{t}}\right)$$

uniformly for $\left\|\frac{\kappa}{\sqrt{t}}\right\| < \theta'$. Consequently, the integral corresponding to II tends to 0 as $t \to \infty$. Finally, I vanishes by the definition of $P(0)$ and the bound (iii) of Corollary 3.6 applies to IV.

5. Application to the Lorentz Process

First we summarize the necessary information on the Lorentz process. There is given a periodic set of disjoint, strictly convex, closed scatterers on the Euclidean plane. We assume that the congruent finite sets ($\mathscr{S}$) of the scatterers are labelled by $x \in \mathbb{Z}^2 : \mathscr{S}_x$.

A point particle moves uniformly among the scatterers with elastic collisions at the scatterers. We suppose that the free path length is bounded. It is convenient to discretize the time and to study the Poincaré map of the original continuous time process. This map $\mathscr{T}$ is defined on the two-dimensional phase space $\Omega = \bigcup_{x \in \mathbb{Z}^2} \tilde{\mathscr{S}}_x = \bigcup_{x \in \mathbb{Z}^2}$ boundary $(\mathscr{S}_x) \otimes$angle (in the moments of reflections).

This map $\mathscr{T}$ has an invariant measure, periodic with respect to $\mathbb{Z}^2$. It has the form $\mu_L = \delta \otimes \tilde{\mu}$, where δ is a uniform measure on $\mathbb{Z}^2$ and $\tilde{\mu}$ is a normed measure on $\tilde{\mathscr{S}}_0$. We suppose that the particle starts from $\tilde{\mathscr{S}}_0$ with the initial measure $\tilde{\mu}$. The motion of this particle is called (the discretized) Lorentz-process.

Let $\mathfrak{M}_0 = \{B_{01}, \ldots, B_{0k}, \ldots\}$ be a countable, measurable partition of $\tilde{\mathscr{S}}_0$. $\mathfrak{M}_x = \{B_{x1}, \ldots, B_{xk}, \ldots\}$ denotes the congruent image of $\mathfrak{M}_0$ on $\tilde{\mathscr{S}}_x$. We can associate a symbolic dynamics $\xi_L(t)$ to $\mathscr{T}$ acting on $\bigcup_{x \in \mathbb{Z}^2} \mathfrak{M}_x$ $\Big($the union of partitions $\mathfrak{M}_x$ on $\bigcup_{x \in \mathbb{Z}^2} \mathscr{S}_x\Big)$ as follows: for μ_L almost every $\omega \in \Omega$ $\xi_L(t) = (x, j) \Leftrightarrow \mathscr{T}^t \omega \in B_{xj}$. $\Big($We suppose that $\bigcup_{x \in \mathbb{Z}^2} \mathfrak{M}_x$ is a generating partition for $\mathscr{T}$, i.e. the above correspondence is μ_L-a.e. uniquely defined.$\Big)$

Roughly speaking the symbolic dynamics $\xi_L(t) = (n_L(t), \varepsilon_L(t))$ can be regarded as a (non-Markovian) random walk with countably many internal degrees of freedom.

Let us introduce some further notations:

$$\mathfrak{M}_x^n = \left(\bigvee_{j=0}^{n-1} \mathscr{T}^{-j}\left(\bigcup_{y \in \mathbb{Z}^2} \mathfrak{M}_y\right)\right) \cap \tilde{\mathscr{S}}_x .$$

($\bigvee$ denotes the refinement of partitions.) Analogously to $\xi_L(t)$ the partition $\bigcup_{x\in\mathbb{Z}^2} \mathfrak{M}_x^n$ defines a symbolic dynamics $\xi_L^n(t)$ as follows:

$$\xi_L^n(t)=(x,j) \Leftrightarrow \mathcal{T}^t\omega\in B_{x,j}^n,$$

where $B_{x,j}^n\in\mathfrak{M}_x^n$. (The reader should be cautious: though η_L^n will approximate η_L in some sense, ε_L^n has only auxiliary character.)

Let $\{B_{01}^n,\dots,B_{0v}^n\}$ be a finite family of sets belonging to $\mathfrak{M}_0^n$ and let $\{B_{x1}^n,\dots,B_{xv}^n\}$ be its congruent image on $\tilde{\mathcal{S}}_x$ (v will depend on n which will correspond to the notation n_v of Sect. 4).

The transition probabilities

$$q_{jk}^{(v)}(y-x)=\frac{\mu_L((\xi_L^n(t)=(y,k))\cap(\xi_L^n(t-1)=(x,j)))}{\mu_L(\xi_L^n(t-1)=(x,j))} * \left\{\sum_{y\in\mathbb{Z}^2}\sum_{k=1}^{v}\mu_L((\xi_L^n(t)=(y,k))\cap(\xi_L^n(t-1)=(x,j)))\right\}^{-1}$$

define a random walk $\xi^{(v)}(t)$ with v internal degrees of freedom.

Let $P^{(v)}(\cdot)$ denote the stationary measure for $\xi^{(v)}(t)$ whose unicity follows from the nice properties of the Markov partition $\mathfrak{M}_0$. In fact in B–S (1981)] it is proved that

(i) For every n we can choose an appropriate family $\{B_{01}^n,\dots,B_{0v}^n\}\subset\mathfrak{M}_0^n$ in such a way that

$$\sum_{\substack{1\leqq j\leqq v\\ 0\leqq\tau\leqq t}}|P^{(v)}(\xi^{(v)}(\tau)=(\cdot,j_\tau),0\leqq\tau\leqq t)-\mu_L(\xi_L^n(\tau)=(\cdot,j_\tau),0\leqq\tau\leqq t)|\leqq t\lambda^n, \tag{5.1}$$

where $0<\lambda<1$.

(ii) The sequence of random walks $\xi^{(v)}(t)$ satisfies Conditions 4.2–4.4 with $n_v=4n$, $v=n^n$.

(iii) For some $0<\alpha<1$,

$$|E\eta_L(0)\eta_L(t)|\leqq\text{const}\,e^{-t^\alpha}.$$

Corollary of the Main Theorem. *For the Lorentz process introduced in the beginning of this section the distribution of $t^{-1/2}\eta_L(t)$ tends to a non-degenerate Gaussian one.*

Proof. Choose $n=t^{1/5}$ and $v=n^n$. Then, by Statement (i), $\eta_L(t)$ can be approximated by a family of random walks with v internal degrees of freedom satisfying Conditions 2–4 and the error is less than $t\lambda^{t^{1/5}}$. To apply the main theorem we should only check the condition $\lim_{v\to\infty}\sigma^{(v)}=\sigma$. But it is a consequence of (iii), the Doeblin Condition formulated for $\xi^{(v)}(t)$ and the inequality (5.1).

The non-degeneracy of the limit distribution can be proved by standard methods of ergodic theory (cf. [B–S (1981)]).

Remark. Property (iii) itself can be derived simply from (5.1) and the Doeblin condition formulated for $\xi^{(v)}(t)$.

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