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On a C^*-Algebra Approach to Phase Transition in the Two-Dimensional Ising Model

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Abstract. We investigate the state on the C^*-algebra of Pauli spins on a one-dimensional lattice (infinitely extended in both directions) which gives rise to the thermodynamic limit of the Gibbs ensemble in the two-dimensional Ising model (with nearest neighbour interaction). It is shown that the representation of the Pauli spin algebra associated with the state is factorial above and at the known critical temperature, while it has a two-dimensional center below the critical temperature. As a technical tool, we derive a general criterion for a state of the Pauli spin algebra corresponding to a Fock state of the Fermion algebra to be primary. We also show that restrictions of two quasifree states of the Fermion algebra to its even part are equivalent if and only if the projection operators E_1 and E_2 (on the direct sum of two copies of the basic Hilbert space) satisfy the following two conditions: (1) $E_1 - E_2$ is in the Hilbert–Schmidt class, (2) $E_1 \wedge (\mathbf{1} - E_2)$ has an even dimension, where the even-oddness of dim $E_1 \wedge (\mathbf{1} - E_2)$ is called $\mathbb{Z}_2$-index of E_1 and E_2 and is continuous in E_1 and E_2 relative to the norm topology.

1. Main Results

We consider the two-dimensional Ising model with the Hamiltonian (with the free boundary condition)

$$H^{LM}(\xi) = -\left(\sum_{i=-L}^{L-1} \sum_{j=-M}^{M} J_1 \xi_{ij}\xi_{i+1,j} + \sum_{i=-L}^{L} \sum_{j=-M}^{M-1} J_2 \xi_{ij}\xi_{i,j+1} \right), \tag{1.1}$$

where $\xi_{ij} = \pm 1$ is the (classical) spin at the lattice site $(i,j) \in \mathbb{Z}^2$, and J_1 and J_2 are positive constants. We are interested in the thermodynamic limit ($L,M \to \infty$) of the Gibbs ensemble average

$$\langle F \rangle_{LM} = Z_{LM}^{-1} \sum_{\xi} F(\xi) e^{-\beta H^{LM}(\xi)}, \tag{1.2}$$

$$Z_{LM} = \sum_{\xi} e^{-\beta H^{LM}(\xi)},$$

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where the sum is over all configurations $\xi = \{\xi_{ij}\}$, $\beta \geqq 0$, and F is a function of $\xi_{ij}, |i| \leqq l, |j| \leqq m$ for some $l \leqq L$ and $m \leqq M$.

There is the following transfer matrix method [9] of expressing it in terms of a state on the C^*-algebra $\mathfrak{A}^P$ generated by Pauli spin matrices $\sigma_x^{(i)}, \sigma_y^{(i)}, \sigma_z^{(i)}$ on sites $i \in \mathbb{Z}$ of the one-dimensional lattice $\mathbb{Z}$.

By identifying a function of $\xi_j = \pm 1$ and $\xi'_j = \pm 1$ as the (ξ_j, ξ'_j) matrix element of a linear combination of 2×2 matrices $\mathbf{1}$, $\sigma_x^{(j)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_y^{(j)} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ and $\sigma_z^{(j)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have the identification

$$(T_M)_{\xi,\xi'} = \exp\left\{ \frac{K_2}{2} \sum_{j=-M}^{M-1} (\xi_j \xi_{j+1} + \xi'_j \xi'_{j+1}) + K_1 \sum_{j=-M}^{M} \xi_j \xi'_j \right\}, \tag{1.3}$$

$$\begin{aligned} T_M = (2\sinh 2K_1)^{M+(1/2)} \exp\left\{ \frac{K_2}{2} \sum_{j=-M}^{M-1} \sigma_x^{(j)} \sigma_x^{(j+1)} \right\} \\ \times \exp\left\{ K_1^* \sum_{j=-M}^{M} \sigma_z^{(j)} \right\} \exp\left\{ \frac{K_2}{2} \sum_{j=-M}^{M-1} \sigma_x^{(j)} \sigma_x^{(j+1)} \right\}, \end{aligned} \tag{1.4}$$

where $K_j = \beta J_i (j = 1,2)$,

$$K_1^* = (1/2)\log\{\coth K_1\}, \tag{1.5}$$

and $\xi = (\xi_j)$, $\xi' = (\xi'_j)$. Then

$$Z_{LM} = \|(T_M)^L \Omega_M\|^2, \tag{1.6}$$

$$\Omega_M(\xi) = \exp\left\{ \frac{K_2}{2} \sum_{j=-M}^{M-1} \xi_j \xi_{j+1} \right\}, \tag{1.7}$$

$$\langle F \rangle_{LM} = (T_M^L \Omega_M, F_{\beta M} T_M^L \Omega_M)/Z_{LM}, \tag{1.8}$$

where $F_{\beta M} \in \mathfrak{A}^P$. (For example, if $F = \prod_{i=-l}^{l} F_i$ with $F_i = F_i(\xi_{i,-m} \dots \xi_{im})$, then $F_{\beta M} = T_M^{-l} \hat{F}_{-l} T_M \hat{F}_{-l+1} \dots T_M \hat{F}_l T_M^{-l}$ with $\hat{F}_i = F_i(\sigma_x^{(-m)} \dots \sigma_x^{m})$.)

The $L \to \infty$ limit selects the unique unit eigenvector $\Omega^M = \Omega^M(\xi)$ $(\Omega^M(\xi) > 0)$ belonging to the largest eigenvalue of T_M.

$$\lim_{L \to \infty} \langle F \rangle_{LM} = (\Omega^M, F_{\beta M} \Omega^M). \tag{1.9}$$

In $M \to \infty$ limit, $F_{\beta M}$ is known to converge to an element F_β of $\mathfrak{A}^P$ (it is $\alpha_{il}^M(\hat{F}_{-l}) \dots \alpha_i^M(\hat{F}_{-1}) \hat{F}_0 \alpha_{-i}^M(\hat{F}_1) \dots \alpha_{-il}^M(\hat{F}_l)$ for $\alpha_{il}^M(x) = T_M^{-l} x T_M^l$, which converges for all integers l, see [2]), whilst the limit of the vector state by Ω^M

$$\varphi_\beta(a) = \lim_{M \to \infty} (\Omega^M, a \Omega^M) \tag{1.10}$$

will be explicitly given in Sect. 2, so that

$$\varphi_\beta(F_\beta) = \lim_{M\to\infty} \lim_{L\to\infty} \langle F \rangle_{LM}. \tag{1.11}$$

Our main result is the following:

Theorem 1. *The cyclic representation of* $\mathfrak{A}^P$ *associated with* φ_β *is factorial if* $0 \leqq \beta \leqq \beta_c$, *whilst it is non-factorial with 2-dimensional center (for the weak closure) if* $\beta > \beta_c$, *where* β_c *is the unique value of* $\beta > 0$ *for which* $K_1^* = K_2$.

The two-dimensional Ising model with nearest neighbour interactions was shown to have a phase transition in [13]. There is a critical (inverse) temperature β_c such that the spontaneous magnetisation $m^* = 0$ for $\beta < \beta_c$, and $m^* \neq 0$ for $\beta > \beta_c$. Moreover for $\beta < \beta_c$, a unique equilibrium state exists, whilst for $\beta > \beta_c$, there are exactly two distinct extremal equilibrium states [1, 5]. Schultz, Mattis and Lieb [15] reformulated the Onsager–Kaufman transfer matrix treatment [7, 8] using a Fermion algebra $\mathfrak{A}^F$. Now the even subalgebras of $\mathfrak{A}^P$ and $\mathfrak{A}^F$ are canonically isomorphic, even though the algebras $\mathfrak{A}^P$ and $\mathfrak{A}^F$ themselves are not (for the system infinitely extended to all directions), and so one has a correspondence between even states on the Pauli spin algebra $\mathfrak{A}^P$ and even states on the Fermion algebra $\mathfrak{A}^F$. For a two dimensional lattice, infinitely extended in all directions, the Gibbs state in the thermodynamic limit induces a pure (hence primary) Fock state on the Fermion algebra $\mathfrak{A}^F$ [14, 11, 16]. In [11] Lewis and Sisson discussed how the phase transition manifests itself by a jump in the index of a Fredholm operator associated with the Fock state. Subsequently, Lewis and Winnink [12] showed that the phase transition also reveals itself in the restricted state on the Fermion algebra $\mathfrak{A}^F([1,\infty))$ of a half-line (regarded as a subalgebra of $\mathfrak{A}^F$). The restricted state is a non-Fock quasi-free state. It is primary for $\beta < \beta_c$, and non-primary for $\beta > \beta_c$. Again this involves the computation of an index (mod 2) of a Fredholm operator. For a half lattice, the Pauli spin algebra $\mathfrak{A}^P([1,\infty))$ is canonically isomorphic to $\mathfrak{A}^F([1,\infty))$, and Kuik in [10] showed that (for periodic boundary conditions) the thermodynamic limit of the Gibbs state induces precisely the above restricted state on $\mathfrak{A}^P([1,\infty))$. Hence for a half lattice, the manifestation of the phase transition is apparent.

In the present case of a two dimensional lattice, infinitely extended to all directions (in contrast to a half lattice), the state for the Fermion algebra is pure (Fock state) for all β, but our Theorem 1 shows that the phase transition manifests itself in the state for the Pauli spin algebra, which is primary for $0 \leqq \beta \leqq \beta_c$ and non-primary for $\beta > \beta_c$.

2. Correspondence of Pauli and Fermion Algebras

Following [4], we consider the C^*-algebra $\mathfrak{A}^F$ (the Fermion algebra) generated by annihilation and creation operators c_i and c_i^* ($i \in \mathbb{Z}$) satisfying the canonical anticommutation relations (CAR's):

$$[c_i, c_j]_+ = [c_i^*, c_j^*]_+ = 0, [c_i, c_j^*]_+ = \delta_{ij}\mathbf{1}. \tag{2.1}$$

Defining the automorphism Θ_- of $\mathfrak{A}^F$ satisfying

$$\Theta_- c_i = \begin{cases} c_i & \text{if } i \geqq 1 \\ -c_i & \text{if } i \leqq 0, \end{cases} \tag{2.2}$$

we construct the crossed product $\hat{\mathfrak{A}} = \mathfrak{A}^F \times_{\Theta_-} \mathbb{Z}_2$, which is generated by $\mathfrak{A}^F$, and $T \in \hat{\mathfrak{A}}$ satisfying

$$T^2 = \mathbf{1},\ T^* = T, \tag{2.3}$$

$$Ta = \Theta_-(a)T \quad (a \in \mathfrak{A}^F). \tag{2.4}$$

The Pauli spin algebra $\mathfrak{A}^P$ can be identified with a C^*-subalgebra of $\hat{\mathfrak{A}}$ generated by

$$\sigma_z^{(j)} = 2c_j^* c_j - 1, \tag{2.5}$$

$$\sigma_x^{(j)} = TS_j(c_j + c_j^*), \quad \sigma_y^{(j)} = TS_j i(c_j - c_j^*), \tag{2.6}$$

$$S_j = \begin{cases} \prod_{k=1}^{j-1} \sigma_z^{(k)} & \text{if } j > 1, \\ \mathbf{1} & \text{if } j = 1, \\ \prod_{k=0}^{j} \sigma_z^{(k)} & \text{if } j < 1. \end{cases} \tag{2.7}$$

Let Θ be the automorphism of $\hat{\mathfrak{A}}$ satisfying

$$\Theta(c_i) = -c_i, \quad \Theta(T) = T. \tag{2.8}$$

We have

$$\Theta(\sigma_z^{(j)}) = \sigma_z^{(j)}, \Theta(\sigma_x^{(j)}) = -\sigma_x^{(j)}, \Theta(\sigma_y^{(j)}) = -\sigma_y^{(j)}. \tag{2.9}$$

The Fermion algebra $\mathfrak{A}^F$ is split into even and odd parts:

$$\mathfrak{A}^F = \mathfrak{A}_+^F + \mathfrak{A}_-^F, \mathfrak{A}_\pm^F = \{a \in \mathfrak{A}^F : \Theta(a) = \pm a\}. \tag{2.10}$$

Accordingly

$$\hat{\mathfrak{A}} = \sum_{\sigma, \sigma' = \pm} \hat{\mathfrak{A}}_{\sigma\sigma'} \supset \mathfrak{A}^P = \mathfrak{A}_+^P + \mathfrak{A}_-^P, \tag{2.11}$$

$$\hat{\mathfrak{A}}_{\sigma+} = \mathfrak{A}_\sigma^F, \hat{\mathfrak{A}}_{\sigma-} = T\mathfrak{A}_\sigma^F (\sigma = \pm), \tag{2.12}$$

$$\mathfrak{A}_+^P = \hat{\mathfrak{A}}_{++} = \mathfrak{A}_+^F, \mathfrak{A}_-^P = \hat{\mathfrak{A}}_{--} = T\mathfrak{A}_-^F. \tag{2.13}$$

By the relation (2.5) and

$$\sigma_x^{(i)} \sigma_x^{(i+1)} = (c_i - c_i^*)(c_{i+1} + c_{i+1}^*), \tag{2.14}$$

T_M is given, up to a positive coefficient, by

$$T'_M = e^{(K_2/2)(B^M, H_2 B^M)} e^{K_1^*(B^M, H_1 B^M)} e^{(K_2/2)(B^M, H_2 B^M)}, \tag{2.15}$$

where

$$H_2 = \frac{1}{2} \begin{pmatrix} -(U^* + U) & U^* - U \\ U - U^* & U^* + U \end{pmatrix}, \quad H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.16}$$

$$(Uf)_k = f_{k+1}, (U^*f)_k = f_{k-1} (f = (f_k) \in l_2(\mathbb{Z})), \tag{2.17}$$

and we have used the following notation of [3]:

$$B^M(h) \equiv \sum_{j=-M}^{M} (c_j^* f_j + c_j g_j) \left(h = \begin{pmatrix} f \\ g \end{pmatrix}, f = (f_j), g = (g_j) \right), \tag{2.18}$$

$$(B^M, HB^M) = \sum_{mn} B^M(e_m)(e_m, He_n) B^M(e_n)^*, \tag{2.19}$$

for an orthonormal basis $\{e_n\}$ of

$$\mathscr{K}_M = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in l_2 \oplus l_2 ; f_j = g_j = 0 \quad \text{if} \quad |j| > M \right\}. \tag{2.20}$$

We note that H_j $(j = 1, 2)$ satisfy

$$H_j^* = H_j, \quad \Gamma H_j + H_j \Gamma = 0, \tag{2.21}$$

where $\Gamma \begin{pmatrix} f \\ g^* \end{pmatrix} \equiv \begin{pmatrix} g \\ f^* \end{pmatrix}$ and $B^M(h)^* = B^M(\Gamma h)$.

Let H^M be the unique selfadjoint operator satisfying

$$e^{2H^M} = e^{K_2 H_2^M} e^{2K_1^* H_1^M} e^{K_2 H_2^M}, \tag{2.22}$$

where H_j^M is the restriction of H_j to $l_2(\{-M, \ldots, M\})$. Due to $\mathrm{tr}\sigma_x^{(j)} = \mathrm{tr}\sigma_z^{(j)} = 0$, we have $\det e^{H^M} = 1$. Since the right-hand side of (2.22) is analytic in K_1^* and K_2, so is H for all real K_1^* and K_2. By repeated use of (8.48) in [3] (also see the definition (8.23)), we have

$$e^{(B^M, H^M B^M)} = T'_M, \tag{2.23}$$

where the possible sign ± 1 multiplying T'_M (according to the use of (8.48)) is $+1$ because it is $+1$ for $K_1^* = K_2 = 0$, and both sides of (2.23) are real analytic in K_1^* and K_2.

3. The State φ_β

Let E_-^M, E_0^M and E_+^M be the spectral projection of H^M for $(-\infty, 0)$, $\{0\}$ and $(0, \infty)$. Due to (2.21), $\Gamma E_\pm^M \Gamma = E_\mp^M$, $[\Gamma, E_0] = 0$. Let $\{e_\nu\}$ be a complete orthonormal basis of $E_+^M \mathscr{K}_M$, satisfying $H^M e_\nu = \lambda_\nu e_\nu (\lambda_\nu > 0)$, and set $b_\nu = B(e_\nu)$. Due to $B(h)^* = B(\Gamma h)$ and

$$[B(h_1)^*, B(h_2)]_+ = (h_1, h_2)\mathbf{1}, \tag{3.1}$$

$\{b_\nu\}$ satisfy CAR's among themselves. (Note that $\{\Gamma e_\nu\}$ is an orthonormal basis of $E_-^M \mathscr{K}_M$ satisfying $H^M \Gamma e_\nu = -\lambda_\nu \Gamma e_\nu$.) Furthermore

$$(B^M, H^M B^M) = \sum \lambda_\nu (b_\nu b_\nu^* - b_\nu^* b_\nu) = \sum_\nu \lambda_\nu (1 - 2b_\nu^* b_\nu). \tag{3.2}$$

Therefore the eigensubspace of $(B^M, H^M B^M)$ belonging to the largest eigenvalue consists of vectors Ω (in $l^2(\{1, -1\}^{2M+1})$) satisfying $b_\nu \Omega = 0$ for all ν and has the dimension $2^{\dim E_0^M}$.

For finite K_1^*, all matrix elements of $e^{K_1^*(B^M,H_1B^M)}$ are strictly positive and hence the same is true for T_M. By the Perron–Frobenius theorem, the largest eigenvalue of T_M is non-degenerate as is claimed in Sect. 1.

In the finite dimensional case, one can view the Pauli algebra $\mathfrak{A}^{PM}$ generated by $\sigma_x^{(j)}, \sigma_y^{(j)}, \sigma_z^{(j)}, |j| \leqq M$ as the Fermion algebra generated by $\bar{c}_j$ and $\bar{c}_j^*$ related to σ's by (2.5) $\sim$ (2.7) (c's replaced by $\bar{c}$'s) with the difference that $T = \prod_{j=0}^{-M} \sigma_z^{(j)} = \prod_{j=0}^{-M} (2\bar{c}_j\bar{c}_j^* - \mathbf{1})$. Still we obtain the same expression for T_M, and hence we conclude that the largest eigenvalue of $(B^M, H^M B^M)$ must be non-degenerate, which implies $E_0^M = 0$ and

$$\mathbf{1} - E_-^M = E_+^M = \Gamma E_-^M \Gamma. \tag{3.3}$$

We now see that the vector state $\varphi_M^+(a) = (\Omega^M, a\Omega^M)$ of $\mathfrak{A}_+^{PM}$ has the property that

$$\varphi_M^+(B^M(h)^* B^M(h)) = 0 \quad \text{if} \quad h \in E_+^M \mathscr{K}_M. \tag{3.4}$$

Any state φ on $\mathfrak{A}_+^{PM}$ has a (unique) extension to a Θ-invariant state $\hat{\varphi}$ on the Fermion algebra $\mathfrak{A}^{FM}$ (generated by c_j and c_j^*, $|j| \leqq M$) by $\hat{\varphi}(a) = \varphi(a_+)$, $a_\pm = (a \pm \Theta(a))/2$, $a \in \mathfrak{A}^{FM}$. The extended state $\hat{\varphi}$ must be the Fock state, for which b_ν and b_ν^* act as annihilation and creation operators due to (3.4). Such a state has been denoted as φ_P in [3] with $P = E_-^M$ satisfying property (3.3) for a "basis projection."

In the limit of $M \to \infty$, U^M and $(U^*)^M$ (restrictions of U and U^* to the subspace $\mathscr{K}_M$, which are no longer unitary) tend strongly to U and U^*. Hence H_j^M tends to $H_j (j = 1, 2)$ and H^M to H defined by

$$e^{2H} = e^{K_2 H_2} e^{2K_1^* H_1} e^{K_2 H_2}, \quad H^* = H. \tag{3.5}$$

Due to the property $\Gamma H_j \Gamma = -H_j$, H also satisfies $\Gamma H \Gamma = -H$.

We need the following result:

Lemma 3.1. *H does not have a point spectrum at 0.*

Before going into its proof, we describe its consequence. Since H does not have the point spectrum at 0, Lemma 3.3 at the end of the section implies

$$\lim_{M \to \infty} E_\pm^M = E_\pm, \tag{3.6}$$

where E_- and E_+ are spectral projections of H for $(-\infty, 0)$ and $(0, \infty)$. This means that the limit of φ_M^+ as $M \to \infty$ is the restriction of the Fock state φ_{E_-} on $\mathfrak{A}^F$ to $\mathfrak{A}_+^F = \mathfrak{A}_+^P$.

The automorphism Θ on $\mathfrak{A}^{PM}$ can be implemented by a unitary operator $U(\Theta) = \prod_{j=-M}^{M} \sigma_z^{(j)}$. By Θ-invariance of H^M, Ω^M is $U(\Theta)$ invariant and hence $(\Omega^M, a\Omega^M) = 0$ if $\Theta(a) = -a$, $a \in \mathfrak{A}^{PM}$. Therefore φ_β is also Θ-invariant and is determined by its restriction to $\mathfrak{A}_+^P$, i.e. $\varphi_\beta(a) = \varphi_\beta(a_+)$, $a_+ = (a + \Theta(a))/2 \in \mathfrak{A}_+^P$ for any $a \in \mathfrak{A}^P$. Therefore we obtain the following

Lemma 3.2. *φ_β is the unique Θ-invariant extension, to $\mathfrak{A}^P$, of a state φ_β^+ on $\mathfrak{A}_+^P = \mathfrak{A}_+^F$, where φ_β^+ is the restriction, to $\mathfrak{A}_+^F$, of the Fock state φ_{E_-} on $\mathfrak{A}^F$ and E_- is the spectral projection of H (given by (3.5)) for $(-\infty, 0)$.*

We now prove Lemma 3.1. The unitary operator U has the spectral decomposition $U = \int_0^{2\pi} e^{i\theta} dE_U(\theta)$ with a simple Lebesgue spectral measure on $[0, 2\pi]$, as is well-known from the theory of Fourier series. Thus $H = \int H(\theta) dE_U(\theta)$ with θ-dependent 2×2 matrix $H(\theta)$, given by

$$2H(\theta) = -\gamma(\theta)V(\theta), \tag{3.7a}$$

$$V(\theta) = \begin{pmatrix} \cos\vartheta(\theta) & -i\sin\vartheta(\theta) \\ i\sin\vartheta(\theta) & -\cos\vartheta(\theta) \end{pmatrix}, \tag{3.7b}$$

where $\gamma(\theta) \geqq 0$ is determined by

$$\cosh 2K_1^* \cosh 2K_2 - \sinh 2K_1^* \sinh 2K_2 \cos\theta = \cosh\gamma(\theta), \tag{3.8}$$

and $\delta(\theta) \equiv \vartheta(\theta) - \theta$ is determined by

$$\cos\delta(\theta) = (\sinh\gamma(\theta))^{-1}(\cosh 2K_1^* \sinh 2K_2 - \sinh 2K_1^* \cosh 2K_2 \cos\theta), \tag{3.9a}$$

$$\sin\delta(\theta) = (\sinh\gamma(\theta))^{-1} \sinh 2K_1^* \sin\theta. \tag{3.9b}$$

(See [14, 11, 16].) The right-hand side of (3.8) is not 1 except for discrete values of θ satisfying $\cos\theta = 1$ (and only if $K_1^* = K_2$) due to

$$\begin{aligned} |\sinh 2K_1^* \sinh 2K_2 \cos\theta| + 1 &\leqq \sinh 2K_1^* \sinh 2K_2 + 1 \\ &\leqq (\sinh^2 2K_1^* + 1)^{1/2}(\sinh^2 2K_2 + 1)^{1/2} = \cosh 2K_1^* \cosh 2K_2. \end{aligned} \tag{3.10}$$

Thus $\delta(\theta)$ is well-defined (modulo 2π) by (3.9) for all θ if $K_1^* \neq K_2$ and for all $\theta \neq 0$ (modulo 2π) if $K_1^* = K_2$. (If $K_1^* = K_2$, $\gamma(\theta) = 0$ for $\theta = 0$ and any value of $\vartheta(0)$ leads to the same $H(0)$.) Since the matrix part of (3.7) is selfadjoint unitary, $H(\theta)$ does not have an eigenvalue 0.

Lemma 3.3. *Let* $\lim A_n = A$, E_n *and* E *be the spectral projections of* A_n *and* A *for an (infinite or finite) interval* (b, a) *either including or not including the eigenprojections of* a *and/or* b. *If* a *and* b *are not eigenvalues of* A, *then*

$$\lim E_n = E. \tag{3.11}$$

Proof. It is enough to treat the case of $b = -\infty$, a finite because the case of $a = \infty, b$ finite will follow by considering $-A_n \to -A$ and the case of a and b finite will follow by taking the product of projections for the two infinite cases.

For any given vector Ψ and $\varepsilon > 0$, there exists Φ belonging to the A-spectral subspace for the complement of the interval $(a-\delta, a+\delta)$ for some $\delta > 0$ and satisfying $\|\Psi - \Phi\| < \varepsilon$. Since E_n is uniformly bounded, it is enough to prove

$$\lim E_n \Phi = E\Phi \tag{3.12}$$

for such Φ.

However, on the A-spectral subspace for the complement of the interval $(a-\delta, a+\delta)$, we can apply Theorem 2 of [6] to the characteristic function of the interval (a, b) (the endpoints a and/or b included or not included according to the definition of E_n) and obtain (3.12).

4. A General Criterion for Triviality of Center

In this section, we develop a method of deciding whether φ_β is factorial from a property of the projection E_-.

We consider the following general situation. Let $\mathfrak{A}$ be a unital C^*-algebra with two automorphisms α and β satisfying

$$\alpha^2 = \beta^2 = \mathbf{1}, \quad \alpha\beta = \beta\alpha, \tag{4.1}$$

and with a unitary element U satisfying

$$\alpha(U) = -U. \tag{4.2}$$

Let $\hat{\mathfrak{A}}$ be the crossed product of $\mathfrak{A}$ by β action of $\mathbb{Z}_2$, which is generated by $\mathfrak{A}$ and $T \in \hat{\mathfrak{A}}$ satisfying

$$T^2 = \mathbf{1}, \quad T^* = T, \quad Ta = \beta(a)T \quad (a \in \mathfrak{A}). \tag{4.3}$$

Let

$$\mathfrak{A}_\pm = \{a \in \mathfrak{A} : \alpha(a) = \pm a\}. \tag{4.4}$$

Let

$$\mathfrak{B} = \mathfrak{A}_+ + T\mathfrak{A}_-. \tag{4.5}$$

It is a C^*-subalgebra of $\hat{\mathfrak{A}}$. (Note that $\mathfrak{A}_\pm$ are β-invariant.) We extend α and β to $\hat{\mathfrak{A}}$ by

$$\hat{\alpha}(a + Tb) = \alpha(a) + T\alpha(b), \tag{4.6}$$

$$\hat{\beta}(a + Tb) = \beta(a) + T\beta(b), \tag{4.7}$$

where $a, b \in \mathfrak{A}$. For an α-invariant state φ of $\mathfrak{A}$, there corresponds a unique $\hat{\alpha}$ invariant state $\varphi^{\mathfrak{B}}$ of $\mathfrak{B}$ which is an extension of the restriction of φ to $\mathfrak{A}_+$.

In our application $\mathfrak{A} = \mathfrak{A}^F$, $\alpha = \Theta$, $\beta = \Theta_-$, $\mathfrak{B} = \mathfrak{A}^P$, $\varphi^{\mathfrak{B}} = \varphi_\beta$ and $U = 2^{-1/2}(c_i + c_i^*)$ for any fixed i. Our claim in the above general situation is the following:

Theorem 2. *Assume that φ is an α-invariant pure state of $\mathfrak{A}$. Then $\varphi^{\mathfrak{B}}$ is not pure if and only if*

(1) φ and $\varphi \circ \beta$ are equivalent and

(2) $\varphi_+ = \varphi | \mathfrak{A}_+ \circ \beta$ are not equivalent.

If $\varphi^{\mathfrak{B}}$ is not pure, it is a mixture of two non-equivalent pure states.

The proof is divided into several lemmas.

Lemma 4.1. *If φ is α-invariant and pure, then φ_+ is pure.*

Proof. Let $p = (\mathbf{1} + \alpha)/2$. Then $\varphi_+ \to \varphi = \varphi_+ \circ p$ and $\varphi \to \varphi_+ = \varphi | \mathfrak{A}_+$ yield a bijective affine map between states φ_+ of $\mathfrak{A}_+$ and α-invariant states φ of $\mathfrak{A}$. Hence φ_+ is pure if and only if φ is an extremal α-invariant state. In particular, if φ is α-invariant and pure, then φ_+ is pure.

Lemma 4.2. *If an α-invariant state φ of $\mathfrak{A}$ is pure, then φ_+ and $\varphi_- \equiv \varphi_+ \circ \mathrm{Ad}\, U$ are disjoint. (The disjointness refers to the associated cyclic representations of $\mathfrak{A}_+$.)*

(Note that $UaU^* \in \mathfrak{A}_+$ if $a \in \mathfrak{A}_+$ due to $\alpha(U) = -U$.)

Proof. Let $\mathscr{H}_\varphi$, π_φ and ξ_φ be the (GNS) triplet of a Hilbert space, a representation of $\mathfrak{A}$ and a cyclic vector associated with φ. Due to α-invariance of φ,

$$\mathscr{H}_\varphi = \mathscr{H}_+ \oplus \mathscr{H}_-, \quad \mathscr{H}_\pm = \overline{\pi_\varphi(\mathfrak{A}_\pm)\xi_\varphi}. \tag{4.8}$$

Then $\mathscr{H}_{\varphi_\pm}$, $\pi_{\varphi_\pm}$ and $\xi_{\varphi_\pm}$ can be identified with $\mathscr{H}_\pm$, $\pi_\pm = (\pi_\varphi|\mathfrak{A}_+)|\mathscr{H}_\pm$ and $\xi_+ = \xi_\varphi$, $\xi_- = \pi_\varphi(U)\xi_\varphi$. By Lemma 3.1, φ_+ and hence $\varphi_- = \varphi_+ \circ \operatorname{Ad} U$ are pure. Therefore $\pi_\pm$ are both irreducible. We now derive a contradiction assuming that they are equivalent.

Since $\pi_\pm$ are irreducible, there exists a unitary $W_0 \in \pi_+(\mathfrak{A}_+)''$, $\operatorname{Ad} W_0$ on $\pi_+(\mathfrak{A}_+)$ implementing $\operatorname{Ad} U$ on $\mathfrak{A}_+$. Since $\pi_+ \sim \pi_-$, there exists a unitary $W \in \pi_\varphi(\mathfrak{A}_+)''$, $\operatorname{Ad} W$ on $\pi_\varphi(\mathfrak{A}_+)$ implementing $\operatorname{Ad} U$ on $\mathfrak{A}_+$. Since $(\operatorname{Ad} W)W = W$, W has to commute with $\pi_\varphi(U)$. For $a \in \mathfrak{A}_-$, $aU^* \in \mathfrak{A}_+$, and

$$\begin{aligned}(\operatorname{Ad} W)\pi_\varphi(a) &= (\operatorname{Ad} W)(\pi_\varphi(aU^*)\pi_\varphi(U)) \\ &= \{(Ad\, W)\pi_\varphi(aU^*)\}\pi_\varphi(U) \\ &= \pi_\varphi(\{(\operatorname{Ad} U)(aU^*)\}U) = \pi_\varphi((\operatorname{Ad} U)a). \end{aligned} \tag{4.9}$$

Therefore $\operatorname{Ad} W$ coincides with $\operatorname{Ad} \pi_\varphi(U)$ on $\pi_\varphi(\mathfrak{A})$ and hence on $\pi_\varphi(\mathfrak{A})''$. Since π_φ is irreducible, $W = c\,\pi_\varphi(U)$ for some complex number c of modulus 1.

On the other hand, by α-invariance of φ, α can be extended to an isomorphism $\bar{\alpha}$ of $\pi_\varphi(\mathfrak{A})''$. By $W \in \pi_\varphi(\mathfrak{A}_+)''$, we have $\bar{\alpha}(W) = W$, whilst $\alpha(U) = -U$. This contradicts with $W = c\pi_\varphi(U)$, $W^*W = 1$.

Lemma 4.3. *Let φ be an α-invariant pure state of $\mathfrak{A}$. The states $\varphi_+ = \varphi|\mathfrak{A}_+$ and $\varphi_+ \circ \beta \circ \operatorname{Ad} U$ of $\mathfrak{A}_+$ are equivalent if and only if*

(1) *φ is equivalent to $\varphi \circ \beta$ and*

(2) *φ_+ is not equivalent to $\varphi_+ \circ \beta$.*

Proof. If $\varphi_+ \sim \varphi_+ \circ \beta \circ \operatorname{Ad} U$, then states $\varphi = \varphi_+ \circ p$ and $\varphi \circ \beta \circ \operatorname{Ad} U = \varphi_+ \circ \beta \circ \operatorname{Ad} U \circ p$ of $\mathfrak{A}$ are equivalent due to a result of Stratila and Voiculescu (Sect. 2.7 of [17]). Since $\operatorname{Ad} U$ is inner on $\mathfrak{A}$, we obtain $\varphi \sim \varphi \circ \beta$. (Note that $[\alpha, \operatorname{Ad} U] = 0$ due to $\alpha(U) = -U$.)

Now assume that $\varphi \sim \varphi \circ \beta$. Since $\pi_\varphi|\mathfrak{A}_+ = \pi_+ \oplus \pi_-$ and $\pi_\pm$ are disjoint by Lemmas 4.1 and 4.2, we have two alternatives:

(A) $\pi_+ \sim \pi_+ \circ \beta$ disjoint from $\pi_- \sim \pi_- \circ \beta$,

(B) $\pi_+ \sim \pi_- \circ \beta$ disjoint from $\pi_- \sim \pi_+ \circ \beta$.

The desired conclusion $\pi_+ \sim \pi_- \circ \beta (\sim \pi_+ \circ \beta \circ \operatorname{Ad} U)$ holds in and only in case (B), which is characterized by $\varphi \sim \varphi \circ \beta$ and π_+ disjoint from $\pi_+ \circ \beta$.

Lemma 4.4. *If φ_+ and $\varphi_+ \circ \beta \circ \operatorname{Ad} U$ are disjoint, then $\varphi^{\mathfrak{B}}$ is pure.*

Proof. Let $\mathscr{H}_\varphi^\beta$, π_φ^β and ξ_φ^β be the GNS triplet for the state $\varphi \circ \beta$. Let (4.10)

$$\hat{\mathscr{H}}_\varphi = \mathscr{H}_\varphi \oplus \mathscr{H}_\varphi^\beta,$$

$$\begin{aligned}&\hat{\pi}_\varphi(a + Tb)[\pi_\varphi(a_1)\xi_\varphi \oplus \pi_\varphi^\beta(a_2)\xi_\varphi^\beta] \\ &\quad = \pi_\varphi(aa_1 + \beta(ba_2))\xi_\varphi \oplus \pi_\varphi^\beta(aa_2 + \beta(ba_1))\xi_\varphi^\beta. \end{aligned} \tag{4.11}$$

Then the restriction of $\hat{\pi}_\varphi(\mathfrak{B})$ to

$$\mathscr{H}_\varphi^{\mathfrak{B}} = \mathscr{H}_+ \oplus \mathscr{H}_-^\beta \quad (\mathscr{H}_\pm^\beta \equiv \overline{\pi_\varphi^\beta(\mathfrak{A}_\pm)\xi_\varphi^\beta}) \tag{4.12}$$

(denoted $\pi_\varphi^{\mathfrak{B}}$) yields the cyclic representation of $\mathfrak{B}$ associated with $\varphi^{\mathfrak{B}}$.

Since $\pi_\varphi^{\mathfrak{B}}|\mathfrak{A}_+ \sim \pi_+ \oplus \pi_- \circ \beta$ and the assumption implies the disjointness of π_+ and $\pi_- \circ \beta$ (due to $\pi_- \circ \beta = \pi_+ \circ \mathrm{Ad}\, U \circ \beta = \pi_+ \circ \beta \circ \mathrm{Ad}\,\beta(U) \sim \pi_+ \circ \beta \circ \mathrm{Ad}\, U$ because of $U^*\beta(U) \in \mathfrak{A}_+$, any $X \in \pi_\varphi^{\mathfrak{B}}(\mathfrak{A}_+)'$ is of the form

$$X = \lambda \mathbf{1}_+ \oplus \mu \mathbf{1}_{-,\beta}. \tag{4.13}$$

If $X \in \pi_\varphi^{\mathfrak{B}}(\mathfrak{A})'$, X must commute with $\pi_\varphi^{\mathfrak{B}}(TU)$, which connects $\mathscr{H}_+$ with $\mathscr{H}_-^\beta$ and hence $\lambda = \mu$. This proves that $\pi_\varphi^{\mathfrak{B}}$ is irreducible.

Lemma 4.5. *If φ_+ is equivalent to $\varphi_+ \circ \beta \circ \mathrm{Ad}\; U$, then $\varphi^{\mathfrak{B}}$ is a non-trivial mixture of two non-equivalent pure states.*

Proof. Since π_+ is irreducible and $\varphi_+ \sim \varphi_+ \circ \beta \circ \mathrm{Ad}\; U$, there exists a unitary $W_0 \in \pi_+(\mathfrak{A}_+)''$, Ad W_0 on $\pi_+(\mathfrak{A}_+)$ implementing $\beta \circ \mathrm{Ad}\, U$ on $\mathfrak{A}_+$. By $\pi_+ \sim \pi_+ \circ \beta \circ \mathrm{Ad}$ U, there exists a unitary $W \in \pi_\varphi^{\mathfrak{B}}(\mathfrak{A}_+)''$, such that Ad W on $\pi_\varphi^{\mathfrak{B}}(\mathfrak{A}_+)$ implements $\beta \circ \mathrm{Ad}$ U on $\mathfrak{A}_+$. Since $(\mathrm{Ad}\; W)W = W$, W has to commute with $\pi_\varphi^{\mathfrak{B}}(TU)^*$. Let

$$V = W\pi_\varphi^{\mathfrak{B}}(TU)^*. \tag{4.14}$$

For any $a_- \in T\mathfrak{A}_-$, $a_- = (a_- TU)\,(TU)^*$ and hence

$$\begin{aligned}(\mathrm{Ad}\, W)\pi_\varphi^{\mathfrak{B}}(a_-) &= \{(\mathrm{Ad}\, W)\pi_\varphi^{\mathfrak{B}}(a_- TU)\}\pi_\varphi^{\mathfrak{B}}(TU)^* \\ &= \pi_\varphi^{\mathfrak{B}}(\{(\hat{\beta} \circ \mathrm{Ad}\, U)(a_- TU)\}(TU)^*) \\ &= \pi_\varphi^{\mathfrak{B}}((\hat{\beta} \circ \mathrm{Ad}\, U)(a_-)),\end{aligned} \tag{4.15}$$

because $(\hat{\beta} \circ \mathrm{Ad}\, U)(TU) = \hat{\beta}(UT) = \beta(U)T = TU$. Therefore V is in the center $\pi_\varphi^{\mathfrak{B}}(\mathfrak{B})' \cap \pi_\varphi^{\mathfrak{B}}(\mathfrak{B})''$. It is non-trivial because it connects $\mathscr{H}_+$ and $\mathscr{H}_+^\beta$. Thus $\varphi^{\mathfrak{B}}$ is not pure.

Since the restriction of $\pi_\varphi^{\mathfrak{B}}$ to $\mathfrak{A}_+$ is a sum of two equivalent irreducible representations, $\pi_\varphi^{\mathfrak{B}}(\mathfrak{A}_+)'$ is isomorphic to the algebra of 2×2 matrices and $\pi_\varphi^{\mathfrak{B}}(\mathfrak{B})'$ has to be its non-trivial proper $*$-subalgebra, which must be a two-dimensional commutative algebra, coinciding with the center of $\pi_\varphi^{\mathfrak{B}}(\mathfrak{B})''$. Then $\varphi^{\mathfrak{B}}$ is a non-trivial mixture of two non-equivalent pure states.

5. Application of the General Criterion

For φ_β discussed in Sect. 3, which is a Fock state φ_E, the state $\varphi_\beta \circ \Theta_-$ is the Fock state $\varphi_{\theta_- E_- \theta_-}$, where

$$(\theta_- f)_j = \begin{cases} f_j & j \geqq 1 \\ -f_j & j \leqq 0. \end{cases} \tag{5.1}$$

By a general criterion for two Fock states to be equivalent (for example, see Theorem 1 of [3]), the condition (1) of Theorem 2 is satisfied if and only if $E_- - \theta_- E_- \theta_-$ is in the Hilbert–Schmidt class. We have

$$\begin{aligned}\|E_- - \theta_- E_- \theta_-\|_{\mathrm{HS}}^2 &= \operatorname{tr}((\mathbf{1}-E_-)\theta_- E_- \theta_-(\mathbf{1}-E_-) \\ &\quad + E_-\theta_-(\mathbf{1}-E_-)\theta_- E_-) = \|E_-\theta_-(\mathbf{1}-E_-)\|_{\mathrm{HS}}^2 \\ &\quad + \|(\mathbf{1}-E_-)\theta_- E_-\|_{\mathrm{HS}}^2 = 2\|E_-\theta_-(\mathbf{1}-E_-)\|_{\mathrm{HS}}^2,\end{aligned} \tag{5.2}$$

where the last equality is due to $\Gamma\theta_-\Gamma=\theta_-$ and $\Gamma E_-\Gamma=(\mathbf{1}-E_-)$. By evaluating this quantity in Sect. 6, we obtain the following:

Lemma 5.1. *φ_{E_-} and $\varphi_{\theta_- E_- \theta_-}$ are equivalent if $\beta \neq \beta_c$. They are not equivalent if $\beta=\beta_c$.*

To deal with the second condition of Theorem 2, we introduce the following $\mathbb{Z}_2$ index between two basis projections E_1 and E_2 for which $E_1 - E_2$ is in the Hilbert–Schmidt class:

$$\sigma(E_1,E_2) = (-1)^{\dim E_1 \wedge (1-E_2)}. \tag{5.3}$$

By $\Gamma\{E_1 \wedge (\mathbf{1}-E_2)\}\Gamma = (\mathbf{1}-E_1)\wedge E_2$, σ is symmetric in E_1 and E_2. We prove the following in Sect. 7:

Theorem 3. *$\sigma(E_1,E_2)$ is continuous in E_1 and E_2 with respect to the norm topology of E's.*

The criterion for the equivalence of the restriction of Fock states to $\mathfrak{A}_+$ is given by the following theorem to be proved in Sect. 7.

Theorem 4. *The restrictions of Fock states φ_{E_1} and φ_{E_2} of $\mathfrak{A}^F$ to the even part $\mathfrak{A}^F_+$ are equivalent if and only if*

(i) $E_1 - E_2$ is in the Hilbert–Schmidt class and

(ii) $\sigma(E_1,E_2)=1$.

We shall show the following result in Sect. 6 using this criterion.

Lemma 5.2. *$\sigma(E_-,\theta_- E_- \theta_-)$ is 1 if $\beta<\beta_c$ and -1 if $\beta>\beta_c$.*

Proof of this lemma in Sect. 6 consists of two parts, one proving the norm continuity of E_- in the positive parameters K_1^* and K_2 as long as $K_1^* \neq K_2$. Theorem 3 then implies the constancy of $\sigma(E_-,\theta_- E_- \theta_-)$ for $K_2 < K_1^*$ and for $K_1^* < K_2$. Finally $\sigma(E_1,E_2)=1$ is explicitly shown for $K_2=0$ and $\sigma(E_1,E_2)=-1$ for $K_1^*=0$. ($\beta \gtrless \beta_c$ if and only if $K_2 \gtrless K_1^*$.)

Theorem 2, Theorem 4, Lemma 5.1 and Lemma 5.2 imply Theorem 1.

6. Equivalence of φ_β and $\varphi_\beta \circ \Theta_-$ (Proof of Lemma 5.1)

(1) *Equivalence for $K_1^* \neq K_2$*

Denoting $q_\pm = (\mathbf{1} \pm \theta_-)/2$, we have

$$\begin{aligned}\|E_-\theta_-(\mathbf{1}-E_-)\|_{\mathrm{HS}}^2 &= 4\|(\mathbf{1}-E_-)q_+E_-\|_{\mathrm{HS}}^2 \\ &= 4\lim_{\varepsilon\to 0}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}|1-e^{i(\theta_1-\theta_2)-\varepsilon}|^{-2}\operatorname{tr}(E_-(\theta_1)E_+(\theta_2)),\end{aligned} \tag{6.1}$$

where $E_{\pm}(\theta) = (1 \pm V(\theta))/2$ are eigenprojections of (3.7b) belonging to eigenvalues ± 1 and

$$\begin{aligned} \operatorname{tr}(E_-(\theta_1)E_+(\theta_2)) &= \{2 - \operatorname{tr} V(\theta_1)V(\theta_2)\}/4 \\ &= \{1 - \cos(\vartheta(\theta_1) - \vartheta(\theta_2))\}/2. \end{aligned} \tag{6.2}$$

If $K_1^* \neq K_2$, then $\gamma(\theta)$ defined by (3.8) is a strictly positive, real analytic function of θ with a period 2π due to the strict inequality in (3.10) for $K_1^* \neq K_2$. Therefore $\cos\vartheta(\theta)$ and $\sin\vartheta(\theta)$ are real analytic functions of θ with a period 2π due to (3.9) and $\vartheta(\theta) = \theta + \delta(\theta)$. Therefore the integrand of (6.1) is uniformly bounded even at $\varepsilon = 0$ and (6.1) is finite.

(2) *Non-equivalence for* $K_1^* \equiv K_2 \equiv K$

For $\cos\theta \neq 1$ (i.e. $\theta \neq 0 \bmod 2\pi$), $\cos\vartheta(\theta)$ and $\sin\vartheta(\theta)$ are real analytic in θ, and hence the integrand of (6.1) is integrable except possibly near the point $\theta_1 = \theta_2 = 0$. As $\theta \to 0, \gamma(\theta) \to +0$ with

$$\lim_{\theta \to 0} (\gamma(\theta)/\theta)^2 = \sinh^2 2K. \tag{6.3}$$

Therefore $\delta(\theta) \to \pm\pi/2$ (mod 2π) as $\theta \to \pm 0$ by (3.9). Thus the contribution from $\theta_1\theta_2 > 0$ in (6.1) in a neighbourhood of $\theta_1 = \theta_2 = 0$ is finite while the contribution from $\theta_1\theta_2 < 0$ is $+\infty$ due to

$$\int_0^{\alpha} d\theta_1 \int_{-\alpha}^{0} d\theta_2 \{(1 - \cos(\theta_1 - \theta_2))^2 + \sin^2(\theta_1 - \theta_2)\}^{-1} = \infty. \tag{6.4}$$

Therefore φ_β and $\varphi_\beta \circ \Theta_-$ are disjoint at $\beta = \beta_c$.

(3) We now compute the $\mathbb{Z}_2$ index. We consider K_1^* and K_2 as parameters. For $K_1^* \neq K_2$, φ_β and $\varphi_\beta \circ \Theta_-$ are equivalent.

Furthermore the continuity of E_- on the parameters K_1^* and K_2 in the region $K_1^* \neq K_2$ can be seen as follows.

Let the functions δ, ϑ, V and the projections $E_{\pm}$ for another parameter $K_1'^*$ and K_2' be denoted by δ', ϑ', V' and $E'_{\pm}$. Then

$$\begin{aligned} \|E_- - E'_-\| &= \sup_\theta \|(1 - V(\theta)) - (1 - V'(\theta))\|/2 \\ &= 2^{-1} \sup_\theta \|V(\theta) - V'(\theta)\|. \end{aligned} \tag{6.5}$$

For $K_1^* \neq K_2$, $\gamma(\theta)$ is real analytic in K_1^*, K_2 and θ by (3.8) and so are $\cos\delta(\theta)$ and $\sin\delta(\theta)$ due to $\gamma(\theta) \neq 0$. Hence $V(\theta)$ is uniformly continuous in K_1^*, K_2 and θ over a compact set. In particular, (6.5) tends to 0 as $(K_1'^*, K_2')$ tends to (K_1^*, K_2), and hence E_- is continuous in the parameter K_1^* and K_2 relative to the norm topology except for $K_1^* = K_2$.

Hence, by Theorem 3, we have only to find the $\mathbb{Z}_2$-index for (A) $K_2 = 0$, $K_1^* > 0$ and (B) $K_1^* = 0$, $K_2 > 0$.

Case (A). From (3.8) and (3.9), we have $\gamma(\theta) = 2K_1^*$, $\delta(\theta) = \pi - \theta$, $\vartheta(0) = \pi$, $V(\theta) =$

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. (Or $H = K_1^* H_1$ by (3.5).) Therefore $E_- = \theta_- E_- \theta_-$ and $\sigma(E_-, \theta_- E_- \theta_-)$ $= 1$ for $K_2 = 0$ and hence for $K_1^* > K_2$.

Case (*B*). We have $H = K_2 H_2$ with the selfadjoint unitary H_2 given by (2.16). By $E_\pm = (\mathbf{1} \pm H_2)/2$, $h = \begin{pmatrix} f \\ g \end{pmatrix} \in (E_- \wedge \theta_-(\mathbf{1} - E_-)\theta_-)\mathscr{K}$ has to satisfy

$$2(\mathbf{1} - E_-)h = (\mathbf{1} + H_2)h = 0, \tag{6.6}$$

$$2\theta_-(\theta_- E_- \theta_-)h = (\mathbf{1} - H_2)\theta_- h = 0. \tag{6.7}$$

To solve this simultaneous equation, let $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \equiv h_1 = vh$ with $v = 2^{-1/2}$ $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Since $v^2 = \mathbf{1}$ and

$$vH_2 v = -\begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix}, \tag{6.8}$$

we obtain

$$Ug_1 = f_1, U^* f_1 = g_1, \tag{6.9}$$

$$U\theta_- g_1 = -\theta_- f_1, U^* \theta_- f_1 = -\theta_- g_1. \tag{6.10}$$

In particular, we have

$$(U + \theta_- U \theta_-)g_1 = 0, (U^* + \theta_- U^* \theta_-)f_1 = 0. \tag{6.11}$$

These equations mean that all components of g_1 except for the $n = 1$ component and all components of f_1 except for the $n = 0$ component vanish. Furthermore, the $n = 1$ component of g_1 and the $n = 0$ component of f_1 are equal. Conversely such f_1 and g_1 satisfy (6.9) and (6.10). Therefore

$$\dim(E_- \wedge \theta_-(\mathbf{1} - E_-)\theta_-) = 1. \tag{6.12}$$

Hence $\sigma(E_-, \theta_- E_- \theta_-) = -1$ *for* $K_1^* < K_2$.

7. Equivalence Criterion for the Fock Representation of the Even Part $\mathfrak{A}_+^F$

Proof of Theorem 4. Let φ_{1+} and φ_{2+} denote the restrictions of φ_{E_1} and φ_{E_2} to $\mathfrak{A}_+^F$. If φ_{1+} and φ_{2+} are equivalent, then φ_{E_1} and φ_{E_2} must be equivalent due to a result of Stratila and Voiculescu (Sect. 2.7 of [17]). In view of Theorem 1 of [3], this implies that the condition (i) is a necessary condition.

We now assume that (i) holds. We follow the proof of Lemma 9.4 in [3]. There exists a Bogoliubov transformation U which does not change $E_j(\mathbf{1} - E_{\pi/2})$ $(j = 1, 2)$ and changes $E_2 E_{\pi/2}$ to $E_1 E_{\pi/2}$, where

$$E_{\pi/2} = \{E_1 \wedge (\mathbf{1} - E_2)\} + \{(\mathbf{1} - E_1) \wedge E_2\}, \tag{7.1}$$

namely

$$E_2' \equiv U^* E_2 U = E_2(\mathbf{1} - E_{\pi/2}) + E_1 E_{\pi/2}. \tag{7.2}$$

The Bogoliubov automorphism of $\mathfrak{A}^F$ induced by this U is inner and the implementing unitary operator $\hat{Q}(U)$ belongs to $\mathfrak{A}^F_\sigma$, $\sigma = +$ or $-$ according to whether the $\mathbb{Z}_2$-index (5.3) is even or odd. Under the condition (i), there is another Bogoliubov transformation $R(E'_2/E_1)$, which brings E'_2 to E_1

$$R(E'_2/E_1)^* E'_2 R(E'_2/E_1) = E_1, \tag{7.3}$$

and which is implemented in the Fock representation π_1 associated with φ_{E_1} by a unitary operator $Q(R(E'_2/E_1))$ in $\pi_1(\mathfrak{A}^F_+)''$. (In the proof of Lemma 9.4 of [3], it is implemented by a limit of a complex multiple of unitary elements $Q(g_n)$, in the notation of Lemma 9.3, which is of a form $\pi_1(\exp i(B,HB)/2)$ according to Sect. 8 of [3] and belongs to $\pi_1(\mathfrak{A}^F_+)$.) Therefore the cyclic representation π_{2+} associated with φ_{2+} (which is the restriction of the cyclic representation π_2 of $\mathfrak{A}^F$ associated with φ_{E_2} to the subalgebra $\mathfrak{A}^F_+$ and to the subspace $(\mathfrak{A}^F_+\xi_2)$, where ξ_2 is the cyclic vector for φ_{E_2}) is equivalent to the representation π_{1+} or π_{1-} through the unitary operator $Q(R(E'_2/E_1))\pi_1(\hat{Q}(U))$ according to whether the $\mathbb{Z}_2$-index is $+1$ or -1, where $\pi_{1\pm}$ are the restrictions of $\pi_1|\mathfrak{A}^F_+$ to the closure of $\pi_1(\mathfrak{A}^F_\pm)\xi_1$, ξ_1 being the cyclic vector for φ_{E_1}. In view of Lemma 4.2, π_{2+} is equivalent to or disjoint from π_{1+} according to whether the $\mathbb{Z}_2$-index is $+1$ or -1.

Proof of Theorem 3. $E_1 \wedge (\mathbf{1} - E_2)$ is the eigenprojection of $E_1 - E_2$ belonging to the eigenvalue 1. Let $E^0_1 - E^0_2$ be in the Hilbert–Schmidt class and $4\varepsilon\,(>0)$ be the distance of 1 from $\{\mathrm{Spec}\,(E^0_1 - E^0_2)\}\backslash\{1\}$. Then there exists a neighbourhood $\mathcal{N}$ of (E^0_1, E^0_2) such that for any $(E_1, E_2)\in\mathcal{N}$, $\mathrm{Spec}\,(E_1 - E_2) \subset [-1, 1-3\varepsilon] \cup [1-\varepsilon, 1]$. Then

$$E = (2\pi i)^{-1} \int_{|Z-1|=2\varepsilon} (Z - (E_1 - E_2))^{-1} dZ \tag{7.4}$$

must be the spectral projection of $E_1 - E_2$ *for the interval* $[1-\varepsilon, 1]$. *Since it is of* finite dimension for (E^0_1, E^0_2) and is continuous in the norm topology by (7.4), dim E is finite and independent of $(E_1, E_2)\in\mathcal{N}$.

Let $(E_1, E_2)\in\mathcal{N}$ be a pair of basis projections (i.e. $\Gamma E_j \Gamma = 1 - E_j$), such that $E_1 - E_2$ is in the Hilbert–Schmidt class. Theorem 11 of [3] says that the von Neumann algebra generated by E_1 and E_2, restricted to an eigensubspace of $|E_1 - E_2|$ belonging to an eigenvalue not equal to 0 or 1, is isomorphic to the algebra of all 4×4 matrices. This means that the multiplicity of such an eigenvalue of $|E_1 - E_2|$ must be a multiple of 4. Since $\Gamma(E_1 - E_2)\Gamma = (\mathbf{1} - E_1) - (\mathbf{1} - E_2) = -(E_1 - E_2)$, the multiplicity of an eigenvalue of $E_1 - E_2$ belonging to an eigenvalue not equal to 0 or ± 1 must be even. Due to the constancy of the dimension of E, the multiplicity of the eigenvalue 1 of $E_1 - E_2$ is even or odd according to whether the multiplicity of the eigenvalue 1 of $E^0_1 - E^0_2$ is even or odd. Namely we have the constancy of the $\mathbb{Z}_2$-index $\sigma(E_1, E_2) = \sigma(E^0_1, E^0_2)$ in N.

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