Commun. Math. Phys. 88, 113–131 (1983)

Communications in Mathematical Physics

Pure Point Spectrum for Discrete Almost Periodic Schrödinger Operators

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Abstract. The finite difference Schrödinger operator on $\mathbb{Z}^m$ is considered

$$Hu_j = \left(\sum_{\nu=1}^{m} D_\nu^2\right) u_j + \frac{1}{\varepsilon} q_j u_j, \qquad u \in \ell^2(\mathbb{Z}^m),$$

where $\sum_{\nu=1}^{m} D_\nu^2$ is the difference Laplacian in m dimensions. For ε sufficiently small almost periodic potentials q_j are constructed such that the operator H has only pure point spectrum. The method is an inverse spectral procedure, which is a modification of the Kolmogorov-Arnol'd-Moser technique.

1. Introduction

There has been recent interest in the nature of the spectrum of the Schrödinger operator endowed with an almost periodic potential. In contrast to the periodic case, in which there is the classical band structure and the spectrum is all absolutely continuous, there is a wide range of other possibilities. For example the spectrum could be nowhere dense, [1, 12], and pure point or singular continuous spectrum could occur [2]. Somewhat more is known about the spectrum of finite difference Schrödinger operators on $\ell^2(\mathbb{Z})$, [4, 17], especially in the "almost Mathieu" case, in which the potential is given by a pure cosine with period incommensurate with the lattice period. However the existence of pure point spectrum, that is, of $\ell^2(\mathbb{Z})$ eigenvectors, has only been demonstrated in several special cases, for example [4, 16]. In this paper I construct, via an inverse spectral procedure, finite difference Schrödinger operators

$$\begin{aligned} (Hu)_j &= \sum_{|k|=1} u_{j+k} + \lambda q_j u_j, \qquad u_j \in \ell^2(\mathbb{Z}^m), \qquad j \in \mathbb{Z}^m, \\ |j| &= \sum_{\nu=1}^{m} |j_\nu|, \end{aligned} \tag{1.1}$$

which, for λ sufficiently large, have spectrum which is entirely pure point. In these cases the $\ell^2(\mathbb{Z}^m)$ eigenfunctions decay exponentially at a rate given in terms of $\varepsilon = 1/\lambda$, and there is one distinct eigenfunction localized near each lattice site. Examples are given both in which the spectrum (the closure of the set of eigenvalues) is an interval, and in which the spectrum is nowhere dense. Other examples of spectral properties, and properties of the integrated density of states are demonstrated.

There is an intuition that goes along with large coupling constant λ, which states that to a wave function the potential appears as deep wells, separated by high barriers, which should tend to localize solutions. However spatially separated lattice sites with identical values of the potential should resonate, and sites with almost identical values should almost resonate. We start the inverse spectral procedure for the discrete operator (1.1) with a given sequence d_j of eigenvalues as the elements of an infinite diagonal matrix. In order that almost resonant sites be widely separated we ask that

$$|d_j - d_k| > c_1 \Omega(|j-k|), \tag{1.2}$$

where the function $\Omega(s)$ is one of the typical controls of small divisors. For instance [15],

$$\Omega(s) = s^{-\tau}, \quad s \geqq 1, \quad \tau > 0, \tag{1.3}$$

or

$$\Omega(s) = \begin{cases} \exp(-c_0 s/(\log s)^{1+\beta}, & s > e, \quad \beta > 0, \\ \exp(-c_0 e), & 0 \leqq s \leqq e. \end{cases}$$

Although Theorems 1 and 2 could be proven with this condition alone, in order to keep track of the almost periodic nature of the problem we ask the following. For some function $D(x)$ which is 2π periodic in m variables, and for some ω_ℓ, $\ell = 1 \dots m$, independent vectors in $\mathbb{R}^m$, all of whose coefficients are irrational multiples of 2π, the sequence d_j is given by

$$d_j = D(\omega \cdot j), \quad j \in \mathbb{Z}^m, \quad \omega \cdot j = \sum_{\ell=1}^{m} j_\ell \omega_\ell. \tag{1.4}$$

Unfortunately if the function $D(x)$ is continuous the sequence (1.4) violates condition (1.2). Instead we ask that for a certain R-norm to be described in Sect. 4, $\|D(x)\|_R < \infty$. If $m=1$ a possible R-norm is the bounded variation norm. Sequences d_i are not necessarily uniformly almost periodic, but are ℓ^p-almost periodic (Theorem 3), which is a somewhat weaker sense.

Theorem 1 is the main theorem of this paper. Its proof is the construction of a potential, and a convergent infinite product of bounded invertible transformations of $\ell^2(\mathbb{Z}^m)$ which transforms operator (1.1) into diagonal form. At each iteration step there is a loss of decay in the off diagonal direction of these transformations, controlled by requirement (1.3). In addition each matrix multiplication involves an infinite sum, and contributes a loss of decay as well. Both these losses are overcome by the use of a rapidly convergent iteration scheme, which is a variant of the Kolmogorov-Arnol'd-Moser technique. It is a curious fact that we are able to handle the inverse problem, that is spectrum→potential, as the linearized operator is fixed. The more usual potential→spectrum in this case seems more difficult.

Finally, I would like to mention that if one considers the sequence

$$v_k = \varepsilon \sum_{v=1}^{m} \cos(\omega_v k + x_v), \qquad k \in \mathbb{Z},$$

a modification of Theorem 1, [3] and the Aubrey-Andre duality demonstrate that for ε sufficiently small the operator (1.1) on $\mathbb{Z}^m$ with

$$q_j = 4\cos\omega \cdot j, \qquad j \in \mathbb{Z}^m$$

has *some* pure point spectrum.

Note. A preprint of this work has stimulated some additional research which I would like to mention. Pöschel [14] has constructed uniformly almost periodic, limit periodic sequences satisfying condition (1.2) and has constructed by these methods examples of uniformly almost periodic potentials with entirely pure point spectrum. These include cases both where the spectrum is an interval, and where the spectrum forms a Cantor set. Also Bellissard et al. [3] have give examples of functions $D(x)$ satisfying (1.2), (1.4) such that the nonresonant condition (1.2) is preserved under perturbation, so that the forward problem can be done. The localization results of Sarnak [16] and of Fishman et al. [10] are recovered for sufficiently large coupling constant.

2. Main Results

The discrete Schrödinger operator on $\mathbb{Z}^m$ with potential λq_j can be written

$$\sum_{|k|=1} u_{j+k} + \lambda q_j u_j, \qquad j \in \mathbb{Z}^m, \qquad u_j \in \ell^2(\mathbb{Z}^m).$$

We use the notation that

$$|k| = \sum_{k=1}^{m} |k_v|,$$

$$\|k\| = \sup_{k=1\dots m} |k_v|.$$

For $\varepsilon = 1/\lambda$ we multiply through to express the spectral problem

$$(Hu)_j = (\varepsilon M u)_j + (Qu)_j = Eu_j, \tag{2.1}$$

where

$$M_{ij} = \sum_{|k|=1} \delta_{i,j+k}, \qquad Q_{ij} = q_j \delta_{ij} \quad \text{diagonal}.$$

The principal result of this paper is that we are able to construct potentials Q with entirely pure point spectrum via an inverse spectral procedure. The method is to fix a diagonal matrix $D_{ij} = d_j \delta_{ij}$, where the sequence d_j satisfies

$$|d_j - d_i| > c_1 \Omega(|i-j|), \tag{2.2}$$

$$d_j = D(\omega \cdot j), \tag{2.3}$$

for some function $D(x)$ 2π-periodic in m variables. An additional condition is imposed on $D(x)$ so that the sequence d_j will be almost periodic, that is we ask that $\|D(x)\|_R < \infty$ for an R-norm described in Sect. 4. An example of the R-norm for $m=1$ is given by the total variation of the function $D(x)$. The points $\omega \cdot j = \sum_{v=1}^{m} \omega_v j_v$ form an irrational lattice in $\mathbb{R}^m$; ω_v are mutually independent vectors all of whose coefficients are irrational multiples of 2π. Anticipating the potential $q_j = d_j + z_j$ for some sequence z_j to be determined,

$$z_j = Z(\omega \cdot j), \quad Z(x)\ 2\pi\text{-periodic}, \tag{2.4}$$

we construct a unitary transformation of $\ell^2(\mathbb{Z}^m)$ such that (2.1) is transformed into the diagonal matrix D.

Theorem 1. *Given a matrix D satisfying* (2.2), (2.3), *with $\|D(x)\|_R < \infty$, for ε sufficiently small there exist G unitary on $\ell^2(\mathbb{Z}^m)$ and Z diagonal satisfying* (2.4) *such that*

$$D = G^{-1}(D + Z + \varepsilon M)G. \tag{2.5}$$

Furthermore

$$\|Z(x)\|_R \leqq \varepsilon^2 c_0,$$

and the matrix elements of G satisfy

$$|g_{ij} - \delta_{ij}| \leqq \varepsilon c_2 e^{-\sigma|i-j|},$$

where $\sigma = -\log 2m\varepsilon - c_3 > 0$. The constants are independent of ε.

The proof of the theorem is in Sect. 5.

Using directly the unitary transformation G of Theorem 1, we see that all solutions $\psi \in \ell^2(\mathbb{Z}^m)$ of

$$\varepsilon M\psi + (D + Z)\psi = E\psi \tag{2.6}$$

are given by

$$\psi_k = G\delta_{0k}, \qquad E_k = D(\omega \cdot k).$$

Theorem 2. *The operator* (2.1) *with potential $Q = D + Z$ has spectrum exactly the closure of the set of eigenvalues*

$$E_k = D(\omega \cdot k).$$

The associated $\ell^2(\mathbb{Z}^m)$ eigenvectors ψ_k form a complete orthonormal set. Furthermore ψ_k decay exponentially; they satisfy the estimate

$$|(\psi_k)_j - \delta_{jk}| < \varepsilon c_2 \exp(-\sigma(\varepsilon)|j-k|). \tag{2.7}$$

Theorems 1 and 2 state that there exist almost periodic potentials $q_j = d_j + z_j$ with entirely pure point spectrum. The estimate (2.7) implies that all eigenfunctions are exponentially localized, and that there is one eigenfunction corresponding to each lattice site. However since $D(x)$ cannot be a continuous function without violating the nonresonance condition (2.2), the sequence $q_j = D(\omega \cdot j) + Z(\omega \cdot j)$ is not uniformly almost periodic. On the other hand, if the vectors ω are sufficiently irrational, then q_j is ℓ^p-almost periodic. (For the reader's convenience the standard definition is stated in Sect. 4.)

Theorem 3. *Assume that for each* $v = 1 \dots m$ *the vectors* ω *satisfy*

$$|(\omega \cdot j)_v \bmod 2\pi| > c_1 |j|^{-r} \tag{2.8}$$

for $0 < r < m/(m-1)$. *If* $\|D(x)\|_R < \infty$ *the sequence*

$$q_j = D(\omega \cdot j) + Z(\omega \cdot j)$$

is ℓ^p*-almost periodic.*

The proof of the theorem is essentially Lemma 4.4. In one dimension if $\|D(x)\|_R < \infty$ it is easy to show that q_j is ℓ^p-almost periodic without condition (2.8), and the only requirement imposed upon ω is that (2.2) must hold.

With the discrete Schrödinger operator (2.1) we may compute the integrated density of states. A convenient definition is

$$k(E) = \lim_{L \to \infty} \frac{1}{(2L)^m} \operatorname{tr}(\chi_L P_{(-\infty, E]}(H)),$$

where $P_{(-\infty, E]}(H)$ is the spectral resolution of H, and χ_L is the projection

$$a_j \to \chi_L a_j = \begin{cases} a_j & \text{if } \quad \|j\| \leqq L, \\ 0 & \text{otherwise}. \end{cases}$$

For $\varepsilon = 0$ the quantity

$$k_0(E) = \lim_{L \to \infty} \frac{1}{(2L)^m} \operatorname{tr}(\chi_L P_{(-\infty, E]}(D))$$

is particularly easy to compute; for D satisfying (2.3),

$$k_0(E) = \mu\{x \in T^m;\ D(x) \leqq E\}, \tag{2.9}$$

where μ is normalized Lebesgue measure on the m dimensional torus T^m.

Theorem 4. *For* $H = \varepsilon M + (D + Z)$ *the operator of Theorem 1, we know* $k(E) = k_0(E)$.

This is a corollary of a general fact about self adjoint operators on $\ell^2(\mathbb{Z}^m)$ possessing a complete set of exponentially localized eigenfunctions.

Lemma 2.1. *Suppose for* H *self adjoint that* $G^{-1}HG = D$, *for* G *unitary and* D *diagonal. Suppose further that the matrix elements of* G *satisfy*

$$|g_{ij} - \delta_{ij}| \leqq c_1 e^{-\sigma|i-j|}.$$

Then for any f *a bounded measurable function on the spectrum of* H,

$$\lim_{L \to \infty} \frac{1}{(2L)^m} \operatorname{tr}(\chi_L f(H)) = \lim_{L \to \infty} \frac{1}{(2L)^m} \operatorname{tr}(\chi_L f(D)).$$

Proof. It suffices to consider smooth functions f. Then

$$\begin{aligned}\operatorname{tr}(\chi_L f(H)) &= \operatorname{tr}(\chi_L G f(D) G^{-1})\\ &= \operatorname{tr}(G^{-1}\chi_L G f(D))\\ &= \operatorname{tr}(\chi_L f(D) + [G^{-1},\chi_L] G f(D)).\end{aligned}$$

To finish the proof we demonstrate that

$$\operatorname{tr}[G^{-1},\chi_L] G f(D) = O(L^{m-1}).$$

Compute the matrix elements

$$[G^{-1},\chi_L]_{ij} = \begin{cases} 0 & \text{if } \|i\| \text{ and } \|j\| > L, \\ 0 & \text{if } \|i\| \text{ and } \|j\| \leqq L, \\ g_{ji} & \text{if } \|i\| \leqq L \text{ and } \|j\| > L, \\ -g_{ji} & \text{if } \|i\| > L \text{ and } \|j\| \leqq L. \end{cases}$$

We denote the elements $(Gf(D))_{ij} = c_{ij}$, and use only that $\|c_{ij}\|_{\ell^\infty}$ is bounded

$$|[G^{-1},\chi_L] G f(D)_{ij}| \leqq \begin{cases} \sum_{\|\ell\| > L} |g_{\ell i} c_{\ell j}| & \text{if } \|i\| \leqq L, \\ \sum_{\|\ell\| \leqq L} |g_{\ell i} c_{\ell j}| & \text{if } \|i\| > L. \end{cases}$$

The decay of the terms g_{ij} allows us the estimate

$$\begin{aligned}|\operatorname{tr}[G^{-1},\chi_L] G f(D)| &\leqq \|c_{\ell i}\|_{\ell^\infty} \sum_{\|i\| \leqq L} \sum_{\|\ell\| < L} c_1 e^{-\sigma|i-\ell|} + \|c_{\ell i}\|_{\ell^\infty} \sum_{\|i\| > L} \sum_{\|\ell\| < L} c_1 e^{-\sigma|i-\ell|}\\ &\leqq c_2 L^{m-1}. \quad \square\end{aligned}$$

3. Examples

Since the inverse spectral procedure of Theorem 1 produces a potential q_j given a spectrum d_j, it is straightforward to generate potentials with varying spectral properties. Here are some examples.

Example 1. On the m-dimensional torus T^m consider the function

$$D(x) = \begin{cases} \sum_{v=1}^{m} x_v, & 0 \leqq x_v < 2\pi, \\ \text{periodically} & \text{continued}. \end{cases}$$

It is easy to check for any $1 \leqq p < \infty$ that $\|D(x)\|_R < \infty$. Furthermore

$$|D(x+\omega\cdot j) - D(x)| = \left| \sum_{v=1}^{m} (x_v + (\omega\cdot j)_v) \bmod 2\pi - x_v \bmod 2\pi \right|.$$

Denote the components of ω_v by $\omega_{v\ell}$; if the sums $\sum_v \omega_{v\ell}$, $\ell = 1 \dots m$, are sufficiently rationally independent, we have

$$\left| \sum_\ell j_\ell \Big(\sum_v \omega_{v\ell} \Big) \bmod 2\pi \right| \geqq c_1 \Omega(|j|),$$

thus

$$|D(x+\omega\cdot j)-D(x)|\geqq c_1\Omega(|j|),$$

and condition (2.2) is satisfied. By Theorems 1 and 2 we may construct a potential $q_j=Q(\omega\cdot j)$ such that (2.1) has pure point spectrum which, being the closure of the set of eigenvalues, is the interval $[0,2\pi m]$. Now $k(E)$ can be computed from (2.9), it is $m-1$ times differentiable, and strictly increasing on $(0,2\pi m)$.

Example 2. Given $C(y)$ a Cantor function on $[0,1]$, increasing, and normalized such that $C(0)=0$, $C(1)=2\pi m$, define $(C^{-1})(t)$ so that at possible jump discontinuities it takes on some value between its left and right limits. Now set

$$D(x)-(C^{-1})\left(\sum_{\nu=1}^{m} x_\nu \bmod 2\pi\right).$$

By a simple argument it can be shown that for any increasing bounded function $B(t)$, $t\in[0,2\pi m]$, $B\left(\sum_{\nu=1}^{m} x_\nu \bmod 2\pi\right)$ has finite R-norm.

Suppose now that $C(y)$ were Hölder-α continuous (the usual Cantor function involving removal of the middle thirds of intervals has $\alpha=\log 2/\log 3$). Then

$$\begin{aligned}|D(x)-D(x+\omega\cdot j)|&\geqq\left|\sum_{\nu} x_\nu \bmod 2\pi-(x_\nu+(\omega\cdot j)_\nu) \bmod 2\pi\right|^{1/\alpha}\\ &\geqq c_1\Omega^{1/\alpha}(|j|),\end{aligned}$$

if again $\sum_{\nu}\omega_{\nu\ell}$ are sufficiently rationally independent. For $\alpha>0$, $D(x)$ satisfies condition (2.2) with an admissible $\Omega(s)$, and the hypotheses of Theorem 1 are satisfied.

The spectrum in this case is nowhere dense; it is a Cantor set, the compliment of the open intervals of constancy of $C(y)$. Furthermore, for $m=1$ the integrated density of states is

$$k(E)=\begin{cases}0, & E<0,\\ \dfrac{1}{2\pi}C(E), & 0\leqq E\leqq 1,\\ 1, & 1<E.\end{cases}$$

The spectrum in this example may have either zero or positive Lebesgue measure.

It is known [11, 5] for uniformly almost periodic potentials in one dimension that in any interval of constancy of $k(E)$, the value of $k(E)$ is in the frequency module,

$$k(E)=\frac{\omega j}{2\pi} \bmod 1$$

for some integer j. That this is not necessarily the case for potentials q_j which are almost periodic only in a weaker sense is demonstrated by the following.

Example 3 (a counterexample to the gap labeling theorem). For any $0<b<2\pi$ set

$$D(x)=\begin{cases} x\,, & 0\leqq x<b\,,\\ x+1\,, & b\leqq x<2\pi\,,\\ \text{periodically} & \text{continued}\,. \end{cases}$$

Again, for ω satisfying (2.1) Theorem 1 is applicable, and one constructs a ℓ^p-almost periodic potential such that the spectrum, which is entirely pure point, consists of two intervals, $[0,b]\cup[b+1,2\pi+1]$. For $b<E<b+1$, $k(E)=b/2\pi$.

Remark. Bellissard and Scoppola have given another counterexample to the gap labeling theorem [6].

By modifying Example 3, setting $b<D(b)<b+1$ for $b=\omega k$ for some $k\in\mathbb{Z}$, potentials with isolated eigenvalues are constructed. However under translations on the hull $D(x)\to D(x+\alpha)$, the essential spectrum of (2.1) is preserved; while for only Haar measure zero of such translates α will there exist this isolated eigenvalue.

It is known [8] in one dimension, (and suspected in more than one) that the integrated density of states is at least log-Hölder continuous. That is, for $|E-E'|<\frac{1}{2}$

$$|k(E)-k(E')|\leqq\frac{-c_0}{\ln 2|E-E'|}\,. \tag{3.1}$$

In both the discrete and continuous periodic cases, $k(E)$ is actually Hölder-$\frac{1}{2}$. By using Rüssmann's approach to the control of small divisors, where

$$\Omega(s)=\begin{cases}\exp(-c_0 s/(\log s)^{1+\beta}\,, & s\geqq e\,,\\ \exp(-c_0 e)\,, & 0<s<e\,,\end{cases}$$

almost periodic potentials can be constructed for which $k(E)$ is not Hölder continuous for any α. This is one of the conclusions of Theorem 5.

A function $k(E)$ is Hölder-α continuous, $0<\alpha<1$, if for every E,E',

$$|k(E)-k(E')|^{1/\alpha}\leqq c_1|E-E'|\,. \tag{3.2}$$

Similarly the definition (3.1) of log-Hölder continuity may be restated; for $|E-E'|<\frac{1}{2}$

$$\exp\left(\frac{-c_0}{|k(E)-k(E')|}\right)\leqq 2|E-E'|\,. \tag{3.3}$$

These are to be compared with (3.4) in the following.

Theorem 5. *Let $k(E)$ be an increasing function on $[0,1]$, normalized so that $k(0)=0$, $k(1)=1$, and satisfying the following continuity assumption; for $|E-E'|<\frac{1}{2}$*

$$c_1\exp\left(\frac{-c_0}{|k(E)-k(E')|(-\log|k(E)-k(E')|)^{1+\beta}}\right)\leqq|E-E'|\,. \tag{3.4}$$

Then $k(E)$ is the integrated density of states for an almost periodic Schrödinger operator in $\mathbb{Z}^m$.

Proof. Set

$$D(x)=\begin{cases}k^{-1}(x_1/2\pi)\\ \text{continued } 2\pi\text{-periodically in } x\,,\end{cases}$$

where at possible jumps of $k^{-1}(t)$ assign a value between its left and right limits. If ω_ν are irrational vectors satisfying

$$|(\sum\omega_\nu j_\nu)\,\mathrm{mod}\,2\pi|\geqq c_1|j|^{-1}\,, \tag{3.5}$$

then for $|j|>1$,

$$|D(x+\omega\cdot j)-D(x)|\geqq c_1\exp\left(\frac{-c_0|j|}{(\log|j|)^{1+\beta}}\right),$$

and (2.2) is satisfied. Furthermore, since $k(E)$ is increasing, $D(x)$ can be shown to have finite R-norm, and Theorem 1 through 4 are applicable. □

4. Several Lemmata

The proof of Theorem 1 involves the construction of a convergent iteration scheme for the matrices G and Z. Since we wish to keep track of the almost periodic structure of these matrices induced by the periodic nature of $D(x)$, the problem is rewritten in terms of functions on the m torus T^m. In this section we present the notation, and prove several lemmata about almost periodic matrices, including estimates on products and inverses.

Recall first that p-summable norms of almost periodic sequences are defined in the following manner [7]. For $j\in\mathbb{Z}^m$, $|j|=\sum_{\nu=1}^{m}|j_\nu|$,

$$\|a_j\|_{\ell^p}=\lim_{L\to\infty}\left(\frac{1}{(2L)^m}\sum_{|j|\leqq L}|a_j|^p\right)^{1/p}.$$

A sequence a_j is ℓ^p-almost periodic if given ε there exists a relatively dense set of translation vectors τ such that

$$\|a_{j+\tau}-a_j\|_{\ell^p}<\varepsilon\,.$$

Function Spaces

We consider 2π-periodic functions of m variables for which evaluation on an irrational lattice defines a ℓ^p-almost periodic sequence. Take the case $m=1$ first for simplicity. For $A(x)$ a function on S^1 and Δ finite partitions of S^1, define

$$\|A(x)\|_R=\|A(x)\|_{L^\infty}+\sup_\Delta\left(\sum_{x_q\in\Delta}|A(x_{q+1})-A(x_q)|^p\right)^{1/p}.$$

For $p=1$ this is the bounded variation norm.

Lemma 4.1. $\|AB(x)\|_R\leqq\|A(x)\|_R\|B(x)\|_R$.

Proof.

$$\|AB(x)\|_R \leqq \|AB(x)\|_{L^\infty} + \sup_\Delta \Big(\sum_{x_q \in \Delta} |AB(x_{q+1}) - AB(x_q)|^p \Big)^{1/p}$$
$$\leqq \|A(x)\|_{L^\infty} \|B(x)\|_{L^\infty} + \|A(x)\|_{L^\infty} \sup_\Delta (\sum |B(x_{q+1}) - B(x_q)|^p)^{1/p}$$
$$+ \|B(x)\|_{L^\infty} \sup_\Delta (\sum |A(x_{q+1}) - A(x_q)|^p)^{1/p}. \quad \square$$

Given ω, and $A(x)$ of bounded R-norm, we define the sequence $a_j = A(\omega j)$.

Lemma 4.2. *a_j is ℓ^p-almost periodic.*

Proof. Given ℓ choose translation numbers τ such that

$$|\omega\tau \bmod 2\pi| < \inf_{|j| < \ell} |\omega j \bmod 2\pi| .$$

Kronekker's theorem insures that the set of such τ is relatively dense. For $L = M\ell$

$$\frac{1}{L} \sum_{j=1}^{L} |a_{j+\tau} - a_j|^p = \frac{1}{M} \sum_{k=1}^{M} \frac{1}{\ell} \sum_{j=(M-1)k}^{Mk} |A(\omega(j+\tau)) - A(\omega j)|^p$$
$$\leqq \frac{1}{\ell} \|A(x)\|_R^p .$$

For ℓ large the right hand side is small. $\square$

We are given a function $D(x)$ and an irrational ω such that the sequence $d_j = D(\omega j)$ satisfies the nonresonance condition

$$\inf_{x \in S^1} |D(x) - D(x + \omega j)| \geqq c_1 \Omega(|j|) .$$

The following lemma is useful.

Lemma 4.3. *If $c_1 \Omega(|j|) \leqq \inf_{x \in S^1} |D(x) - D(x + \omega j)|$, then*

$$\left\| \frac{1}{D(x) - D(x + \omega j)} \right\|_R \leqq \frac{2\|D(x)\|_R}{c_1^2 \Omega^2(|j|)} .$$

Proof.

$$\sum_{x_p \in \Delta} \left| \frac{1}{D(x_{p+1}) - D(x_{p+1} + \omega j)} - \frac{1}{D(x_p) - D(x_p + \omega j)} \right|^p$$
$$= \sum_{x_p \in \Delta} \left| \frac{D(x_p) - D(x_p + \omega j) - D(x_{p+1}) + D(x_{p+1} + \omega j)}{(D(x_{p+1}) - D(x_{p+1} + \omega j))(D(x_p) - D(x_p + \omega j))} \right|^p$$
$$\leqq \frac{1}{c_1^{2p} \Omega^{2p}(|j|)} \sum_{x_p \in \Delta} |D(x_p) - D(x_p + \omega j) - D(x_{p+1}) + D(x_{p+1} + \omega j)|^p . \quad \square$$

For the case $m > 1$ the R-norm is a little more detailed. Take any finite set of points x_k of the period cell P of $A(x)$ in $\mathbb{R}^m$, and consider all hyperplanes in the coordinate directions through each x_k. Denote by Δ the set of all m-dimensional open right rectangles R_j defined by these hyperplanes, such that $\bigcup_j \bar{R}_j = P$, $R_j \cap R_k = \emptyset$ if $k \neq j$. Denoting the volume $|R_j|$ and any adjacent vertices x_q, y_q, define the R-norm of the periodic function $A(x)$ to be

$$\|A(x)\|_R = \|A(x)\|_{L^\infty} + \sup_{\Delta}\left(\sum_{R_j \in \Delta} |R_j| \sum_{x_q y_q \in R_j} \frac{|A(x_q) - A(y_q)|^p}{|x_q - y_q|}\right)^{1/p}.$$

The norm is multiplicative, for Lemma 4.1 and its proof remain virtually unchanged.

Define an irrational lattice by fixing a set of m independent vectors $\omega_1 \dots \omega_m$ all of whose components are irrational multiples of 2π. For integer vectors $j \in \mathbb{Z}^m$, we consider the lattice

$$\omega \cdot j = \omega_1 j_1 + \dots + \omega_m j_m .$$

Given $A(x)$ of bounded R-norm we form the sequence

$$a_j = A(\omega \cdot j).$$

If an assumption is made on the rational independence of ω, this sequence is ℓ^p-almost periodic.

Lemma 4.4. *Assume for each* $v = 1 \dots m$ *that*

$$|(\omega \cdot j)_v \bmod 2\pi - (\omega \cdot k)_v \bmod 2\pi| > c_1 |j-k|^{-r}, \qquad 0 < r < m/(m-1).$$

If $\|A(x)\|_R < \infty$, *the sequence* $a_j = A(\omega \cdot j)$ *is* ℓ^p*-almost periodic.*

Proof. Given $\ell > 0$, translation vectors τ are chosen so that for each $v = 1 \dots m$

$$|(\omega \cdot \tau)_v \bmod 2\pi| < \inf_{|j|, |k| \leqq \ell} |(\omega \cdot (j-k))_v \bmod 2\pi| . \tag{4.1}$$

Then

$$\frac{1}{(2\ell)^m} \sum_{|j| \leqq \ell} |a_{j+\tau} - a_j|^p \leqq \frac{c_2}{(2\ell)^m} \sum_{|j| \leqq \ell} \frac{\sup_v |(\omega \cdot \tau)_v \bmod 2\pi|}{\prod_\mu |(\omega \cdot \tau)_\mu \bmod 2\pi|} \cdot \sum_{v=2}^{m} \prod_\mu |(\omega \cdot \tau)_\mu \bmod 2\pi|$$

$$\cdot \frac{\left|A\left(\omega \cdot j + \sum_{\mu=1}^{v} (\omega \cdot \tau)_\mu\right) - A\left(\omega \cdot j + \sum_{\mu=1}^{v-1} (\omega \cdot \tau)_\mu\right)\right|^p}{|(\omega \cdot \tau)_v \bmod 2\pi|} .$$

Consider the points $\omega \cdot (j + \tau)$ and $(\omega \cdot j)$, $|j| \leqq \ell$ as the point x_k defining Δ. Under the hypothesis on the vectors ω_ℓ, we may choose a relatively dense set of translations τ satisfying (4.1) and as well

$$\frac{c_1}{2}(2\ell)^{-r} < \inf_v |(\omega \cdot \tau)_v \bmod 2\pi| < c_1 (2\ell)^{-r} .$$

Hence

$$\frac{1}{(2\ell)^m} \frac{\sup_v |(\omega \cdot \tau)_v \bmod 2\pi|}{\prod_\mu |(\omega \cdot \tau)_\mu \bmod 2\pi|} \leqq \frac{1}{(c_1/2)^{m-1} (2\ell)^{m - r(m-1)}},$$

and the exponent $m-r(m-1)$ is positive. The quantity

$$\sum_{|j|\leqq \ell}\prod|(\omega\cdot\tau)_\mu \bmod 2\pi|\sum_{v=2}^{m}\frac{|A(x_q)-A(y_q)|^p}{|(\omega\cdot\tau)_v \bmod 2\pi|}\leqq \|A(x)\|_R^p,$$

and the lemma is proven. □

Finally, if $D(x)$ and ω are such that

$$\inf_{x\in T^m}|D(x+\omega\cdot j)-D(x)|\geqq c_1\Omega(|j|),$$

then the conclusion of Lemma 4.3 is true for $m>1$, by virtue of the same proof.

Matrix Multiplication

We consider infinite matrices A mapping sequences on $\mathbb{Z}^m$ to sequences on $\mathbb{Z}^m$. Given an irrational lattice $\omega\cdot j$, $j\in\mathbb{Z}^m$, we say that A is covariant with respect to translation by ω if there exist functions $A_j(x)$ on T^m with bounded R-norm such that

$$A_{\ell j}=A_{j-\ell}(\omega\cdot\ell),$$

that is, the ℓ^{th} row is generated from the zero$^{\text{th}}$ row by translation by $\omega\cdot\ell$. Matrix multiplication of two covariant matrices takes the form

$$AB_j(x)=\sum_k A_k(x)B_{j-k}(x+\omega\cdot k).$$

We are concerned with matrices $A_j(x)$ whose coefficients decay in norm as $|j|$ increases. Typically

$$\|A_j(x)\|_R\leqq c_1e^{-\varrho|j|}.$$

The following lemmata will be used to control this decay.

Lemma 4.5. *Assume that*

$$\|A_j(x)\|_R\leqq c_1e^{-\varrho|j|},$$
$$\|B_j(x)\|_R\leqq c_2e^{-\sigma|j|}.$$

Then

(i) *if* $\varrho\neq\sigma$,

$$\|(AB)_j(x)\|_R\leqq c_1c_2e^{-\inf(\varrho,\sigma)|j|}\left(\frac{2}{\varrho+\sigma}+\frac{2}{|\varrho-\sigma|}\right)^m,$$

(ii) *if* $\varrho=\sigma$, $0<\gamma\leqq\varrho$,

$$\|(AB)_j(x)\|_R\leqq c_1c_2e^{-\varrho|j|}\left(\frac{1}{\varrho}+|j|\right)^m$$
$$\leqq c_1c_2e^{-(\varrho-\gamma)|j|}\left(\frac{1}{\varrho}+\frac{1}{\gamma e}\right)^m.$$

Proof.

$$\|(AB)_j(x)\|_R \leqq \sum_k \|A_k(x)\|_R \|B_{j-k}(x)\|_R$$
$$\leqq c_1 c_2 \sum_k e^{-\varrho|k|} e^{-\sigma|j-k|}.$$

An integration completes the proof. □

Remark. There is polynomial loss of decay in the off diagonal direction for each multiplication. This as well as the loss due to small divisors must be overcome in the iteration procedure of Sect. 5.

Lemma 4.6. *Assume* $\|A_j(x)\|_R \leqq c_1 e^{-\varrho|j|}$, *where* $c_1 < 1$, $\varrho + \sigma > 1$, $0 < \varrho - \sigma < 1$. *Then*

(i) $$\|A_j^n(x)\|_R \leqq c_1^n \left(\frac{4}{\varrho-\sigma}\right)^{m(n-1)} e^{-\varrho|j|}.$$

If we further ask that $c_1\left(\frac{4}{\varrho-\sigma}\right)^m < \frac{1}{2}$, *then*

(ii) $$\|(I+A)_j^{-1}(x) - \delta_{0j}\|_R \leqq c_1 \sum_{n=0}^{\infty} c_1^n \left(\frac{4}{\varrho-\sigma}\right)^{n(m-1)} e^{-\sigma|j|} \leqq 2c_1 e^{-\sigma|j|}.$$

Proof. (i) uses Lemma 4.5 repeatedly. (ii) follows by applying (i) to the Neumann series for $(I+A)^{-1}$. □

Lemma 4.7. *If* $A_j(x) = 0$ *for* $|j| \neq 1$, *and* $\|A_j(x)\|_R \leqq \varepsilon < 1/2m$, *then*

$$\|(I+A)_j^{-1}(x) - \delta_{0j}\|_R \leqq \frac{1}{2m(1-2m\varepsilon)} e^{-\varrho(\varepsilon)|j|},$$

where

$$\varrho(\varepsilon) = -\ln 2m\varepsilon.$$

Proof. $(I+A)_j^{-1}(x) = \delta_{0j} + \sum_{n=1}^{\infty} (-A)_j^n(x)$. $A_j^n(x) = 0$ unless $n \geqq |j|$, hence

$$\|(I+A)_j^{-1}(x) - \delta_{0j}\|_R \leqq \frac{1}{2m} \sum_{n=|j|}^{\infty} (2m\varepsilon)^n. \quad \square$$

5. Proof of Theorem 1

The proof involves the convergence of an iteration scheme similar to that of the Kolmogorov-Arnol'd-Moser theorem. The unitary matrix G is successively approximated by solutions of a linear equation, with a quadratic error term. In this case the linearized equation for G is a commutator relation whose solution involves small divisors introduced by the quantities $(D(x) - D(x+\omega \cdot j))^{-1}$. The effect of the small divisors is a loss of decay in the terms $|g_{ij}|$ in the off diagonal directions. Because each matrix multiplication involves infinite sums, each multiplication introduces an additional loss of decay. Both are controlled using the rapid convergence of the iteration. Since we are doing the inverse spectral problem, $D(x)$ spectral generating function $\to Q(x) = D(x) + Z(x)$ potential generating function, we know the linearized operator at the solution, namely

$$\mathscr{L}(\cdot) = [D, \cdot].$$

The forward problem, in which $\mathscr{L}$ varies, seems more difficult to handle by these methods.

The First Iteration Step

Consider $Z(x)$ arbitrary, with $\|Z(x)\|_R < \infty$, we seek a transformation

$$G^{(1)} = I + W^{(1)},$$

approximately diagonalizing the operator

$$D + Z + \varepsilon M, \tag{5.1}$$

where M is the matrix $\sum_{|k|=1} \delta_{j,j+k}$, and Z is the diagonal matrix with elements

$$z_j = Z(\omega \cdot j).$$

We find that if $W^{(1)}$ satisfies the commutator relation

$$[W^{(1)}, D] = \varepsilon M, \tag{5.2}$$

and if $(I + W^{(1)})$ is invertible, then

$$D + \varepsilon M^{(1)} + (G^{(1)})^{-1} Z G^{(1)} = (G^{(1)})^{-1}(D + Z + \varepsilon M) G^{(1)},$$

with

$$M^{(1)} = (G^{(1)})^{-1} M W^{(1)}.$$

There is the obvious compatibility condition $M_0(x) = 0$ for (5.1), which is satisfied, and we write

$$W_j^{(1)}(x) = \begin{cases} \dfrac{\varepsilon}{D(x) - D(x + \omega \cdot j)}, & \text{if} \quad |j| = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Using Lemma 4.7, and assuming for $|j| = 1$ that

$$\inf_{x \in T^m} |D(x) - D(x + \omega \cdot j)| > c_1 \Omega(1),$$

$$\varepsilon < c_1^2 \Omega^2(1)/4m \|D(x)\|_R,$$

we estimate

$$\|W_j^{(1)}(x)\|_R \leqq \frac{1}{2m}, \quad \text{when} \quad |j| = 1,$$

$$\|(I + W^{(1)})_j^{-1}(x)\|_R \leqq \frac{1}{2m} \frac{1}{1 - 2m\varepsilon} e^{-\sigma(1)|j|},$$

where

$$\sigma(1) = -\ln(2m\varepsilon).$$

This establishes the exponential decay with respect to $|j|$ which will be used to compensate for the small divisors in further iterations. Finally

$$\|M_j^{(1)}(x)\|_R \leqq c_2(1)e^{-\sigma(1)|j|}.$$

Subsequent Iteration Steps

Assume that after the ν^{th} iteration the Hamiltonian has the form

$$D + \varepsilon M^{(\nu)} + (G^{(\nu)})^{-1} Z^{(\nu)} G^{(\nu)}, \tag{5.3$_\nu$}$$

satisfying the following estimates:

$$\begin{aligned} &\|M_j^{(\nu)}(x)\|_R \leqq c_2(\nu)e^{-\sigma(\nu)|j|}, \\ &\|G_j^{(\nu)}(x) - \delta_{0j}\|_R \leqq c_3(\nu)e^{-\varrho(\nu)|j|}, \\ &\|(G^{(\nu)})_j^{-1}(x) - \delta_{0j}\|_R \leqq c_4(\nu)e^{-\sigma(\nu)|j|}, \\ &\varrho(\nu) > \sigma(\nu), \end{aligned} \tag{5.4$_\nu$}$$

and $Z^{(\nu)}(x)$ is a diagonal matrix, with

$$\|Z^{(\nu)}(x)\|_R \leqq c_5(\nu).$$

To construct a transformation $(I + W^{(\nu+1)})$ approximately diagonalizing $(5.3)_\nu$ we solve the commutator relation

$$[W^{(\nu+1)}, D] = \varepsilon M^{(\nu)} + (G^{(\nu)})^{-1} A^{(\nu+1)} G^{(\nu)} \tag{5.5$_\nu$}$$

for $W_j^{(\nu+1)}(x)$ and $A^{(\nu+1)}(x)$ with finite R-norm. Given that a solution exists, the new operator has the form

$$\begin{aligned} &D + \varepsilon M^{(\nu+1)} + (G^{(\nu+1)})^{-1}(Z^{(\nu)} - A^{(\nu+1)})G^{(\nu+1)} \\ &\quad = (I + W^{(\nu+1)})^{-1}(D + \varepsilon M^{(\nu)} + (G^{(\nu)})^{-1} Z^{(\nu)} G^{(\nu)})(I + W^{(\nu+1)}), \end{aligned} \tag{5.3$_{\nu+1}$}$$

where we have defined

$$\begin{aligned} &G^{(\nu+1)} = G^{(\nu)}(I + W^{(\nu+1)}), \\ &\varepsilon M^{(\nu+1)} = \varepsilon(I + W^{(\nu+1)})^{-1} M^{(\nu)} W^{(\nu+1)} + (G^{(\nu+1)})^{-1} A^{(\nu+1)} G^{(\nu)} W^{(\nu+1)}. \end{aligned}$$

The next two lemmata estimate solutions to Eq. $(5.5)_\nu$. The first shows the existance of a diagonal matrix $A^{(\nu+1)}$ such that the right hand side of $(5.5)_\nu$ satisfies the compatibility condition

$$\varepsilon M_0^{(\nu)}(x) + (G^{(\nu)})^{-1} A^{(\nu+1)} G_0^{(\nu)}(x) = 0.$$

The second lemma bounds $W_j^{(\nu+1)}(x)$ itself.

Lemma 5.1. *Given* $(5.4)_\nu$ *and assume*

$$c_3(\nu)+c_4(\nu)+4^m\cdot\frac{c_3(\nu)c_4(\nu)}{\sigma(\nu)^m}<\tfrac{1}{2}. \tag{5.6}$$

Then there exists a diagonal matrix $A^{(\nu+1)}(x)$ *such that*

(i) $$\varepsilon M_0^{(\nu)}(x)+(G^{(\nu)})^{-1}A^{(\nu+1)}G_0^{(\nu)}(x)=0,$$

(ii) $$\|A^{(\nu+1)}(x)\|_R\leqq 2\varepsilon c_2(\nu).$$

Proof. Use Lemmata 4.5 and 4.6 to find that for a diagonal matrix $A(x)$,

$$\|(G^{(\nu)})^{-1}AG_0^{(\nu)}(x)-A(x)\|_R\leqq\left(c_3(\nu)+c_4(\nu)+4^m\frac{c_3(\nu)c_4(\nu)}{\sigma(\nu)^m}\right)\|A(x)\|_R.$$

Since the quantity

$$\left(c_3(\nu)+c_4(\nu)+4^m\frac{c_3(\nu)c_4(\nu)}{\sigma(\nu)^m}\right)<\frac{1}{2},$$

a contraction argument can be used to solve for a function $A(x)$ satisfying (i) and (ii). □

Lemma 5.2. *There exists a covariant* $W^{(\nu+1)}$ *satisfying Eq.* $(5.5)_\nu$. *If inductively*

$$\varrho(\nu)+\sigma(\nu)>1,$$
$$\varrho(\nu)-\sigma(\nu)<1,$$
$$c_3(\nu)+c_4(\nu)<1,$$

then $W_j^{(\nu+1)}(x)$ *admit the estimates*

$$\|W_j^{(\nu+1)}(x)\|_R\leqq c_6\frac{\varepsilon c_2(\nu)}{(\varrho(\nu)-\sigma(\nu))^m}\frac{e^{-\sigma(\nu)|j|}}{\Omega^2(|j|)} \tag{5.7$_\nu$}$$

for c_6 *dependent only on* m *and* $\|D(x)\|_R$.

Proof. The solution is given by

$$W_j^{(\nu+1)}(x)=\frac{1}{D(x)-D(x+\omega\cdot j)}(\varepsilon M_j^{(\nu)}(x)+(G^{(\nu)})^{-1}A^{(\nu+1)}G_j^{(\nu)}(x)),\quad j\neq 0,$$
$$W_0^{(\nu+1)}=0.$$

Using that $\|A^{(\nu+1)}(x)\|_R\leqq 2\varepsilon c_2(\nu)$, we find from multiplication Lemma 4.5 that for $j\neq 0$

$$\|(G^{(\nu)})^{-1}A^{(\nu+1)}G_j^{(\nu)}(x)\|_R$$
$$\leqq 2\varepsilon c_2(\nu)\cdot\left(c_3(\nu)+c_4(\nu)+c_3(\nu)c_4(\nu)\left[\frac{2}{\varrho(\nu)+\sigma(\nu)}+\frac{2}{\varrho(\nu)-\sigma(\nu)}\right]^m\right)e^{-\sigma(\nu)|j|}.$$

An application of Lemma 4.3 bounds the small divisor loss, and completes the proof. □

We sacrifice some exponential decay to overcome the small divisors. Let $\sigma(v+1)<\varrho(v+1)<\sigma(v)$ be new decay rates, to be made explicit later. Using $(5.7)_v$ we estimate the terms of the transformed operator in $(5.3)_{v+1}$:

$$\|W_j^{(v+1)}(x)\|_R \leqq c_6 \frac{\varepsilon c_2(v)}{(\varrho(v)-\sigma(v))^m} \cdot \sup_{|j|>0}\left(\frac{e^{-(\sigma(v)-\varrho(v+1))|j|}}{\Omega^2(|j|)}\right) \cdot e^{-\varrho(v+1)|j|}. \tag{5.8i}$$

Denote the constant on the right hand side by $c_7(v)$:

$$\|(I+W^{(v+1)})_j^{-1}(x)-\delta_{0j}\|_R \leqq 2c_7(v)e^{-\sigma(v+1)|j|}, \tag{5.8ii}$$

where we inductively assume $\varrho(v+1)$ and $\sigma(v+1)$ have been chosen so that

$$\begin{aligned} c_7(v)\cdot\frac{4^{m+1}}{\varrho(v+1)-\sigma(v+1))^m}&<1,\\ \varrho(v+1)+\sigma(v+1)&>1,\\ \varrho(v+1)-\sigma(v+1)&<1.\end{aligned} \tag{5.9}$$

$$\|(I+W^{(v+1)})^{-1}M^{(v)}W_j^{(v+1)}(x)\|_R \leqq \frac{c_2(v)c_7(v)}{(\sigma(v)-\varrho(v+1))^m}e^{-\sigma(v+1)|j|}, \tag{5.8iii}$$

where we also use (5.9).

$$\begin{aligned}&\|(I+W^{(v+1)})^{-1}(G^{(v)})^{-1}A^{(v+1)}G^{(v)}W^{(v+1)}\|_R\\ &\leqq \varepsilon 4^{m+2}\frac{c_2(v)c_7(v)}{(\varrho(v)-\sigma(v))^m(\sigma(v)-\varrho(v+1))^m}\cdot e^{-\sigma(v+1)|j|},\end{aligned} \tag{5.8iv}$$

where we use (5.6), (5.9), and $\sigma(v)-\varrho(v+1)<1$ as inductive assumptions.

$$\begin{aligned}\|G_j^{(v+1)}(x)-\delta_{0j}\|_R &\leqq \left(c_3(v)+c_7(v)+4^m\frac{c_3(v)c_7(v)}{(\varrho(v)-\varrho(v+1))^m}\right)e^{-\varrho(v+1)|j|},\\ \|(G^{(v+1)})_j^{-1}(x)-\delta_{0j}\|_R &\leqq c_4(v)+2c_7(v)+\frac{4^{m+1}c_4(v)c_7(v)}{(\sigma(v)-\sigma(v+1))^m}\cdot e^{-\sigma(v+1)|j|},\end{aligned} \tag{5.8v}$$

$$\|Z^{(v+1)}(x)\|_R = \|Z^{(v)}(x)-A^{(v+1)}(x)\|_R \leqq \|Z^{(v)}(x)\|_R+2\varepsilon c_2(v). \tag{5.8vi}$$

We define

$$\begin{aligned} c_2(v+1)&=\frac{c_2(v)c_7(v)}{(\sigma(v)-\varrho(v+1))^m}\frac{4^{m+3}}{(\varrho(v)-\sigma(v))^m},\\ c_3(v+1)&=c_3(v)+c_7(v)+4^m\frac{c_3(v)c_7(v)}{(\varrho(v)-\varrho(v+1))^m},\\ c_4(v+1)&=c_4(v)+2c_7(v)+4^{m+1}\frac{c_4(v)c_7(v)}{(\sigma(v)-\sigma(v+1))^m},\end{aligned}$$

following (5.8i) and (5.8iii)–(5.8v). The iteration scheme converges if there are successive choices of the decay rates $\sigma(\nu)$, $\varrho(\nu)$ such that

$$\lim_{\nu\to\infty}\sigma(\nu)=\lim_{\nu\to\infty}\varrho(\nu)=\sigma(\infty)>0\,, \tag{5.10i}$$

$$\begin{aligned}&\lim_{\nu\to\infty}c_2(\nu)=0\,, \qquad \sum_{\nu=1}^{\infty}c_2(\nu)<\infty\,,\\ &\lim_{\nu\to\infty}c_7(\nu)=0\,.\end{aligned} \tag{5.10ii}$$

The inductive assumptions (5.6), (5.9) as well as
$\sigma(\nu)-\varrho(\nu+1)<1$ hold for all ν. (5.10iii)

For the function $\Omega(s)=\exp(-s/(\log s)^{1+\beta})$, it is known [11] that one may take

$$\varrho(\nu)-\sigma(\nu)=\sigma(\nu)-\varrho(\nu+1)=c_0\nu^{-(1+\beta)}\,, \quad \text{where} \quad c_0>\frac{2}{(\log 2)^{1+\beta}}\,.$$

If ε is chosen sufficiently small, (5.9) may be satisfied.

The matrices $G^{(\nu)}$ converge to a covariant matrix $G^{(\infty)}$ which, however, is not necessarily unitary. Its rows are eigenvectors of (2.6), and being self-adjoint with distinct eigenvalues, all rows are orthogonal. If we normalize via a diagonal matrix

$$T_{ij}=\Big(\sum_{k\in\mathbb{Z}^m}|G_k^{(\infty)}(\omega\cdot i)|^2\Big)^{1/2}\delta_{ij}\,,$$

the transformation of Theorem 1 is given by $G=G^{(\infty)}T$.

To complete the proof of Theorem 1 we set $Z(x)=\sum_{\nu=1}^{\infty}A^{(\nu)}(x)$, and the desired form of the operator (2.5) is achieved. The estimate on $\|Z(x)\|_R$ arises from the fact that the correction $A^{(1)}(x)=0$.

Acknowledgements. I would like to thank many people for valuable discussions; especially B. Simon and J. Avron, who interested me in almost periodic potentials, J. Moser, and J. Bellissard, from whom I took the translations on the torus.

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Communicated by B. Simon

Received June 3, 1982; in revised form September 1, 1982