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Nonlinear Schrödinger Equations and Sharp Interpolation Estimates

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Abstract. A sharp sufficient condition for global existence is obtained for the nonlinear Schrödinger equation

$$\text{(NLS)} \qquad 2i\phi_t + \Delta\phi + |\phi|^{2\sigma}\phi = 0, \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}^+,$$

in the case $\sigma = 2/N$. This condition is in terms of an exact stationary solution (nonlinear ground state) of (NLS). It is derived by solving a variational problem to obtain the "best constant" for classical interpolation estimates of Nirenberg and Gagliardo.

I. Introduction

The "best constant" of an interpolation estimate among various norms often has an analytical or geometrical significance [2, 23].

The main objective of this paper is to present a relationship between the best constant for a classical interpolation inequality due to Nirenberg and Gagliardo, and a sharp criterion for the existence of global solutions to the nonlinear Schrödinger equation:

$$2i\frac{\partial\phi}{\partial t} + \Delta\phi + |\phi|^{2\sigma}\phi = 0, \quad \phi = \phi(x,t), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}^+, \quad \phi(x,0) = \phi_0(x). \tag{I.1}$$

in the critical case $\sigma = 2/N$.

We will use the notation $\|f\|_p \equiv \left(\int_{\mathbb{R}^N} |f(x)|^p dx\right)^{1/p}$.

In Sect. II we determine the best constant $C_{\sigma,N}$ for the interpolation estimate [12, 13, 22]:

$$\|f\|_{2\sigma+2}^{2\sigma+2} \leqq C_{\sigma,N}^{2\sigma+2} \|\nabla f\|_2^{\sigma N} \|f\|_2^{2+\sigma(2-N)}, \quad \text{if} \quad 0 < \sigma < \frac{2}{N-2}, \quad N \geqq 2. \tag{I.2}$$

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We find

$$C_{\sigma,N}=\left(\frac{\sigma+1}{\|\psi\|_2^{2\sigma}}\right)^{\frac{1}{2\sigma+2}}, \tag{I.3}$$

where ψ is the ground state solution of

$$\frac{\sigma N}{2}\Delta\psi-\left(1+\frac{\sigma}{2}(2-N)\right)\psi+\psi^{2\sigma+1}=0. \tag{I.4}$$

The results (I.3)–(I.4) are evident from the following considerations: To compute $C_{\sigma,N}$, it will suffice to minimize the functional:

$$J^{\sigma,N}(f)=\frac{\|\nabla f\|_2^{\sigma N}\|f\|_2^{2+\sigma(2-N)}}{\|f\|_{2\sigma+2}^{2\sigma+2}}. \tag{I.5}$$

In Sect. II, we show that the minimum is attained at some H^1 function ψ^*. By scaling we can take $\|\nabla\psi^*\|_2=1$ and $\|\psi^*\|_2=1$. Computing the Euler-Lagrange equation leads to (I.3) and (I.4).

Estimate (I.2) and the constant (I.3) were obtained in the case $N=1$ by Nagy [20]. Partial results for the case $N=2$, $\sigma=2$, were obtained by Levine [17]. The case $N=2$, $\sigma=1$, arose in the work of Payne [24]. He proves $C_{1,2}\leqq\frac{1}{2^{1/4}}$. Levine proves $C_{1,2}\leqq\frac{1}{\pi^{1/4}}$, using an estimate of Federer [10] and Fleming and Rishel [11]. We have computed the ground state solution of $\Delta\psi-\psi+\psi^3=0$ in $\mathbb{R}^2$ for which we obtain $\|\psi\|_2^2=(1.86225\ldots)(2\pi)$. By expression (I.3) we have $C_{1,2}=\left(\frac{1}{\pi\cdot(1.86225\ldots)}\right)^{1/4}$. Our methods enable one to answer the analogous question for any of the inequalities (I.2).

In proving (I.2), we demonstrate the existence of a positive, radial and H^1 solution of

$$\Delta u-u+u^{2\sigma+1}=0 \quad \text{if} \quad 0<\sigma<\frac{2}{N-2}. \tag{I.6}$$

Many other authors have obtained results on the existence of solutions to semilinear equations of the form,

$$\Delta u-u+f(u)=0. \tag{I.7}$$

The case of a power nonlinearity [as in Eq. (I.6)] has been studied by Synge [30], Nehari [21], Ryder [26], Berger [6], and Coffman [9]. The most general results for a general nonlinear term have been obtained by Strauss [27] and by Berestycki and Lions [4].

When $\sigma=2/N$, Eq. (I.1) is known to have global solutions for any $\phi_0\in H^1$ with $\|\phi_0\|_2$ sufficiently small. In Sect. III, we give an answer to the question: "How small?" The answer is simple:

Theorem A. *Let $\phi_0\in H^1(\mathbb{R}^N)$. For $\sigma=2/N$, a sufficient condition for global existence in the initial-value problem* (I.1) *is:*

$$\|\phi_0\|_2<\|\psi\|_2. \tag{I.8}$$

Here ψ is a positive solution of the equation

$$\Delta u - u + u^{\frac{4}{N}+1} = 0 \tag{I.9}$$

of minimal L^2 norm (the ground state), and $\psi e^{it/2}$ is an exact solution of (I.1).

In Sect. IV results of Glassey [16] and Tsutsumi [31] on the "blow up" of solutions to (I.1) are summarized. We then discuss the distinguished role played by solutions of (I.9), in particular the "ground state," in this context. Instability Theorem C expresses the sense in which (I.8) is a sharp condition.

In Sect. V, we present a brief account of numerical results [18, 29, 34] for the "critical case": $\sigma=1$, $N=2$. These observations indicate, at least in the axially symmetric case, that the structure of blowing up solutions is largely self-similar and dominated by the profile of the ground state ψ.

Related qualitative results as well as a detailed account of numerical experiments on the nature of blowing up solutions will be presented in [18, 29, 33].

II. Solution of a Variational Problem

We begin by studying $J^{\sigma,N}$ [see (I.5)], the nonlinear functional naturally associated with the estimate (I.2). By estimate (I.2), $J^{\sigma,N}$ is defined on $H^1(\mathbb{R}^N)$ for $0<\sigma<\frac{2}{N-2}$.

Theorem B. *For* $0<\sigma<\frac{2}{N-2}$,

$$\alpha \equiv \inf_{u\in H^1(\mathbb{R}^N)} J^{\sigma,N}(u)$$

is attained at a function ψ with the following properties:

(1) *ψ is positive and a function of $|x|$ alone.*

(2) $\psi \in H^1(\mathbb{R}^N)\cap C^\infty(\mathbb{R}^N)$

(3) *ψ is a solution of Eq.* (I.4) *of minimal L^2 norm (the ground state).*

In addition,

$$\alpha = \frac{\|\psi\|_2^{2\sigma}}{\sigma+1}.$$

Corollary 2.1. *The best (smallest) constant for which the interpolation estimate* (I.2) *holds is given by expression* (I.3), *where ψ is the ground state of Eq.* (I.4).

Corollary 2.2. *Let* $0<\sigma<\frac{2}{N-2}$. *Then, Eq.* (I.6) *has a positive, radial solution of class* $H^1(\mathbb{R}^N)$.

Remark. McLeod and Serrin [19] have obtained results on the uniqueness of decaying positive solutions for a class of semilinear equations including (I.6). For Eq. (I.6) their results imply uniqueness of the ground state (the H^1 positive solution) in the ranges:

$$0<\sigma<\infty \quad \text{for} \quad 1\leqq N\leqq 2,$$

$$0<\sigma<\frac{2}{N-2} \quad \text{for} \quad 2\leqq N\leqq 4,$$

$$0<\sigma<\frac{8-N}{2N} \quad \text{for} \quad 4<N<8.$$

Note that these results do not cover the entire range in which a ground state is known to exist.

In the proof of Theorem B, we follow Strauss [27] in using a compactness property of functions in $H^1_{\text{radial}}(\mathbb{R}^N)$. We summarize this technique in the following

Compactness Lemma. *For* $0<\sigma<\frac{2}{N-2}$, *the imbedding*

$$H^1_{\text{radial}}(\mathbb{R}^N)\to L^{2\sigma+2}(\mathbb{R}^N) \text{ is compact}.$$

Proof. The lemma will follow from the interpolation estimate

$$\|u\|_{2\sigma+2}^{2\sigma+2}\leqq C\|u\|_{H^1}^{\sigma N}\|u\|_2^{2+\sigma(2-N)}, \qquad 0<\sigma<\frac{2}{N-2},$$

valid for $u\in H^1(\Omega)$, where Ω is a bounded domain, if we can show that a bounded sequence in $H^1_{\text{radial}}(\mathbb{R}^N)$ is uniformly small at infinity. This is a consequence of an estimate due to Strauss [27]: If $u\in H^1_{\text{radial}}(\mathbb{R}^N)$, then

$$|u(x)|\leqq \frac{C}{|x|^{(N-1)/2}}\|u\|_{H^1}. \quad \square$$

Proof of Theorem B. First note that if we set $u^{\lambda,\mu}(x)\equiv\mu u(\lambda x)$, then

(i) $$J^{\sigma,N}(u^{\lambda,\mu})=J^{\sigma,N}(u),$$

(ii) $$\|u^{\lambda,\mu}\|_2^2=\lambda^{-N}\mu^2\|u\|_2^2,$$

(iii) $$\|\nabla_x u^{\lambda,\mu}\|_2^2=\lambda^{2-N}\mu^2\|\nabla u\|_2^2.$$

Since $J^{\sigma,N}(u)\geqq 0$, there exists a minimizing sequence $u_\nu\in H^1(\mathbb{R}^N)\cap L^{2\sigma+2}(\mathbb{R}^N)$, i.e. $\alpha=\inf J^{\sigma,N}(u)=\lim_{\nu\uparrow\infty} J(u_\nu)<\infty$. We can assume $u_\nu>0$, and by symmetrization [2, 25, 27] we can take $u_\nu=u_\nu(|x|)$.

Choosing $\lambda_\nu=\|u_\nu\|_2/\|\nabla u_\nu\|_2$ and $\mu_\nu=\|u_\nu\|_2^{N/2-1}/\|\nabla u_\nu\|_2^{N/2}$, we obtain a sequence $\psi_\nu(x)=u^{\lambda_\nu,\mu_\nu}(x)$ with the following properties:

(a) $$\psi_\nu\geqq 0, \quad \psi_\nu=\psi_\nu(|x|),$$

(b) $$\psi_\nu\in H^1(\mathbb{R}^N),$$

(c) $$\|\psi_\nu\|_2=1 \quad\text{and}\quad \|\nabla\psi_\nu\|_2=1,$$

(d) $$J^{\sigma,N}(\psi_\nu)\downarrow\alpha \quad\text{as}\quad \nu\to\infty.$$

Since the sequence ψ_ν is bounded in $H^1(\mathbb{R}^N)$, some subsequence has a weak H^1 limit ψ^*. Since ψ_ν are radial and uniformly bounded in $H^1(\mathbb{R}^N)$, it follows from the compactness lemma that we can take ψ_ν strongly convergent to ψ^* in $L^{2\sigma+2}(\mathbb{R}^N)$ for $0<\sigma<2/(N-2)$. By weak convergence, $\|\psi^*\|_2 \leqq 1$ and $\|\nabla\psi^*\|_2 \leqq 1$. Hence,

$$\alpha \leqq J^{\sigma,N}(\psi^*) \leqq \frac{1}{\int |\psi^*|^{2\sigma+2} dx} = \lim_{\nu\uparrow\infty} J(\psi_\nu) = \alpha .$$

It follows that $\|\nabla\psi^*\|_2^{\sigma N} \|\psi^*\|_2^{2+\sigma(2-N)} = 1$ and therefore $\|\psi^*\|_2 = \|\nabla\psi^*\|_2 = 1$, so $\psi_\nu \to \psi^*$ strongly in H^1. This proves parts (1) and (2).

Part (3) follows from the fact that ψ^*, the minimizing function, is in $H^1(\mathbb{R}^N)$ and satisfies the Euler-Lagrange equation:

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} J^{\sigma,N}(\psi^* + \varepsilon\eta) = 0 \quad \text{for all} \quad \eta \in C_0^\infty(\mathbb{R}^N).$$

Taking into account that $\|\psi^*\|_2 = 1$ and $\|\nabla\psi^*\|_2 = 1$, we have

$$\frac{\sigma N}{2} \Delta\psi^* - \left(1 + \frac{\sigma}{2}(2-N)\right)\psi^* + \alpha(\sigma+1)(\psi^*)^{2\sigma+1} = 0 .$$

The smoothness of ψ^* follows from results in [5]. Let $\psi^* = [\alpha(\sigma+1)]^{-1/2\sigma}\psi$. Then, ψ satisfies Eq. (1.4) and $\alpha = \|\psi\|_2^{2\sigma}/(\sigma+1)$. This completes the proof of Theorem B.

III. Global Existence for the Initial-Value Problem in the Critical Case $\sigma = 2/N$

In this section we use our "best constant" results of Sect. II to prove Theorem A.

The following result is a consequence of the more general theory of Ginibre and Velo [14]. The special case: $\sigma = 1$, $N = 2$, was studied previously by Baillon et al. [1].

Theorem 3.1. *Let* $\phi_0 \in H^1(\mathbb{R}^N)$.

(i) *If* $0<\sigma<2/N$, *then there exists a unique solution* $\phi \in C([0,\infty); H^1(\mathbb{R}^N))$, *of the initial-value problem* (I.1) *in the sense of the equivalent integral equation.*

(ii) *If* $\sigma = 2/N$, *then for* $\|\phi_0\|_2$ *sufficiently small, the conclusion of* (i) *holds.*

(iii) *As long as* $\phi(x,t)$ *remains in* $H^1(\mathbb{R}^N)$,

$$\mathcal{N}(\phi) \equiv \int |\phi(x,t)|^2 dx$$

and

$$\mathcal{H}(\phi) \equiv \int \left(|\nabla\phi(x,t)|^2 - \frac{1}{\sigma+1} |\phi(x,t)|^{2\sigma+2} \right) dx \tag{III.1}$$

are constants in time.

Remark. If $\sigma \geqq 2/N$, solutions may develop singularities in finite time; hence the term "critical" for the case $\sigma = 2/N$.

In the local existence theorem [14], which holds for $0<\sigma<\dfrac{2}{N-2}$, it is shown that the length T, of the interval of existence $[t_0, t_0+T]$, can be taken to depend only on $\|\phi(t_0)\|_{H^1}$. It follows that if $\phi(x,t)$ is a maximally defined solution on $[t_0, t_{\max}]$, then either

(i) $$t_{\max} = +\infty,$$

or

(ii) $$\lim_{t\uparrow t_{\max}} \|\phi(t)\|_{H^1} = +\infty.$$

The heart of the global existence proof is the use of the invariants (III.1) to get an *a priori* bound of the following type:

$$\|\phi(t)\|_{H^1(\mathbb{R}^N)} \leqq C(\mathcal{N}, \mathcal{H}). \tag{III.2}$$

Ginibre and Velo [14] show that this "H^1-control" of the solution is sufficient for global existence in H^1.

To establish Theorem A, we prove a particular version of (III.2). We proceed as follows:

By the constants of motion and interpolation estimate (I.2):

$$\|\nabla\phi(t)\|_2^2 \leqq \mathcal{H} + \frac{C_{\sigma,N}^{2\sigma+2}}{\sigma+1} \|\phi_0\|_2^{2+\sigma(2-N)} \|\nabla\phi(t)\|_2^{\sigma N}. \tag{III.3}$$

If $0<\sigma<2/N$, the estimate (III.2) follows easily from (III.3). If $\sigma=2/N$, we find

$$\left(1 - \frac{C_N^{4/N+2}}{2/N+1} \|\phi_0\|_2^{4/N}\right) \|\nabla\phi(t)\|_2^2 \leqq \mathcal{H}. \tag{III.4}$$

Corollary 2.1 implies the estimate:

$$\left(1 - \left(\frac{\|\phi_0\|_2}{\|\psi\|_2}\right)^{4/N}\right) \|\nabla\phi(t)\|_2^2 \leqq \mathcal{H}. \tag{III.5}$$

Taking $\|\phi_0\|_2 < \|\psi\|_2$, we get a time-independent bound on $\|\nabla\phi(t)\|_2$. Noting that the scaling $f(x) \to \lambda^{1/\sigma} f(\lambda x)$ leaves the L^2 norm unchanged when $\sigma = \frac{2}{N}$, we find that ψ can be taken to solve Eq. (I.9), from which Theorem A follows.

IV. Blowing Up Solutions

Theorem 4.1. *Let* $|x|\phi_0(x) \in L^2$, *and let* $\phi(x,t)$ *be an* H^1 *solution of Eq.* (I.1) *for* $0 \leqq t \leqq T$. *Then, for* $0 \leqq t \leqq T$

(1) $$\frac{d}{dt}\int \left\{ |x\phi - it\nabla\phi|^2 - \frac{t^2}{\sigma+1}|\phi|^{2\sigma+2} \right\} dx = t \cdot \frac{\sigma N - 2}{\sigma+1} \int |\phi|^{2\sigma+2} dx,$$

(2) $$\frac{d^2}{dt^2}\int |\phi|^2 |x|^2 dx = 2\mathcal{H}(\phi_0) + \frac{N}{\sigma+1}\left(\frac{2}{N} - \sigma\right) \int |\phi|^{2\sigma+2} dx.$$

Remarks. (i) The identity (2) was discovered by Vlasov et al. [32]. It is actually a combination of the identity (1) and the conservation of $\mathcal{H}$, which were derived rigorously for H^1 solutions by Ginibre and Velo [14]. Identity (1) is referred to as the "pseudoconformal conservation law".

(ii) The version of identity (1) obtained for nonlinear Schrödinger equations with the "repulsive interaction" $-|u|^{2\sigma}u$ (instead of $+|u|^{2\sigma}u$), plays an important role in the scattering theory of such equations [15]. In the critical case $\sigma=2/N$, this identity is in fact a conservation law. Scattering results particular to the critical case have been obtained by Strauss [28].

Glassey [16] proved a result on finite time blow up of solutions to (I.1) when $\sigma \geqq 2/N$. We give the proof of a strengthening of Glassey's result due to Tsutsumi [31].

Lemma. *Let $|x|f$ and ∇f belong to $L^2(\mathbb{R}^N)$. Then, f is in $L^2(\mathbb{R}^N)$ and the following estimate holds:*

$$\|f\|_2^2 \leqq \frac{2}{N}\|\nabla f\|_2 \|xf\|_2 . \tag{IV.1}$$

Proof. Note that $-N\int |f|^2 dx = 2\,\mathrm{Re}\int x\cdot \nabla f \bar{f}\, dx$ and apply the Cauchy-Schwarz inequality. □

We remark that "$2/N$" is the best constant for the estimate (IV.1), with equality holding for the functions $f(x)=\exp\{-\frac{1}{2}|x|^2\}$.

We assume for the remainder of this section that $\frac{2}{N} \leqq \sigma < \frac{2}{N-2}$.

Theorem 4.2. *Let either*

(i) $\mathscr{H}(\phi_0)<0$

(ii) $\mathscr{H}(\phi_0)=0$ *and* $\mathrm{Im}\int x\cdot\bar{\phi}_0\nabla\phi_0 dx<0$,

or

(iii) $\mathscr{H}(\phi_0)>0$ *and* $\mathrm{Im}\int x\cdot\bar{\phi}_0\nabla\phi_0 dx \leqq -2\sqrt{\mathscr{H}(\phi_0)}\,\|x\phi_0\|_2$.

Then, there exists $0<T<\infty$ such that

$$\lim_{t\uparrow T}\|\nabla\phi(t)\|_2=+\infty .$$

Proof. Under hypothesis (i), (ii) or (iii), part (2) of Theorem 4.1 implies that if ϕ remains in $H^1(\mathbb{R}^N)$, then there is a $t^*<\infty$ such that

$$\lim_{t\uparrow t^*}\int |\phi|^2|x|^2 dx=0 .$$

We have used that Eq. (I.1) implies that

$$\left.\frac{d}{dt}\right|_{t=0}\int |\phi|^2|x|^2 dx = 2\,\mathrm{Im}\int x\cdot\bar{\phi}_0\nabla\phi_0 dx .$$

By the preceding lemma,

$$\|\phi_0\|_2^2 \leqq \frac{2}{N}\|\nabla\phi(t)\|_2\|x\phi(t)\|_2 .$$

Thus, $\lim_{t\uparrow t^*}\|\nabla\phi(t)\|_2=+\infty$. By the discussion following Theorem 3.1, $t_{\max}\leqq t^*$ and $\lim_{t\uparrow t_{\max}}\|\nabla\phi(t)\|_2=+\infty$. □

In the critical case $\sigma=2/N$, identity (2) of Theorem 4.1 simplifies:

$$\frac{d^2}{dt^2}\int |\phi|^2|x|^2 dx = 2\mathscr{H}(\phi_0) . \tag{IV.2}$$

Consider now the particular solutions $\Phi(x,t)=e^{it/2}R(x)$, where $R(x)$ is an H^1 function satisfying Eq. (I.9). By identity (IV.2),

$$\mathscr{H}(R)=0. \tag{IV.3}$$

Remark. Identity (IV.3) is also a consequence of the Pohozaev identity [4, 27], which can be derived directly from (I.9).

A consequence of (IV.3) is the following instability result in the case $\sigma=2/N$, which also expresses the sense in which the condition (I.8) is sharp.

Instability Theorem C. *Let $\sigma=2/N$. The nontrivial H^1 solutions of Eq.* (I.9) *are unstable for the nonlinear Schrödinger equation* (I.1) *in the following sense:*

Let $R\in H^1$ $(R\not\equiv 0)$ solve Eq. (I.9). *Then for any $\delta>0$, there is a function ζ, with $\|\zeta-R\|_2<\delta$, such that for $\phi(x,t)$ the solution of* IVP (I.1) *with $\phi_0=\zeta$*

$$\lim_{t\to T^-}\|\nabla\phi(t)\|_2=\infty,$$

for some $0<T<\infty$.

Proof. Let $\zeta^\varepsilon(r)=(1+\varepsilon)R(r)$. Then for ε positive, $\|\zeta^\varepsilon\|_2^2=(1+\varepsilon)^2\|R\|_2^2>\|R\|_2^2$ since $R\not\equiv 0$. By (IV.2) $\mathscr{H}(\zeta^\varepsilon)=-2\varepsilon\|R\|_2^2+O(\varepsilon^2)<0$. The result now follows from Theorem 4.2. □

The following picture emerges in the critical case $\sigma=2/N$.

(1) *If* $\phi_0\in H^1(\mathbb{R}^N)$ and $\|\phi_0\|_2<\|\psi\|_2$, where ψ is the ground state of Eq. (I.9) (a positive, radial and H^1 solution of minimal L^2 norm, see Theorem A), then the initial-value problem for Eq. (I.1) has a unique global solution $\phi(x,t)$ of class $C([0,\infty);H^1(\mathbb{R}^N))$.

(2) If $\mathscr{H}(\phi_0)<0$, then the solution $\phi(x,t)$ of Eq. (I.1) blows up in finite time in $H^1(\mathbb{R}^N)$.

(3) By Theorem 4.2 $\mathscr{H}(\phi_0)\geqq 0$ is not sufficient for global existence.

(4) If $\|\phi_0\|_2\leqq\|\psi\|_2$, then $\mathscr{H}(\phi_0)\geqq 0$, by estimate (III.5).

(5) If R is a nontrivial H^1 solution of Eq. (I.8), then $Re^{it/2}$ is an exact solution of Eq. (I.1), and $\mathscr{H}(Re^{it/2})=0$. These solutions are unstable in the sense of Theorem C.

Remarks. The regime defined by

(i) $\mathscr{H}(\phi_0)\geqq 0$ and $\|\phi_0\|_2\geqq\|\psi\|_2$, [$\psi$, the ground state of (I.9)] is currently under numerical investigation [18, 29].

(ii) Berestycki and Cazenave [3] have proved, in the supercritical case $\sigma>2/N$, that the stationary solutions $Re^{it/2}$, where R is a nontrivial H^1 solution of Eq. (I.9), are unstable.

(iii) Cazenave and Lions [7] have proved "orbital stability" of these stationary solutions in the subcritical case $\sigma<2/N$.

V. Numerical Observations and Open Questions

The observations concluding Sect. IV may lead one to suspect that the "zero energy" ($\mathscr{H}=0$) modes $e^{it/2}R(x)$, and in particular the "ground state" $e^{it/2}\psi(|x|)$, play a fundamental role in the structure of solutions with finite time singularities in the critical case $\sigma=2/N$.

We close with a brief account of numerical results obtained in the critical case $\sigma=1$, $N=2$ [18, 29, 34], which corroborate this view. Equation (I.1) reduces to

$$2i\phi_t + \Delta\phi + |\phi|^2\phi = 0, \qquad t\in\mathbb{R}^+, \qquad x\in\mathbb{R}^2. \tag{V.1}$$

This equation arises in modeling the propagation of a thin electromagnetic beam through a medium with an index of refraction dependent on the field intensity (see for example [8]).

For a variety of axially symmetric initial data $\phi_0(|x|)$, the field $\phi(|x|,t)$, as the time of "blow up" is approached, is observed to have the dominant behavior:

$$\phi(|x|,t) \sim \left[\frac{1}{a(t)}\psi\left(\frac{|x|}{a(t)}\right)+P\right]\exp\left\{\frac{i}{2}\int^t \frac{ds}{a^2(s)}\right\}.$$

Here, ψ is the ground state of $\Delta\psi-\psi+\psi^3=0$. P denotes a plateau or a slowly decaying part as $|x|\to\infty$, which is not as prominent in the supercritical case $\sigma=1$, $N=3$ and $a(t)\sim c\cdot(t^*-t)^{2/3}$ as $t\to t^*$. These observations were first made by Zakharov and Synakh [34].

It remains an open problem to establish analytically the sense in which the "ground state" is the state to which blowing up solutions are attracted. Theorem A is, to the author's knowledge, the only known analytical result which displays a connection between the nature of blow up in the critical case and the ground state solution, ψ, of Eq. (I.9).

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References

1. Baillon, J.-B., Cazenave, T., Figueira, M.: Équation de Schrödinger nonlinéaire. C.R. Acad. Sci. Paris **284**, 869–872 (1977)
2. Bandle, C.: Isoperimetric inequalities and applications. London: Pitman 1980
3. Berestycki, H., Cazenave, T.: Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires. C.R. Acad. Sci. **293**, 489–492 (1981)
4. Berestycki, H., Lions, P.L.: Existence of stationary states in nonlinear scalar field equations. In: Bifurcation phenomena in mathematical physics and related topics, pp. 269–292, Bardos, C., Bessis, D. (eds.). Dordrecht, Boston, London: Reidel 1976
5. Berestycki, H., Lions, P.L.: Existence d'ondes solitaires dans des problemes nonlinéaires du type Klein-Gordon. C.R. Acad. Sci. Paris **288**, 395–398 (1979)
6. Berger, M.S.: On the existence and structure of stationary states for a nonlinear Klein-Gordon equation. J. Funct. Anal. **9**, 249–261 (1972)
7. Cazenave, T., Lions, P.L.: Orbital stability of standing waves for some nonlinear Schrödinger equations. Commun. Math. Phys. **85**, 549–561 (1982)
8. Chiao, R.Y., Garmire, E., Townes, C.H.: Self-trapping of optical beams. Phys. Rev. Lett. **13**, 479–482 (1964)
9. Coffman, C.V.: Uniqueness of the ground state solution for $\Delta u-u+u^3=0$ and a variational characterization of other solutions. Arch. Rat. Mech. Anal. **46**, 81–95 (1972)
10. Federer, H.: Curvature measure. Trans. Am. Math. Soc. **93**, 418–490 (1959)
11. Fleming, W., Rishel, R.: An integral formula for total gradient variation. Arch. der Math. **11**, 218–220 (1960)

12. Gagliardo, E.: Proprieta di alcune classi di funzioni in piu varibili. Ricerche di Math. **7**, 102–137 (1958)
13. Gagliardo, E.: Ulteriori proprieta di alcune classi di funzioni in piu variabili. Ricerche di Math. **8**, 24–51 (1959)
14. Ginibre, J., Velo, G.: On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case. J. Funct. Anal. **32**, 1–32 (1979)
15. Ginibre, J., Velo, G.: On a class of nonlinear Schrödinger equations. II. Scattering theory, general case. J. Funct. Anal. **32**, 33–71 (1979)
16. Glassey, R.T.: On the blowing-up of solutions to the Cauchy problem for the nonlinear Schrödinger equation. J. Math. Phys. **18**, 1794–1797 (1977)
17. Levine, H.A.: An estimate for the best constant in a Sobolev inequality involving three integral norms. Ann. Math. Pura Appl. **124**, 181–197 (1980)
18. McLaughlin, D.W., Papanicolaou, G.C., Weinstein, M.I.: Focusing and saturation of nonlinear beams. Siam Rev. (in preparation)
19. McLeod, K., Serrin, J.: Uniqueness of solutions of semilinear Poisson equations. Proc. Natl. Acad. Sci. USA **78**, 6592–6595 (1981)
20. Nagy, B.V.Sz.: Über Integralgleichungen zwischen einer Funktion und ihrer Ableitung. Acta Sci. Math. (Szeged) **10**, 64–74 (1941)
21. Nehari, Z.: On a nonlinear differential equation arising in nuclear physics. Proc. R. Irish Acad. Sci. **62**, 117–135 (1963)
22. Nirenberg, L.: Remarks on strongly elliptic partial differential equations. Commun. Pure Appl. Math. **8**, 648–674 (1955)
23. Osserman, R.: The isoperimetric inequality. Bull. Am. Math. Soc. **84**, 1182–1238 (1978)
24. Payne, L.E.: Uniqueness criteria for steady state solutions of Navier Stokes equations. In: Simpos. Internoz. Appl. Anal. Fix. Mat. (Cagliari-Sassari, 1964), pp. 130–153. Roma: Edizioni Cremonese 1965
25. Polya, G., Szegö, G.: Isoperimetric inequalities in mathematical physics. Princeton: Princeton University Press 1951
26. Ryder, G.H.: Boundary value problems for a class of nonlinear differential equations. Pac. J. Math. **22**, 477–503 (1967)
27. Strauss, W.A.: Existence of solitary waves in higher dimensions. Commun. Math. Phys. **55**, 149–162 (1977)
28. Strauss, W.A.: Nonlinear scattering theory at low energy. J. Funct. Anal. **41**, 110–133 (1981)
29. Sulem, P.L., Sulem, C., Patera, A.: Numerical simulation of the two-dimensional nonlinear Schrödinger equation (to appear)
30. Synge, J.L.: On a certain nonlinear differential equation. Proc. R. Irish Acad. Sci. (1961)
31. Tsutsumi, M.: Nonexistence of global solutions to nonlinear Schrödinger equations (unpublished manuscript)
32. Vlasov, V.N., Petrishchev, I.A., Talanov, V.I.: Izv. Rad. **14**, 1353 (1971)
33. Weinstein, M.I.: Ph.D. thesis, New York University (in preparation)
34. Zakharov, V.E., Synakh, V.S.: The nature of the self-focusing singularity. Sov. Phys. JETP **41**, 465–468 (1976)

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Note added in proof: The author has proved, using the technique of this paper, the following:
Theorem. *Let* $u_0 \in H^2(\mathbb{R}^1)$. *A sufficient condition for the existence of a unique global solution of the* IVP

(GKdV) $\quad u_t + u^4 u_x + u_{xxx} = 0$

$u(x, 0) = u_0(x)$

is $\|u_0\|_2 < \|\psi\|_2$. *Here* $\psi(x-t)$ *is the solitary (traveling) wave solution of* GKdV.

The proof will appear elsewhere.