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The Virasoro Algebra

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Abstract. Three theorems are proved. With suitable hypotheses in each case, characterizations are found for the Virasoro algebra, for some of its representations, and for the Ramond–Neveu–Schwarz superalgebra built around the Virasoro algebra.

1. Introduction

In its centerless version the Virasoro algebra is a Lie algebra with basis u_i, i ranging over the integers, and multiplication table $u_i u_j = (i-j)u_{i+j}$. (I am omitting brackets throughout the paper in the belief that there will be no ambiguity.) It surfaced in the physics literature in the late 1960's and numerous later physics papers have studied it and related algebras.

There was (more or less) an anticipation in the mathematical literature. In the 1930's Witt discovered the Lie algebra that subsequently carried his name; it has the same multiplication table as the Virasoro algebra, but the subscripts range over the integers mod p, p a prime, and the characteristic of the coefficient field is p. Witt did not publish anything; the first reference occurred in the paper [6] of Zassenhaus. In the early 1950's several mathematicians discussed the centerless Virasoro algebra (calling it the "infinite Witt algebra") but nothing was published.

Ramond [4] discovered a way of draping a Lie superalgebra around the Virasoro algebra. Independently, and about at the same time, Neveu and Schwarz [3] did it in a slightly different way. These papers marked the first appearance of Lie superalgebras in the western physics literature.

In this paper I make several modest contributions. I give a certain characterization of the Virasoro algebra; I study some of its representations; and I establish a uniqueness theorem for the Ramond–Neveu–Schwarz construction.

In [5] there is a uniqueness theorem for superalgebras which are allowed to be considerably larger. It is to be observed that they assume outright that the products follow the familiar pattern. The point of this paper is to show, under suitable hypotheses, that this pattern is forced in the "small" case.

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2. A Characterization of the Virasoro Algebra

The centerless Virasoro algebra—call it V for short—is Z-graded, that is, graded by the additive group of all integers, and each homogeneous piece is one-dimensional. A converse can be proved if it is assumed that all products of two different homogeneous constituents are nonzero. Indeed, the hypothesis can be cut down to four products.

Theorem 1. *Let $L = \Sigma L_i$ be a Z-graded Lie algebra in which each L_i has dimension $\leqq 1$. Assume that $L_0 L_1$, $L_1 L_{-1}$, $L_2 L_{-1}$, and $L_{-2} L_1$ are nonzero. Then L is isomorphic to V.*

Proof. The conditions $L_0 L_1 \neq 0$, $L_1 L_{-1} \neq 0$ can be recast as saying that the subalgebra spanned by L_{-1}, L_0, and L_1 is the 3-dimensional simple Lie algebra. We take a standard basis u_{-1}, u_0, u_1.

Let A be the subalgebra spanned by the spaces L_i with $i \geqq -1$. From $L_2 L_{-1} \neq 0$ one deduces that $L_i \neq 0$ for $i \geqq 3$ and that a basis can be completed so as to satisfy $u_i u_j = (i-j) u_{i+j}$. This is known; it is essentially the first step in the structure theory of Cartan's infinite pseudo-groups. A proof from scratch takes only a few lines. Exactly the same remarks of course apply to the subalgebra B spanned by the L_i's with $i \leqq 1$. It remains for us to blend A and B.

The key is to prove $u_2 u_{-2} = 4u_0$. Initally we know only that $u_2 u_{-2} = r u_0$ for some scalar r. We proceed to a circuitous trip through various triples, applying the Jacobi identity repeatedly. From the triple u_1, u_{-2}, u_3 we get $u_4 u_{-2} = 6u_2$. We use this in the triple u_1, u_{-2}, u_4 to get $u_5 u_{-2} = 7u_3$. From the triple u_1, u_{-2}, u_2, however, we only get $u_3 u_{-2}$ evaluated in terms of r as $(9-r)u_1$. But by using the information now available, the triple u_2, u_{-2}, u_3 gives us an *equation*. In detail:

$$u_2 u_3 \cdot u_{-2} + u_3 u_{-2} \cdot u_2 + u_{-2} u_2 \cdot u_3 = 0,$$

$$-u_5 u_{-2} + (9-r) u_1 u_2 - r u_0 u_3 = 0,$$

$$-7u_3 - (9-r) u_3 + 3r u_3 = 0,$$

$$7 + (9-r) - 3r = 0.$$

So $r = 4$ is established. It is now easy to complete the proof of Theorem 1. We have to compute $u_i u_{-j}$ with i and j both $\geqq 2$ and one of them (say i) actually larger than 2. We need only write $u_i = (u_{i-1} u_1)/(i-2)$, apply the Jacobi identity to the triple u_1, u_{i-1}, u_{-j}, and use induction.

3. Virasoro Modules

There is as yet no complete classification of the representations of the Virasoro algebra, or even of its irreducible representations. See [1] and [2] for a study of certain representations. Here I present a theorem concerning a different class of representations—essentially those which occur in the Ramond–Neveu–Schwarz superalgebras.

Let A be a representation space for V. It is reasonable to assume that A shares with V the key properties of being Z-graded and having one-dimensional homogeneous constituents. Thus A has a basis $\{v_j\}$ such that $u_i v_j = t(i,j) v_{i+j}$, with the t's scalars yet to be specified. We put aside the trivial representation where each $u_i v_j = 0$. Then the only representations I know have the form

$$u_i v_j = (a + bi - j) v_{i+j}. \tag{1}$$

For any scalars a and b this is indeed a representation.

Remark. The subscript j is being taken to range over the integers. It could instead range over any coset of the integers in the complex numbers. But a renormalization can switch this coset to the integers, a corresponding change being made in a. In Sect. 4, however, it will make a real difference to have j range over halves of odd integers.

There is a decisive result if it is assumed that u_1 and u_{-1} do not annihilate any of the v's.

Theorem 2. *Let A be a representation space for the centerless Virasoro algebra V. Suppose that A has a basis $\{w_j\}$ such that, for all i and j, $u_i w_j$ is a scalar multiple of w_{i+j}. Suppose further that $u_1 w_j$ and $u_{-1} w_j$ are nonzero for all j. Then the w_j's can be replaced by suitable scalar multiples v_j so that (1) holds for appropriate a and b.*

Proof. In the first part of the proof, (1) will be achieved for $i = -1, 0, 1$. This is really a part of well known facts concerning infinite-dimensional representations of the simple three-dimensional Lie algebra. However, since the form of the result as given here is not standard, a sketch will be offered.

The equation $u_0 w_0 = a w_0$ determines a. The commutation relation between u_0 and u_1 then yields $u_0 w_j = (a - j) w_j$ for all j. Write $u_{-1} \cdot u_1 w_j = c_j w_j$, $u_1 \cdot u_{-1} w_j = d_j w_j$. The Jacobi identity yields

$$d_j - c_j = 2(a - j). \tag{2}$$

By evaluating $u_1(u_{-1} \cdot u_1 w_{j-1})$ in two ways we get $u_1 \cdot c_{j-1} w_{j-1} = d_j \cdot u_1 w_{j-1}$ and hence

$$d_j = c_{j-1}. \tag{3}$$

Combining (2) and (3) gives us the recurrence $d_{j+1} = d_j - 2(a - j)$. Let b be either of the solutions of $d_0 = (a - b)(a + b + 1)$, regarded as a quadratic equation in b. Then from the recurrence we can evaluate d_j for all j and find

$$d_j = (a - b - j)(a + b - j + 1). \tag{4}$$

Note that our hypothesis implies $d_j \neq 0$ and hence both factors in (4) are nonzero. We are now ready to define the v's. We take $v_0 = w_0$ and then determine the v's by requiring $u_{-1} v_j = (a - b - j) v_{j-1}$ for all j. It follows from (4) that $u_1 v_j = (a + b - j) v_{j+1}$ for all j.

To complete the proof of Theorem 2 it will suffice to show that u_2 and u_{-2} act on A in the desired fashion, for V is generated by $u_{-2}, u_{-1}, \ldots, u_2$. Our entering wedge is the equation

$$u_2 u_{-1} \cdot v_j = u_2 \cdot u_{-1} v_j - u_{-1} \cdot u_2 v \tag{5}$$

which partially determines the action of u_2. Write $u_2 v_j = e(j) v_{j+2}$. Then (5) becomes

$$3(a + b - j) = e(j - 1)(a - b - j) - e(j)(a - b - j - 2). \tag{6}$$

The desired solution of (6) is $e(j) = a + 2b - j$. We define an "error term" $f(j)$:

$$e(j) = f(j) + a + 2b - j. \tag{7}$$

By inserting (7) into (6) we derive

$$0 = f(j-1)(a-b-j) - f(j)(a-b-j-2). \tag{8}$$

Equation (8) enables us to express $f(1), f(2), \ldots$ in succession in terms of $f(0)$. The result is

$$f(j) = f(0)\frac{(a-b-2)(a-b-1)}{(z-b-2)(z-b-1)}, \tag{9}$$

where we have set $z = a - j$. In parallel fashion we write $u_{-2}v_j = g(j)v_{j-2}$, $g(j) = h(j) + a - 2b - j$ and get

$$h(j) = h(0)\frac{(a+b+2)(a+b+1)}{(z+b+2)(z+b+1)}, \tag{10}$$

The elements u_0, u_2, and u_{-2} span a three-dimensional simple Lie algebra and A is a representation space for it; so is the subspace spanned by the v_j's with j even. Recall that the characteristic roots of u_0 on the even v_j's move with differences of 2. From this it is a standard conclusion that the coefficients $e(j)g(j+2)$ in the map $v_j \to u_{-2} \cdot u_2 v_j = e(j)g(j+2)v_j$ are, for large even j, a quadratic polynomial in j. So the rational function

$$[f(j) + z + 2b][h(j+2) + z - 2b - 2] \tag{11}$$

must actually be a polynomial. We substitute from (9) and (10); then the first factor in (11) becomes

$$\frac{f(0)(a-b-2)(a-b-1) + (z+2b)(z-b-2)(z-b-1)}{(z-b-2)(z-b-1)}, \tag{12}$$

and the second

$$\frac{h(0)(a+b+1)(a+b+2) + (z-2b-2)(z+b)(z+b-1)}{(z+b)(z+b-1)}. \tag{13}$$

For the product of (12) and (13) to be a polynomial, each of the four linear factors in the two denominators must divide the numerator of (12) or the numerator of (13). Let us assume that $f(0)$ and $h(0)$ are both nonzero. Note that our hypothesis that u_1 and u_{-1} do not annihilate any of the v's implies that the coefficients of $f(0)$ and $h(0)$ in (12) and (13) are nonzero. So $z - b - 2$ does not divide the numerator of (12) and it must divide the numerator of (13). The same is true for $z - b - 1$. Now the numerator of (13) is a cubic polynomial in z that leads off $z^3 - 3z \ldots$. Hence we can identify the numerator of (13) as

$$(z-b-2)(z-b-1)(z+2b).$$

(Incidentally, we have thereby evaluated $h(0)$, but this will play no role.) In the same way we find the numerator of (12) to be

$$(z+b)(z+b-1)(z-2b-2).$$

In sum (under the assumption $f(0) \neq 0$, $h(0) \neq 0$) we have proved

$$e(j) = \frac{(z+b)(z+b-1)(z-2b-2)}{(z-b-2)(z-b-1)}$$

and

$$g(j) = \frac{(z-b)(z-b+1)(z+2b+2)}{(z+b+2)(z+b+1)}.$$

It is a fact that these formulas for the action of u_2 and u_{-2} are consistent with all relations inside the set $u_{-2}, u_{-1}, u_0, u_1, u_2$; in other words, we have found a peculiar representation of this "local Lie algebra." So to get our contradiction we have to move higher, say to u_3 and u_{-3}. From $u_3 = u_2 u_1$ we compute the coefficient in $u_3 v_j = (\ ) v_{j+3}$ to be

$$\frac{(z+b)(z+b-1)(z+b-2)(z-3b+3)}{(z-b-2)(z-b-1)(z-b-3)} \tag{14}$$

and similarly for u_{-3} the coefficient is

$$\frac{(z-b)(z-b+1)(z-b+2)(z+3b+3)}{(z+b+2)(z+b+1)(z+b+3)}. \tag{15}$$

Now, just as for u_2 and u_{-2}, the product of (14) and (15) must be a quadratic polynomial in z. It is not. Faced with this contradiction, we retreat to the untenable assumption that $f(0)$ and $h(0)$ are both nonzero, and deny it. By symmetry we may assume that $f(0) = 0$. The product of (12) and (13), i.e. the product of (13) by $z + 2b$, must still be a polynomial. This forces $h(0) = 0$. We have proved that u_2 and u_{-2} act on A as required, and this completes the proof of Theorem 2.

4. Virasoro Superalgebras

We take V and A as in the preceding section and study the possibility of introducing a multiplication $A \times A \to V$ so as to make $V + A$ a Lie superalgebra with V the even part and A the odd part. This is to be done while maintaining the Z-grading; in other words, $v_j v_k$ is to be a scalar multiple of u_{j+k}. There are then two possibilities for the range of j: all integers and all halves of odd integers.

It is possible to set every $v_j v_k$ equal to 0 and get a legal Lie superalgebra. We set aside this trivial case. We can promptly deduce that the parameter a in (1) is 0. For the Jacobi identity on the triple u_0, v_j, v_k reads

$$u_0 \cdot v_j v_k = u_0 v_j \cdot v_k + u_0 v_k \cdot v_j.$$

If $v_j v_k \neq 0$ we deduce

$$-(j+k) = a - j + a - k,$$

whence $a = 0$. From now on $u_i v_j = (b_i - j) v_{i+j}$.

We proceed to prove that $b = 1/2$. Assume the contrary. We have $v_j^2 v_j = 0$ (this is true for any odd element in a Lie superalgebra). We have $u_{2j} v_j = (2jb - j) v_{3j}$ which is nonzero for $j \neq 0$. Hence, temporarily excluding v_0 in case it is present, we have $v_j^2 = 0$. Next take j and k different and nonzero. From the vanishing of $v_j^2 v_k$ and the Jacobi identity we get $v_j \cdot v_j v_k = 0$. Since

$$u_{j+k} v_j = (bj + bk - j) v_{2j+k},$$

we conclude that $v_j v_k = 0$ unless $bj + bk - j = 0$. By using $v_k^2 v_j$ instead we get the same conclusion unless $bj + bk - k = 0$. It is not possible for both of these statements to hold, so $v_j v_k = 0$. This has looked after all products of v's except v_0^2 (if v_0 is present). To catch v_0^2 we use $v_0 v_j = 0$ (for $j \neq 0$) and the Jacobi identity to get $v_0^2 v_j = 0$. Since $u_0 v_j = -j v_j \neq 0$ we deduce $v_0^2 = 0$. So $A^2 = 0$. But this is the trivial case we excluded at the beginning of the discussion. This contradiction shows that $b = 1/2$.

Henceforth $u_i v_j = (i/2 - j) v_{i+j}$. Write $v_j v_k = s(j,k) u_{j+k}$. We proceed to establish that s is actually independent of j and k. By the Jacobi identity

$$v_j^2 v_k + 2 v_j v_k \cdot v_j = 0.$$

From this we get

$$s(j,j) u_{2j} v_k + 2 s(j,k) u_{j+k} v_j = 0,$$

$$s(j,j)(j-k) + s(j,k)(k-j) = 0,$$

so that $s(j,j) = s(j,k)$. From this the independence of s follows at once. We have proved Theorem 3.

Theorem 3. *Suppose given a Lie superalgebra in which the even part V is the centerless Virasoro algebra and the odd part A is one of the representation spaces of Sect. 3. In detail: V has a basis u_i, i ranging over the integers, with $u_i u_j = (i-j) u_{i+j}$; A has a basis v_j, j ranging over the integers or halves of odd integers, with $u_i v_j = (a + bi - j) v_{i+j}$. Suppose further that each $v_j v_k$ is a scalar multiple of u_{j+k}, and that $A^2 \neq 0$. Then $a = 0$, $b = 1/2$, and the scalar s occurring in $v_j v_k = s u_{j+k}$ is independent of j and k.*

Remark 1. We can change basis in the odd part by replacing each v_j by $r v_j$, r a fixed nonzero scalar. This changes s to $r^2 s$. So over the complex numbers, or more generally over any field where every element is a square, we can renormalize s at our pleasure, e.g. to $s = 1$ or (the Ramond–Neveu–Schwarz version) to $s = 2$.

Remark 2. There are parallel versions of the three theorems for the full Virasoro algebra (with center). I leave it to the reader to make the slight changes necessary.

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