Commun. Math. Phys. 85, 517–528 (1982)

Communications in
Mathematical
Physics

Decay of Correlations for Infinite Range Interactions in Unbounded Spin Systems*

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Abstract. In unbounded spin systems at high temperature with two-body potential we prove, using the associated polymer model, that the two-point truncated correlation function decays exponentially (respectively with a power law) if the potential decays exponentially (respectively with a power law). We also give a new proof of the convergence of the Mayer series for the general polymer model.

1. Definitions and Results

In the finite subset Λ of $\mathbb{Z}^d$ we consider the collection of random variables $S_\Lambda = \{S_x \in \mathbb{R}^\nu, x \in \Lambda\}$ distributed with the Gibbs probability measure, i.e.,

$$Z_\Lambda^{-1} e^{-\beta \sum_{\substack{X \subset \Lambda \\ |X| \geqq 2}} \Phi_X(S_X)} W_\Lambda(dS_\Lambda), \tag{1.1}$$

where Φ is a given many-body potential,

$$W_\Lambda(dS_\Lambda) = \prod_{x \in \Lambda} W_x(dS_x),$$

$$W_x(dS_x) = (\int \mu_x(dS_x) \exp - \beta \Phi_x(S_x))^{-1} (\exp - \beta \Phi_x(S_x)) \mu_x(dS_x), \tag{1.2}$$

where μ_x is the *a priori* single spin distribution and β is the inverse temperature, Z_Λ is the partition function and $|X|$ is the number of points of X.

The finite volume correlation functions are

$$\varrho_\Lambda(S_X) = Z_\Lambda^{-1} \int W_{\Lambda \setminus X}(dS_{\Lambda \setminus X}) \exp - \beta \sum_{\substack{X \subset \Lambda \\ |X| \geqq 2}} \Phi_X(S_X). \tag{1.3}$$

Our first result is the following theorem:

Theorem 1. *Let Φ be a two-body potential such that*

$$|\Phi_{xy}(S_x S_y)| \leqq e^{-\delta(x,y)} J(x,y) v_x(S_x) v_y(S_y), \tag{1.4}$$

* Supported by C.N.R. (G.N.F.M.)

0010-3616/82/0085/0517/$02.40

where δ is a metric on $\mathbb{Z}^d$ such that

$$\sup_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d\setminus\{x\}} e^{-\frac{1}{2}\delta(x,y)}=D, \tag{1.5}$$

$J(x,y)$ satisfies

$$\sup_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d\setminus\{x\}} J(x,y)^{1/3}=J, \tag{1.6}$$

and v_x is such that

$$\sup_{x\in\mathbb{Z}^d}\int W_x(dS_x)\exp\tfrac{1}{2}Jv_x(S_x)^2=I(\beta),$$

$$I(\beta)=O(1),\quad \beta\to 0. \tag{1.7}$$

Then there are two functions, $I_3(\beta)$ and $I_{11}(\beta)$, both $O(\sqrt{\beta})$, $\beta\to 0$ such that, if

$$I(\beta)J\sqrt{\beta}\exp DI_3(\beta)<1, \tag{1.8}$$

$$\beta\sup_{\{x,y\}}J(x,y)^{2/3}<1, \tag{1.9}$$

$$\frac{I(\beta)J^{1/2}\beta^{1/4}}{1-J\beta^{1/2}}<\frac{1}{2e+1}, \tag{1.10}$$

we have for each Λ

$$|\varrho_\Lambda(S_xS_y)-\varrho_\Lambda(S_x)\varrho_\Lambda(S_y)|\leqq e^{\frac{1}{2}Jv_x(S_x)^2+\frac{1}{2}Jv_y(S_y)^2}e^{-\frac{1}{2}\delta(x,y)}I_{11}(\beta). \tag{1.11}$$

This theorem eliminates the finite range assumption on the potential present in a similar theorem in [1], leaving essentially unchanged the other hypothesis. The infinite range case has been already considered in [2, 3], but for special classes of systems. We refer to [1] also for a discussion on the physical meaning of the main hypothesis (1.7) and for the proof of the existence of the infinite volume correlation functions to which, obviously, in a suitable range of β the bound (1.11) applies. For the use of the term $\exp-\delta$ in the potential we refer, for instance, to [4].

The main idea of the proof of Theorem 1 is to use the Mayer expansion for the polymer model associated to our system. Let us recall the polymer model [1, 5]. A polymer is a finite subset of $\mathbb{Z}^d$ and its activity is given by

$$\zeta(R)=\int W_R(dS_R)\zeta(S_R), \tag{1.12}$$

$$\zeta(S_R)=\begin{cases}1 & |R|=1\\ \displaystyle\sum_{K\geqq 1}\frac{1}{K!}\sum^{*}_{(X_1,\dots,X_K)}\prod_{i=1}^{K}(e^{-\beta\Phi_{X_i}(S_{X_i})}-1) & |R|>1,\end{cases}$$

where * means that the sum runs over the K-sequences of subsets of R with $|X_i|\geqq 2$, $X_i\neq X_j$, $\cup X_i=R$ and, denoted by $g(X_1,\dots,X_K)$ the graph on $\{1,\dots,K\}$ that has a line $\{i,j\}$ if and only if $X_i\cap X_j\neq\emptyset$, the graph $g(X_1,\dots,X_K)$ is connected. In force of this definition

$$Z_\Lambda=\sum_{n\geqq 1}\sum_{\{R_1,\dots,R_n\}\in\pi(\Lambda)}\zeta(R_1)\dots\zeta(R_n), \tag{1.13}$$

where $\pi(\Lambda)$ is the set of the partitions of Λ. The correlation functions of the polymer model are

$$\bar{\varrho}_\Lambda(X) = Z_\Lambda^{-1} \sum_{n \geqq 1} \sum_{\{R_1, \ldots, R_n\} \in \pi(\Lambda \setminus X)} \zeta(R_1) \ldots \zeta(R_n) = Z_\Lambda^{-1} Z_{\Lambda \setminus X} \tag{1.14}$$

and, using them, the correlation functions of the system can be conveniently expressed

$$\varrho_\Lambda(S_X) = \sum_{Y \subset \Lambda \setminus X} \bar{\varrho}_\Lambda(X \cup Y) \int W_Y(dS_Y) F_{S_X}(S_Y), \tag{1.15}$$

where

$$F_{S_X}(S_Y) = \sum_{n \geqq 1} \sum_{\substack{\{R_1, \ldots, R_n\} \in \pi(X \cup Y) \\ R_i \cap X \neq \emptyset}} \zeta(S_{R_1}) \ldots \zeta(S_{R_n}). \tag{1.16}$$

The Mayer series for the general polymer model is given by the following theorem in which appears the combinatorial function φ^T (truncated function) that we define on $\bigcup_{n \geqq 1} \mathscr{R}^n$, where $\mathscr{R}$ is the set of the polymers R with $|R| \geqq 2$:

$$\varphi^T(R_1, \ldots, R_n) = \begin{cases} 1 & n = 1 \\ \sum\limits_{g \in C_n} \prod\limits_{\{i, j\} \in g} (\chi(R_i, R_j) - 1) & n > 1, \end{cases} \tag{1.17}$$

where C_n is the set of the connected graphs on $\{1, \ldots, n\}$ and

$$\chi(R_i, R_j) = \begin{cases} 0 & R_i \cap R_j \neq \emptyset \\ 1 & R_i \cap R_j = \emptyset. \end{cases}$$

Theorem 2. *If ζ satisfies, for each integer $K \geqq 2$,*

$$\sup_{x \in \mathbb{Z}^d} \sum_{\substack{x \in R \in \mathscr{R} \\ |R| = K}} |\zeta(\mathrm{R})| \leqq \varepsilon^K \tag{1.18}$$

and

$$\frac{\varepsilon}{1 - \varepsilon} < \frac{1}{2e}, \tag{1.19}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(R_1, \ldots, R_n) \in \mathscr{R}_n \\ \exists R_i = R}} |\varphi^T(R_1, \ldots, R_n) \zeta(R_1) \ldots \zeta(R_n)|$$

$$\leqq |\zeta(R)| \left(1 + |R| e^{|R|} \frac{1}{2} \ln \left(1 - 2e \frac{\varepsilon}{1 - \varepsilon} \right)^{-1} \right), \tag{1.20}$$

and the exponentiation formula holds, i.e.,

$$Z_\Lambda = \exp \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(R_1, \ldots, R_n) \in \mathscr{R}^n \\ R_i \subset \Lambda}} \varphi^T(R_1, \ldots, R_n) \zeta(R_1) \ldots \zeta(R_n). \tag{1.21}$$

The classical method of proof of this theorem goes back to [6] and uses the "algebraic method" and integral equations of Kirkwood-Salsburg type [7]. (In [8, 9] the theorem is explicitly stated for the polymer model.) We present here a new proof of this theorem that shows the peculiar combinatorial aspects of the Mayer series of polymers, making clearer the reasons of its convergence. Non-standard proofs of the convergence of the Mayer series for continuous systems are already known (see, for instance [10, 11]). The proof of Theorem 2 is in Sect. 3. We use it to prove Theorem 1 in the next section.

2. Proof of Theorem 1

We get from (1.15), introducing the shortened notation

$$F_{S_x}(T)=\int W_T(dS_T)F_{S_x}(S_T),$$

$$\varrho_\Lambda(S_xS_y)-\varrho_\Lambda(S_x)\varrho_\Lambda(S_y)=\sum_{T\subset\Lambda\setminus\{x,y\}}\varrho_\Lambda(xyT)F_{S_xS_y}(T)$$

$$-\sum_{T_1\subset\Lambda\setminus\{x\}}\bar{\varrho}_\Lambda(xT_1)F_{S_x}(T_1)\sum_{T_2\subset\Lambda\setminus\{y\}}\bar{\varrho}_\Lambda(yT_2)F_{S_y}(T_2).\quad(2.1)$$

Equation (1.16) gives, putting

$$\zeta(S_xT)=\int W_T(dS_T)\zeta(S_x\cup T),$$

$$F_{S_x}(T)=\zeta(S_xT),\quad(2.2)$$

$$F_{S_xS_y}(T)=\zeta(S_xS_yT)+\sum_{T_1\subset T}\zeta(S_xT_1)\zeta(S_yT\setminus T_1),\quad(2.3)$$

and so

$$\varrho_\Lambda(S_xS_y)-\varrho_\Lambda(S_x)\varrho_\Lambda(S_y)=\sum_{T\subset\Lambda\setminus\{x,y\}}\zeta(S_xS_yT)\bar{\varrho}_\Lambda(xyT)$$

$$+\sum_{T_1\subset\Lambda\setminus\{x,y\}}\sum_{\substack{T_2\subset\Lambda\setminus\{x,y\}\\T_2\cap T_1=\emptyset}}\zeta(S_xT_1)\zeta(S_yT_2)\bar{\varrho}_\Lambda(xyT_1T_2)$$

$$-\sum_{T_1\subset\Lambda\setminus\{x\}}\sum_{T_2\subset\Lambda\setminus\{y\}}\zeta(S_xT_1)\zeta(S_yT_2)\bar{\varrho}_\Lambda(xT_1)\bar{\varrho}_\Lambda(yT_2).\quad(2.4)$$

We perform some obvious manipulations quite similar to the ones in [1] and get

$$\varrho_\Lambda(S_xS_y)-\varrho_\Lambda(S_x)\varrho_\Lambda(S_y)=\Sigma_1+\ldots+\Sigma_6,\quad(2.5)$$

where

$$\Sigma_1=\sum_{T\subset\Lambda\setminus\{x,y\}}\zeta(S_xS_yT)\bar{\varrho}_\Lambda(xyT),$$

$$\Sigma_2=\sum_{T_1\subset\Lambda\setminus\{x,y\}}\sum_{\substack{T_2\subset\Lambda\setminus\{x,y\}\\T_2\cap T_1=\emptyset}}\zeta(S_xT_1)\zeta(S_yT_2)(\bar{\varrho}_\Lambda(xyT_1T_2)-\bar{\varrho}(xT_1)\bar{\varrho}_\Lambda(yT_2)),$$

$$\Sigma_3=-\sum_{T_1\subset\Lambda\setminus\{x,y\}}\sum_{\substack{T_2\subset\Lambda\setminus\{x,y\}\\T_2\cap T_1\neq\emptyset}}\zeta(S_xT_1)\zeta(S_yT_2)\bar{\varrho}_\Lambda(xT_1)\bar{\varrho}_\Lambda(yT_2),$$

$$\Sigma_4 = - \sum_{y \in T_1 \subset \Lambda \setminus \{x\}} \sum_{T_2 \subset \Lambda \setminus \{x,y\}} \zeta(S_x T_1)\zeta(S_y T_2)\bar{\varrho}_\Lambda(xT_1)\bar{\varrho}_\Lambda(yT_2),$$

$$\Sigma_5 = - \sum_{T_1 \subset \Lambda \setminus \{x,y\}} \sum_{x \in T_2 \subset \Lambda \setminus \{y\}} \zeta(S_x T_1)\zeta(S_y T_2)\bar{\varrho}_\Lambda(xT_1)\bar{\varrho}_\Lambda(yT_2),$$

$$\Sigma_6 = - \sum_{y \in T_1 \subset \Lambda \setminus \{x\}} \sum_{x \in T_2 \subset \Lambda \setminus \{y\}} \zeta(S_x T_1)\zeta(S_y T_2)\bar{\varrho}_\Lambda(xT_1)\bar{\varrho}_\Lambda(yT_2). \tag{2.6}$$

We use the following two lemmas to estimate the terms $\Sigma_1, \ldots, \Sigma_6$.

Lemma 1. *In the hypothesis* (1.4), (1.6), (1.7), (1.8), (1.9), *we have*

$$\sup_x \sum_{\substack{x \in R \in \mathscr{R} \\ |R| = K}} |\zeta(R)| \leqq I(\beta)^K (J\sqrt{\beta})^{K-1}(1 - J\sqrt{\beta})^{-1}, \tag{2.7}$$

$$\sup_{\{x,y\}} \sum_{\substack{\{x,y\} \subset R \in \mathscr{R} \\ |R| = K}} |\zeta(R)| \leqq e^{-\delta(x,y)} I(\beta)^K (J\sqrt{\beta})^{K-1}(1 - J\sqrt{\beta})^{-1}, \tag{2.8}$$

$$\sup_{\{x,y\}} \sum_{\substack{T \subset \mathbb{Z}^d \\ |T| = K}} \int W_T(dS_T)|\zeta(S_x S_y S_T)|$$
$$\leqq e^{\frac{1}{2}Jv_x(S_x)^2 + \frac{1}{2}Jv_y(S_y)^2} e^{-\delta(x,y)} I(\beta)^K (J\sqrt{\beta})^{K+1}(1 - J\sqrt{\beta})^{-1}, \tag{2.9}$$

$$\sup_{\{x,y\}} \sum_{\substack{y \in T \subset \mathbb{Z}^d \\ |T| = K}} \int W_T(dS_T)|\zeta(S_x S_T)|$$
$$\leqq e^{\frac{1}{2}Jv_x(S_x)^2} e^{-\delta(x,y)} I(\beta)^K (J\sqrt{\beta})^{K}(1 - J\sqrt{\beta})^{-1}. \tag{2.10}$$

Proof. Our main task is to show (2.7) because the other inequalities follows from obvious modifications of the proof of (2.7). We have

$$\zeta(R) = \int W_R(dS_R) \sum_{g \in C_R} \prod_{\{x,y\} \in g} (e^{-\beta\Phi_{xy}(S_x S_y)} - 1). \tag{2.11}$$

We use (1.4), the inequality $e^{\lambda t} - 1 \leqq t(e^\lambda - 1)$ for $0 \leqq t \leqq 1$, $\lambda \geqq 0$ and (1.9):

$$|\zeta(R)| \leqq \sum_{g \in C_R} \int W_R(dS_R) \prod_{\{x,y\} \in g} (e^{\beta e^{-\delta(x,y)} J(x,y) v_x(S_x) v_y(S_y)} - 1)$$

$$\leqq \sum_{g \in C_R} \left(\prod_{\{x,y\} \in g} \beta e^{-\delta(x,y)} J(x,y)^{2/3} \right) \int W_R(dS_R) e^{\sum_{\{x,y\} \in g} J(x,y)^{1/3} v_x(S_x) v_y(S_y)}.$$

The argument of the exponential, for each $g \in C_R$, is bounded by

$$\tfrac{1}{2} \sum_{\{x,y\} \subset R} v_x(S_x)^2 J(x,y)^{1/3} + \tfrac{1}{2} \sum_{\{x,y\} \subset R} v_y(S_y)^2 J(x,y)^{1/3}$$

and so also by $\frac{1}{2}J \sum_{x \in R} v_x(S_x)^2$ in force of (1.6).

The integral is so bounded by

$$\prod_{x \in R} \int W_x(dS_x) e^{\frac{1}{2}Jv_x(S_x)^2}$$

and, using (1.7), we have

$$|\zeta(R)| \leqq e^{-\delta(R)} I(\beta)^{|R|} \sum_{g \in C_R} \prod_{\{x,y\} \in g} (\beta J(x,y)^{2/3}), \tag{2.12}$$

where $\delta(R)$ is the smallest length of the graphs in C_R.

We observe, as in [8], that to each graph $g \in C_R$ and to each $x \in R$, one can associate at least one sequence $(x_1, \ldots, x_q) \in R^q$ with $q \geqq |R|$ such that $x_1 = x$, $x_i \neq x_{i+1}$, $\{x_1, x_{i+1}\} \in g$, and if $\{y,z\} \in g$ there are one or two labels i such that $\{x_i, x_{i+1}\} = \{y,z\}$. This implies, if (1.9) holds,

$$\sum_{\substack{x \in R \in \mathscr{R} \\ |R| = K}} \sum_{g \in C_R} \prod_{\{x,y\} \in g} (\beta J(x,y)^{2/3}) \leqq \sum_{q=K}^{\infty} \sum_{\substack{(x_1, \ldots, x_q) \in (\mathbb{Z}^d)^q \\ x_1 = x, x_i \neq x_{i+1}}} \prod_{i=1}^{q-1} (\beta^{1/2} J(x_i, x_{i+1})^{1/3})$$

$$\leqq \sum_{q=K}^{\infty} (\beta^{1/2} J)^{q-1},$$

and (2.7) follows remembering (1.8).

Lemma 2. *In the hypothesis of Lemma* 1 *and* (1.10) *there is a function* $I_3(\beta) = O(\sqrt{\beta})$, $\beta \to 0$, such that

$$\left| \frac{\bar{\varrho}_\Lambda(X \cup Y)}{\bar{\varrho}_\Lambda(X) \bar{\varrho}_\Lambda(Y)} - 1 \right| \leqq \exp\left(I_3(\beta) \sum_{x \in X} \sum_{y \in Y} e^{-\delta(x,y)} \right) - 1, \tag{2.13}$$

and, in particular,

$$\left| \frac{\bar{\varrho}_\Lambda(X \cup Y)}{\bar{\varrho}_\Lambda(X) \bar{\varrho}_\Lambda(Y)} - 1 \right| \leqq \exp(I_3(\beta) D \min\{|X|, |Y|\}) e^{-\frac{1}{2}\delta(X,Y)}, \tag{2.14}$$

where $\delta(X,Y) = \min\limits_{\substack{x \in X \\ y \in Y}} \delta(x,y)$.

Proof. The bound (2.7) of Lemma 1 and (1.10) allows us to apply the exponentiation formula (1.21) if we choose

$$\varepsilon = I_1(\beta) = I(\beta) J^{1/2} \beta^{1/4} (1 - J\sqrt{\beta})^{-1}, \tag{2.15}$$

and we get

$$\bar{\varrho}_\Lambda(X) = \exp - \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(R_1, \ldots, R_n) \in \mathscr{R}^n \\ R_i \subset \Lambda \\ \exists R_i \cap X \neq \emptyset}} \varphi^T(R_1, \ldots, R_n) \zeta(R_1) \ldots \zeta(R_n). \tag{2.16}$$

It follows, then,

$$\frac{\bar{\varrho}_\Lambda(X \cap Y)}{\bar{\varrho}_\Lambda(X) \bar{\varrho}_\Lambda(Y)} = \exp \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(R_1, \ldots, R_n) \in \mathscr{R}^n \\ R_i \subset \Lambda \\ \exists i : R_i \cap X \neq \emptyset, R_i \cap Y \neq \emptyset}} \varphi^T(R_1, \ldots, R_n) \zeta(R_1) \ldots \zeta(R_n). \tag{2.17}$$

The argument of the exponential is bounded by

$$\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\substack{R\in\mathscr{R}\\ R\cap X\neq\emptyset,R\cap Y\neq\emptyset}}\sum_{\substack{(R_1,\dots,R_n)\in\mathscr{R}^n\\ \exists i.R_i=R}}|\varphi^T(R_1,\dots,R_n)\zeta(R_1)\dots\zeta(R_n)|$$

$$\leqq\sum_{\substack{x\in X\\ y\in Y}}\sum_{\{x,y\}\subset R\in\mathscr{R}}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\substack{(R_1,\dots,R_n)\in\mathscr{R}^n\\ \exists i\cdot R_i=R}}|\varphi^T(R_1,\dots,R_n)\zeta(R_1)\dots\zeta(R_n)|$$

$$\leqq\sum_{\substack{x\in X\\ y\in Y}}\sum_{\{x,y\}\subset R\in\mathscr{R}}|\zeta(R)|\left(1+|R|e^{|R|}\tfrac{1}{2}\ln\left(1-2e\frac{I_1(\beta)}{1-I_1(\beta)}\right)^{-1}\right)$$

$$=\sum_{\substack{x\in X\\ y\in Y}}\sum_{K=2}^{\infty}\sum_{\substack{\{x,y\}\subset R\in\mathscr{R}\\ |R|=K}}|\zeta(R)|\,(1+Ke^K I_2(\beta)),$$

where we have used (1.20) and we have put

$$I_2(\beta)=\tfrac{1}{2}\ln\left(1-2e\frac{I_1(\beta)}{1-I_1(\beta)}\right)^{-1}. \tag{2.18}$$

The last expression can be bounded, using (2.8), by

$$\sum_{\substack{x\in X\\ y\in Y}}\sum_{K=2}^{\infty}(1+Ke^K I_2(\beta))e^{-\delta(x,y)}I(\beta)^K(J\sqrt{\beta})^{K-1}(1-J\sqrt{\beta})^{-1}$$

$$=\sum_{x\in X}\sum_{y\in Y}e^{-\delta(x,y)}I_3(\beta), \tag{2.19}$$

where $I_3(\beta)$ is defined by the last equation, the series converges by (1.8) and this proves the lemma.

In order to prove Theorem 1 we need to estimate each one of the terms $\Sigma_1,\dots,\Sigma_6$. We have by (2.9)

$$|\Sigma_1|\leqq\sum_{T\subset\Lambda\setminus\{x,y\}}|\zeta(S_xS_yT)|$$

$$\leqq\sum_{K=0}^{\infty}e^{\frac{1}{2}Jv_x(S_x)^2+\frac{1}{2}Jv_y(S_y)^2}e^{-\delta(x,y)}\frac{I(\beta)^K(J\sqrt{\beta})^{K+1}}{1-J\sqrt{\beta}}$$

$$=e^{\frac{1}{2}Jv_x(S_x)^2+\frac{1}{2}Jv_y(S_y)^2}e^{-\delta(x,y)}I_4(\beta), \tag{2.20}$$

where $I_4(\beta)$ is defined by the last equation (in the same way are defined all the functions I in the following).

We have in the second term from Lemma 2

$$|\varrho_\Lambda(xyT_1T_2)-\varrho_\Lambda(xT_1)\varrho_\Lambda(yT_2)|\leqq e^{-\frac{1}{2}\delta(xT_1,yT_2)}\exp DI_3(\beta)(|T_1|+1).$$

For the term in Σ_2 with $T_1\neq\emptyset$ and $T_2\neq\emptyset$ we use the bound

$$\sum_{\substack{t_1,t_2\in\Lambda\setminus\{x,y\}\\ t_1\neq t_2}}\sum_{\substack{t_1\in T_1\subset\Lambda\setminus\{x,y\}\\ \delta(x,T_1)\leqq\delta(x,t_1)}}\sum_{\substack{t_2\in T_2\subset\Lambda\setminus\{x,y\}\setminus T_1\\ \delta(y,T_2)\leqq\delta(y,t_2)}}|\zeta(S_xT_1)\zeta(S_yT_2)|e^{-\frac{1}{2}\delta(xT_1,yT_2)}e^{DI_3(\beta)(|T_1|+1)}$$

$$\leqq \sum_{\substack{t_1,t_2\in\Lambda\setminus\{x,y\}\\ t_1\neq t_2}} \sum_{K_1,K_2=1}^{\infty} \sum_{\substack{t_1\in T_1\subset\Lambda\setminus\{x,y\}\\ \delta(x,T_1)\leqq\delta(x,t_1)\\ |T_1|=K_1}} \sum_{\substack{t_2\in T_2\subset\Lambda\setminus\{x,y\}\setminus T_1\\ \delta(x,T_1)\leqq\delta(x,t_1)\\ |T_2|=K_2}} |\zeta(S_xT_1)\zeta(S_yT_2)|$$

$$\cdot e^{DI_3(\beta)(|T_1|+1)}e^{-\frac{1}{2}\delta(x,y)+\frac{1}{2}\delta(x,t_1)+\frac{1}{2}\delta(y,t_2)}$$

$$\leqq e^{-\frac{1}{2}\delta(x,y)}\sum_{t_1\in\Lambda\setminus\{x,y\}} e^{-\frac{1}{2}\delta(x,t_1)} \sum_{t_2\in\Lambda\setminus\{x,y,t_1\}} e^{-\frac{1}{2}\delta(y,t_2)} \sum_{K_1=1}^{\infty} e^{DI_3(\beta)(K_1+1)}e^{\frac{1}{2}Jv_x(S_x)^2}$$

$$\cdot I(\beta)^{K_1}(J\sqrt{\beta})^{K_1}(1-J\sqrt{\beta})^{-1}\sum_{K_2=1}^{\infty} e^{\frac{1}{2}Jv_y(S_y)^2}I(\beta)^{K_2}(J\sqrt{\beta})^{K_2}(1-J\sqrt{\beta})^{-1}$$

$$\leqq e^{-\frac{1}{2}\delta(x,y)}e^{\frac{1}{2}Jv_x(S_x)^2+\frac{1}{2}Jv_y(S_y)^2}I_5(\beta), \tag{2.21}$$

where we have used (1.8) and $\delta(x,y)\leqq\delta(x,t_1)+\delta(y,t_2)+\delta(xT_1,yT_2)$. The terms with $T_1=\emptyset$ or $T_2=\emptyset$ must be separately estimated. For $T_1=\emptyset$ and $T_2\neq\emptyset$ we have

$$\sum_{t_2\in\Lambda\setminus\{x,y\}}\sum_{t_2\in T_2\subset\Lambda\setminus\{x,y\}} |\zeta(S_yT_2)|e^{-\frac{1}{2}\delta(x,yT_2)}e^{DI_3(\beta)}$$

$$\leqq \sum_{t_2\in\Lambda\setminus\{x,y\}}\sum_{K_2=1}^{\infty}\sum_{\substack{t_2\in T_2\subset\Lambda\setminus\{x,y\}\\ |T_2|=K_2\\ \delta(y,T_2)\leqq\delta(y,t_2)}} |\zeta(S_yT_2)|e^{-\frac{1}{2}\delta(x,y)+\frac{1}{2}\delta(y,t_2)}e^{DI_3(\beta)}$$

$$\leqq e^{-\frac{1}{2}\delta(x,y)}\sum_{t_2\in\Lambda\setminus\{x,y\}} e^{-\frac{1}{2}\delta(y,t_2)}\sum_{K_2=1}^{\infty} e^{-\frac{1}{2}Jv_y(S_y)^2}$$

$$\cdot I(\beta)^{K_2}(J\sqrt{\beta})^{K_2}(1-J\sqrt{\beta})^{-1}e^{DI_3(\beta)}$$

$$\leqq e^{-\frac{1}{2}\delta(x,y)}e^{\frac{1}{2}Jv_y(S_y)^2}I_6(\beta), \tag{2.22}$$

where we have used $\delta(x,y)\leqq\delta(x,yT_2)+\delta(y,t_2)$. The term with $T_1\neq\emptyset$ and $T_2=\emptyset$ give the same contribution, while the term with $T_1=T_2=\emptyset$ gives a contribution less than

$$(e^{I_3(\beta)}-1)e^{-\delta(x,y)}.$$

We so get

$$|\Sigma_2|\leqq e^{-\frac{1}{2}\delta(x,y)}e^{\frac{1}{2}Jv_x(S_x)^2+\frac{1}{2}Jv_y(S_y)^2}I_7(\beta). \tag{2.23}$$

The third term in (2.6) is bounded by

$$\sum_{t\in\Lambda\setminus\{x,y\}}\sum_{K_1,K_2=1}^{\infty}\sum_{\substack{t\in T_1\subset\Lambda\setminus\{x,y\}\\ |T_1|=K_1}}\sum_{\substack{t\in T_2\subset\Lambda\setminus\{x,y\}\\ |T_2|=K_2}} |\zeta(S_xT_1)\zeta(S_yT_2)|$$

$$\leqq \sum_{t\in\Lambda\setminus\{x,y\}}\sum_{K_1,K_2=1}^{\infty} e^{-\delta(x,t)-\delta(y,t)}e^{\frac{1}{2}Jv_x(S_x)^2+\frac{1}{2}Jv_y(S_y)^2}\frac{I(\beta)^{K_1+K_2}(J\sqrt{\beta})^{K_1+K_2}}{(1-J\sqrt{\beta})^2}$$

$$\leqq e^{-\frac{1}{2}\delta(x,y)}e^{\frac{1}{2}Jv_x(S_x)^2+\frac{1}{2}Jv_y(S_y)^2}I_8(\beta). \tag{2.24}$$

The fourth term in (2.6) is bounded by

$$\sum_{K_1=1}^{\infty}\sum_{\substack{y\in T_1\subset\Lambda\setminus\{x\}\\ |T_1|=K_1}} |\zeta(\mathrm{S}_x\mathrm{T}_1)|\sum_{K_2=0}^{\infty}\sum_{t_2\in\Lambda\setminus\{x,y\}}\sum_{\substack{t_2\in T_2\subset\Lambda\setminus\{x,y\}\\ |T_2|=K_2}} |\zeta(\mathrm{S}_y\mathrm{T}_2)|$$

$$\leqq \sum_{K_1=1}^{\infty} e^{-\delta(x,y)} e^{\frac{1}{2}Jv_x(S_x)^2} I(\beta)^{K_1} (J\sqrt{\beta})^{K_1} (1-J\sqrt{\beta})^{-1}$$

$$\cdot \sum_{K_2=0}^{\infty} \sum_{t_2 \in \Lambda\setminus\{x,y\}} e^{-\delta(y,t_2)} e^{\frac{1}{2}Jv_y(S_y)^2} I(\beta)^{K_2} (J\sqrt{\beta})^{K_2} (1-J\sqrt{\beta})^{-1}$$

$$\leqq e^{-\delta(x,y)} e^{\frac{1}{2}Jv_x(S_x)+\frac{1}{2}Jv_y(S_y)^2} I_9(\beta). \tag{2.25}$$

The fifth term gives the same contribution as the fourth and the sixth gives

$$\sum_{K_1=1}^{\infty} \sum_{\substack{y \in T_1 \subset \Lambda\setminus\{x\} \\ |T_1|=K_1}} |\zeta(S_x T_1)| \sum_{K_2=1}^{\infty} \sum_{\substack{x \in T_2 \subset \Lambda\setminus\{y\} \\ |T_2|=K_2}} |\zeta(S_y T_2)|$$

$$\leqq e^{-2\delta(x,y)} e^{\frac{1}{2}Jv_x(S_x)^2+\frac{1}{2}Jv_y(S_y)^2} I_{10}(\beta). \tag{2.26}$$

Collecting the six bounds we finally get (1.11) and it is easy to see that

$$I_{11}(\beta) = O(\sqrt{\beta}), \qquad \beta \to 0.$$

3. Proof of Theorem 2

We rewrite Eq. (1.13) in the form

$$Z_\Lambda = 1 + \sum_{n \geqq 1} \frac{1}{n!} \sum_{\substack{(R_1,\ldots,R_n)\in\mathscr{R}^n \\ R_i \cap R_j = \emptyset, R_i \subset \Lambda}} \zeta(\mathrm{R}_1)\ldots\zeta(\mathrm{R}_n)$$

$$= 1 + \sum_{n \geqq 1} \frac{1}{n!} \sum_{\substack{(R_1,\ldots,R_n)\in\mathscr{R}^n \\ R_i \subset \Lambda}} \zeta(R_1)\ldots\zeta(R_n) \prod_{\{i,j\}\subset\{1,\ldots,n\}} \chi(R_i, R_j), \tag{3.1}$$

and we insert in the last expression the expansion

$$\prod_{\{i,j\}\subset\{1,\ldots,n\}} \chi(R_i, R_j) = \sum_{K=1}^{n} \sum_{\{I_1,\ldots,I_K\}\in\pi(\{1,\ldots,n\})} \varphi^T(R_h, h \in I_1)\ldots\varphi^T(R_h, h\in I_K).$$

So we get, at least formally, the exponentiation formula (1.21) exchanging the order of summation. This exchange can be done if the series

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(R_1,\ldots,R_n)\in\mathscr{R}^n \\ R_i \subset \Lambda}} \varphi^T(\mathrm{R}_1, \ldots, \mathrm{R}_n)\zeta(\mathrm{R}_1)\ldots\zeta(\mathrm{R}_n) \tag{3.2}$$

is absolutely convergent. This follows from (1.20) if we use

$$\sum_{\substack{(R_1,\ldots,R_n)\in\mathscr{R}^n \\ R_i \subset \Lambda}} (\ldots) \leqq \sum_{\substack{R\in\mathscr{R} \\ R\subset\Lambda}} \sum_{\substack{(R_1,\ldots,R_n)\in\mathscr{R}^n \\ \exists R_i = R}} (\ldots)$$

and the bound (1.18). In order to prove (1.20) we observe that

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(R_1,\ldots,R_n)\in\mathscr{R}^n \\ \exists R_1 = R}} |\varphi^T(\mathrm{R}_1, \ldots, \mathrm{R}_n)\zeta(\mathrm{R}_1)\ldots\zeta(\mathrm{R}_n)| \tag{3.3}$$

$$\leqq |\zeta(R)| \left(1 + \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{i=1}^{n} \sum_{\substack{(R_1,\ldots,R_n)\in\mathscr{R}^n \\ R_i = R}} |\varphi^T \mathrm{R}_1, \ldots, \mathrm{R}_n) \prod_{K \neq i} \zeta(\mathrm{R}_K)| \right) \tag{3.4}$$

$$\leqq |\zeta(R)| \left(1 + \sum_{n=2}^{\infty} \frac{n}{n!} \sum_{(R_2,\ldots,R_n)\in\mathscr{R}^{n-1}} |\varphi^T(\mathrm{R}_1, \ldots, \mathrm{R}_n)\zeta(\mathrm{R}_2)\ldots\zeta(\mathrm{R}_n)| \right), \tag{3.5}$$

and so our task is reduced to estimate the sum

$$\sum_{(R_2,\ldots,R_n)\in\mathscr{R}^{n-1}} |\varphi^T(R_1, \ldots, R_n)\zeta(R_2)\ldots\zeta(R_n)|. \tag{3.6}$$

We rewrite this sum as a sum over the connected graphs on $\{1, \ldots, n\}$ using that $\varphi^T(R_1, \ldots, R_n)$ depends only on the graph $g(R_1, \ldots, R_n)$. If we define the function φ on C_n by

$$\varphi(g) = \begin{cases} 1 & n=1 \\ \sum_{\substack{f \subset g \\ f \in C_n}} (-1)^{|f|} & n>1, \end{cases} \tag{3.7}$$

we have

$$\varphi^T(R_1, \ldots, R_n) = \varphi(g(R_1, \ldots, R_n)),$$

and so (3.6) is equal to

$$\sum_{f\in C_n} |\varphi(f)| \sum_{\substack{(R_2,\ldots,R_n)\in\mathscr{R}^{n-1} \\ g(R_1,\ldots,R_n)=f}} |\zeta(R_1)\ldots\zeta(R_n)|. \tag{3.8}$$

We use the following nontrivial bound for $\varphi(f)$ in terms of $N(f)$, the number of trees contained in f.

Proposition

$$|\varphi(f)| \leqq N(f).$$

For the proof we refer to [10] or to [13, 14]. From

$$\sum_{f\in C_n} (\ldots) = \sum_{t\in T_n} \sum_{f\supset t} \frac{1}{N(f)} (\ldots),$$

where T_n is the set of the trees on $\{1, \ldots, n\}$, we get

$$\begin{aligned} (3.8) &\leqq \sum_{t\in T_n} \sum_{f\supset t} \sum_{\substack{(R_2,\ldots,R_n)\in\mathscr{R}^{n-1} \\ g(R_1,\ldots,R_n)=f}} |\zeta(R_2)\ldots\zeta(R_n)| \\ &= \sum_{t\in T_n} \sum_{\substack{(R_2,\ldots,R_n)\in\mathscr{R}^{n-1} \\ g(R_1,\ldots,R_n)\supset t}} |\zeta(R_2)\ldots\zeta(R_n)| \\ &= \sum_{t\in T_n} w(t), \end{aligned} \tag{3.9}$$

where the definition of $w(t)$ is implicit in the last equation. Let us compute $w(t)$, for instance, for the tree on $\{1,2,3,4\}$ made by the lines $\{1,2\}$, $\{2,3\}$, $\{2,4\}$. We have

$$w(t)=\sum_{\substack{R_2\in\mathscr{R}\\ R_2\cap R_1\neq\emptyset}}|\zeta(R_2)|\sum_{\substack{R_3\in\mathscr{R}\\ R_3\cap R_2\neq\emptyset}}|\zeta(R_3)|\sum_{\substack{R_4\in\mathscr{R}\\ R_4\cap R_2\neq\emptyset}}|\zeta(R_4)|\,.$$

The sum over R_4 gives by (1.18) a contribution less than

$$\varepsilon(1-\varepsilon)^{-1}|R_2|$$

and the same does the sum over R_3. We so have

$$w(t)\leqq\left(\frac{\varepsilon}{1-\varepsilon}\right)^2|R_1|\sup_x\sum_{x\in R_2\in\mathscr{R}}|\zeta(R_2)||R_2|^2\,.$$

We are so led to estimate the series

$$\sum_{x\in R\in\mathscr{R}}|\zeta(R)||R|^p$$

for each nonnegative integer p and this can be done using, for instance, the bound

$$\sum_{K=1}^{\infty}\varepsilon^K K^p\leqq p!\frac{\varepsilon}{1-\varepsilon}\tag{3.10}$$

that holds if $\varepsilon(1-\varepsilon)^{-1}<(e-1)^{-1}$ and follows from a simple induction argument. We so find

$$w(t)\leqq\left(\frac{\varepsilon}{1-\varepsilon}\right)^3|R_1|2!$$

Generally, for a tree t such that the degree, i.e. the number of lines containing the point $i\in\{1,\ldots,n\}$ is d_i, we have

$$w(t)\leqq|R_1|^{d_1}\left(\frac{\varepsilon}{1-\varepsilon}\right)^{n-1}\prod_{i=2}^{n}(d_i-1)!\tag{3.11}$$

The number of trees on $\{1,\ldots,n\}$ such that the degree of the point i is d_i is given, by the Cayley formula [12], by

$$\frac{(n-2)!}{\prod_{i=1}^{n}(d_i-1)!}\,.$$

The sum over the trees can be performed summing over the sequences $(d_1,\ldots,d_n)\in I_{n-1}^n$, where $I_n=\{1,\ldots,n\}$, with the constraint $d_1+\ldots+d_n=2(n-1)$. (3.9) is so bounded by

$$\sum_{\substack{(d_1,\ldots,d_n)\in I_{n-1}^n\\ d_1+\ldots+d_n=2(n-1)}}\frac{(n-2)!}{\prod_{i=1}^{n}(d_i-1)!}|R_1|^{d_1}\left(\frac{\varepsilon}{1-\varepsilon}\right)^{n-1}\prod_{i=2}^{n}(d_i-1)!$$

$$=\left(\frac{\varepsilon}{1-\varepsilon}\right)^{n-1}(n-2)!\sum_{d_1=1}^{n-1}\frac{|R_1|^{d_1}}{(d_1-1)!}\sum_{\substack{(d_2,\ldots,d_n)\in I_{n-1}^{n-1}\\ d_2+\ldots+d_n=2(n-1)-d_1}}1.\tag{3.12}$$

We now need a bound of the sum over $(d_2,\ldots,d_n)\in I_{n-1}^{n-1}$. This sum can be bounded, for instance, with the sum over $(d_2,\ldots,d_n)\in I_{2(n-1)-d_1}^{n-1}$ that we can denote

$\Gamma_{n-1}(2(n-1)-d_1)$ if we define for $1 \leqq K \leqq m$

$$\Gamma_K(m) = \sum_{\substack{(q_1, \ldots, q_K) \in I_m^K \\ q_1 + \ldots + q_K = m}} 1 . \tag{3.13}$$

But we have, via a simple induction argument on K

$$\Gamma_K(m) \leqq \frac{m^{K-1}}{(K-1)!} \tag{3.14}$$

and so

$$\Gamma_{n-1}(2(n-1)-d_1) \leqq \frac{(2(n-1)-d_1)^{n-2}}{(n-2)!} \leqq \frac{1}{2}(2e)^{n-1} .$$

Finally (3.6) is less than

$$\frac{1}{2}\left(2e\frac{\varepsilon}{1-\varepsilon}\right)^{n-1}(n-2)!\,|R_1|e^{|R_1|}$$

and Eq. (1.20) follows summing the series (3.5).

Acknowledgement. We are grateful for many discussions with E. Olivieri, M. Cassandro, and D. Capocaccia.

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Communicated by E. Lieb

Received December 10, 1981