Commun. Math. Phys. 85, 419–427 (1982)

Communications in Mathematical Physics

Ergodic Properties of a System in Contact with a Heat Bath: A One Dimensional Model*

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Abstract. We consider a one-dimensional model of a system in contact with a heat bath: A particle (the system or *molecule*) of mass M, confined to the unit interval [0, 1], is surrounded by an infinite ideal gas (the *bath of atoms*) of point particles of mass m with which it interacts via elastic collisions. The atoms are not affected by the walls at 0 and at 1. We obtain "convergence to equilibrium" for the molecule, from essentially any initial distribution on its position and velocity. The infinite composite system of molecule and bath has very good ergodic properties: it is a Bernoulli system.

1. Introduction

A central problem in non-equilibrium statistical mechanics is the time evolution of a system in contact with its surroundings. To be precise we consider a sub-system of a large isolated system. We assume that the microscopic state of the large system – specified by giving the positions and velocities of all the particles – has a well defined deterministic evolution given by Hamiltonian dynamics. This microscopic state is however not what is measured. Rather one looks at some smaller set of variables X which represent some specified part of the system.

The time evolution of X will not be autonomous – it will depend on the state Z of the rest of the large system. The appropriate description of the time evolution of X is therefore probabilistic: we specify an initial distribution μ_X for the "unobserved" Z-variables and this, together with the deterministic evolution of the pair (X, Z), determine a stochastic process for the X-variables. This process will in general not be Markovian. It will be stationary if, as is appropriate in many cases, we consider the Z-variables as a "thermal bath" for the "system X" and thus take for μ_X the Gibbs measure of the bath conditioned on the system variable X. We are then left with the problem of finding the properties of the "X-process". In this note we describe a simple example of such a system-bath complex for which many properties of the system process can be found explicitly. Generalizations of this

* Supported in part by NSF Grants PHY 78-03816 and PHY 78-15920

0010-3616/82/0085/0419/$01.80

example to more complex situations – realistic systems confined to a spacial region Λ, interacting with one or more idealized thermal baths – appear quite feasible.

2. Description of Model and Results

Consider the following dynamical system, describing an infinite number of particles moving in one dimension. A point particle of mass M (the *molecule*) moves on the unit interval [0, 1]. It is in contact with a bath of *atoms*, an ideal gas of point particles of mass m, which interacts with the molecule through eleastic collisions. The molecule also undergoes eleastic reflections from the walls at 0 and at 1. These walls do not directly affect the motion of the atoms. Between collisions of the molecule, all the particles move freely (with constant velocity). For simplicity we consider here the case in which the atoms lie only to the right of the molecule. Our results and arguments apply without essential modification to the case where the bath lies on both sides of the molecule.

Suppose that this infinite system is in equilibrium at temperature $T=(k\beta)^{-1}$ and density ϱ, i.e., that the particles are distributed (in phase space) according to the appropriate infinite volume Gibbs state. It follows easily that the distribution of the position of the molecule in [0, 1] is proportional to the exponential distribution with parameter ϱ, while the distribution of atoms to the right of the molecule is Poisson.

Let Ω denote the phase space and ϕ_t the time evolution of our system. For $\omega\in\Omega$, $\phi_t\omega$ is the phase point (*configuration*) to which ω evolves in time t.[1] We investigate the ergodic properties of the dynamical system (Ω, μ, ϕ_t), which we will denote by $\mathscr{A}$. Similar systems have been studied by de Pazzis [3] (the one dimensional, semi-infinite ideal gas), Sinai and Volkovysskii [4] (the ideal gas) and Aizenman et al. [5] (a one dimensional system of identical hard rods). In these systems the particles are either noninteracting or such interaction as there is (for hard rods) does not alter the velocities present. Nonetheless, these systems have been shown to be (isomorphic to) Bernoulli flows [6]. In the system $\mathscr{A}$ considered here a nontrivial interaction is present, though it is restricted to a single particle confined to an interval. We show that $\mathscr{A}$ is also a Bernoulli flow.

Let us denote by $Q(t)$ the position and $V(t)$ the velocity of the molecule at time t. Then $X(t)=(Q(t), V(t))$ is a random variable on (Ω, μ) and $\{X(t)\}_{t\in\mathbb{R}}$ is a stochastic process with state space $[0, 1]\times\mathbb{R}$, the phase space of the molecule. We denote this process by $\mathscr{B}$. $\mathscr{B}$ is a "factor" of the deterministic process $\mathscr{A}$: It is obtained from $\mathscr{A}$ by observing only the molecule. $\mathscr{B}$ models the evolution of a system (the molecule)

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1 There exist configurations $\omega\in\Omega$ such that $\phi_t\omega$ is not well defined for all t. Our prescription is rendered ambiguous by the occurrence of an infinite number of collisions in a finite amount of time or of multiple collisions – the simultaneous collision of the molecule with two or more atoms or with a wall and an atom. There exists, however, a set $\hat{\Omega}\subset\Omega$ with $\mu(\hat{\Omega})=1$ on which these events do not occur. Thus for $\omega\in\hat{\Omega}$, $\phi_t\omega$ is well defined for all t. (To see this, note that with (μ) probability 1, in a finite time interval only finitely many atoms can be involved in collisions. For a system with only finitely many atoms only a finite number of collisions can occur in a finite amount of time – this follows from the general results of [1]; it may also be easily seen directly – and as in the appendix of [2] it may be shown that the set of (finite) configurations giving rise to multiple collisions has Lebesgue measure zero)

in contact with a heat bath on its right. Physically, the heat bath should drive the system to equilibrium: In the process $\mathscr{B}$, all the $X(t)$'s have the common distribution $\nu(dQ, dV) \propto e^{-\varrho Q} e^{-(1/2)\beta M V^2} dQ dV$. (This is the distribution (Q, V) inherits from μ.) ν is thus the equilibrium distribution for the molecule. We prove that for a.e.[2] $X_0 = (Q_0, V_0)$, the distribution $\nu^t_{X_0}$ of $X(t)$ given $X(0) = X_0$ converges to equilibrium in variation norm, $\|\nu^t_{X_0} - \nu\| \to 0$ as $t \to \infty$, where $\| \ \|$ is the variation norm on measures, 6.5 Eq. 2 of [7]. (For any signed measure π on a measurable space Γ, $\|\pi\| = \sup_{|f| \leqq 1} \int_\Gamma f d\pi$.)

Note that this result does not follow from the Bernoulliness of $\mathscr{B}$, which implies "*weak*" convergence, i.e., on integrable functions, to equilibrium starting from *absolutely continuous* measures. It does follow from the Bernoulliness of $\mathscr{A}$ that $\mathscr{B}$ is a Bernoulli process, i.e., the shift (time translation) on trajectories $X(t)$, $-\infty < t < \infty$, defines a Bernoulli flow (when the space of trajectories is equipped with the process measure.) In fact, $\mathscr{A}$ is isomorphic (in the sense of the ergodic theory of abstract dynamical systems [8]) to the time translations on $\mathscr{B}$, since the configuration $\omega \in \Omega$ can be recovered from knowledge of $(Q(t), V(t))$, $-\infty < t < \infty$. Thus the Bernoulliness of $\mathscr{A}$ is in fact equivalent to the Bernoulliness of $\mathscr{B}$.

Our results are proven by consideration of a system (process) $\mathscr{M}$ intermediate between $\mathscr{A}$ and $\mathscr{B}$. The point is the following. For Markov processes in which "sufficient spreading" occurs, convergence to equilibrium can be established. The process $\mathscr{A}$ is Markov, because it is deterministic, but has no spreading, also because it is deterministic. $\mathscr{B}$ seems to have abundant spreading but it is not Markov, because of the possibility of recollision. The process $\mathscr{M}$ is obtained by observing all particles in [0, 1], not just the molecule. $\mathscr{M}$ is the process $\{Y(t)\}_{t \in \mathbb{R}}$, where $Y(t)$ is the configuration of particles in [0, 1] at time t. The state space of $\mathscr{M}$ is $\Omega[0, 1]$, the set of configurations in [0, 1]. $\mathscr{M}$ is a Markov process because knowledge of $\{Y(t)\}$, $t \leqq t_0$, is equivalent to knowledge of $Y(t_0)$ and of the atoms which have left [0, 1] before t_0, while the atoms entering [0, 1] after t_0 are independent of this information. $\mathscr{M}$ has the stationary distribution $\sigma(dy) = \mu(Y(0) \in dy)$. The major problem in this paper lies in showing that $\mathscr{M}$ has sufficient spreading. Once this is shown, we will have convergence to equilibrium (σ) for $\mathscr{M}$. From this it follows (see Appendix B) that $\mathscr{M}$ is Bernoulli. Since the shift on $\mathscr{M}$ is isomorphic to $\mathscr{A}$ (for the same reason that the shift on $\mathscr{B}$ is) it will follow that $\mathscr{A}$ and $\mathscr{B}$ are also Bernoulli. Since the stochastic process $\mathscr{B}$ is a "factor" of $\mathscr{M}$, convergence to equilibrium for $\mathscr{B}$ follows from convergence to equilibrium for $\mathscr{M}$. We now proceed to the details.

3. Mathematical Details

The notion of "sufficient spreading" which is relevant to our purposes is provided by the concept of an *ergodic, aperiodic Harris chain.*

By a *stationary Markov chain* (Γ, π, P) we mean a discrete time Markov process on the state space Γ, with transition probability $P(\xi, d\eta)$ and stationary probability measure $\pi(d\xi)$; $\pi P(d\eta) \equiv \int_\Gamma \pi(d\xi) P(\xi, d\eta) = \pi(d\eta)$. P acts naturally on functions[3] by

2 Almost everywhere, i.e. with the possible exception of a set of Lebesgue measure 0

3 All sets and functions to which we refer are to be understood as appropriately measurable

$f \to Pf(\xi) = \int P(\xi, d\eta) f(\eta)$. A function f is called *invariant* if $Pf = f$ (π a.e.). A set $S \subset \Gamma$ is *invariant* if its indicator function I_S, $I_S(\xi) = 1$ if $\xi \in S$ and $= 0$ otherwise, is invariant, i.e. if $P(\xi, S) = I_S(\xi)$ (π a.e.). A stationary Markov chain (Γ, π, P) is called ergodic, if it has no nontrivial invariant sets. It is said to have a *cyclically moving class* if there exist nontrivial sets $S_1, S_2, \ldots, S_n$, $n \geqq 2$, such that (π a.e.) $PI_{S_1} = I_{S_2}$, $PI_{S_2} = I_{S_3}, \ldots, PI_{S_n} = I_{S_1}$. If there are no cyclically moving classes, the stationary Markov chain is called *aperiodic*. Finally, (Γ, π, P) is called a *Harris chain* if for π a.e. $\xi \in \Gamma$ there exists a positive integer $n = n(\xi)$ such that the n-step transition probability $P^n(\xi, d\eta)$ has a component absolutely continuous with respect to π.

Proposition [9, 10]. *Let (Γ, π, P) be an ergodic, aperiodic Harris chain. Then for* π a.e. $\xi \in \Gamma$

$$\| P^n(\xi, \cdot) - \pi \| \to 0 \quad \text{as} \quad n \to \infty ,$$

where $\| \ \|$ denotes the variation norm.

Let $P^t(y, dy') = \mu(Y(t) \in dy' \mid Y(0) = y)$ be the transition probability for the process $\mathscr{M}$. (It is easy to see how to define $P^t(y, \cdot)$ for *all* y.) For each $\tau > 0$, let $\mathscr{M}_\tau$ be the Markov chain obtained by observing Y only at times which are integer multiples of τ. $\mathscr{M}_\tau = (\Omega[0, 1], \sigma, \bar{P})$ is a stationary Markov chain with $\bar{P} \equiv P^\tau$. We will show that $\mathscr{M}_\tau$ is an ergodic, aperiodic Harris chain.

Until the remark following Theorem 1, we will assume that $M > m$, though Theorem 1 is also valid for $M \leqq m$.

Let $\mathscr{P}$ denote the measure on path space for the $\mathscr{M}$ process (the *process measure* for $\mathscr{M}$) and for any $y \in \Omega[0, 1]$ let $\mathscr{P}_y$ denote the process measure for $\mathscr{M}_y$, the $\mathscr{M}$ process starting from state y: For any set B of paths

$$\mathscr{P}(B) = \mu(Y \in B) ,$$

$$\mathscr{P}_y(B) = \mu(Y \in B \mid Y(0) = y) ,$$

where $Y = \{Y(t)\}$, $t \geqq 0$, denotes the path for the process.

Let t_1 be the time at which the molecule first hits the wall at 1 and let t_2 be the next such time. Let $V_1 = |V(t_1)|$ and $V_2 = |V(t_2)|$ be the molecular speed at these times. It is shown in Appendix A that t_1 and t_2 are finite μ a.e., so that $t_1 < \infty$ $\mathscr{P}_y$ almost surely (a.s.) for σ a.e. y.

A key observation (Lemma 1 below) is that $\mathscr{P}(dV_2, dt_2)$, the distribution of (V_2, t_2) for the $\mathscr{M}$-process, and the distributions $\mathscr{P}_y(dV_2, dt_2)$ for the $\mathscr{M}_y$ processes are "overlapping". Two measures μ_1 and μ_2 on the same measure space are *overlapping* if μ_1 and μ_2 are not mutually singular, i.e., if $\mu_1(A) = 1$ and $\mu_2(B) = 1$ then $A \cap B \neq \phi$. Note that μ_1 has a component absolutely continuous with respect to μ_2 if and only if μ_1 and μ_2 are overlapping. A family of measures will be called *overlapping* if every pair of measures from the family consists of overlapping measures.

Lemma 1. *There exists a set $\bar{\Omega} \subset \Omega[0, 1]$, $\sigma(\bar{\Omega}) = 1$, such that the family of measures $\{\mathscr{P}_y(dV_2, dt_2),\ y \in \bar{\Omega};\ \mathscr{P}(dV_2, dt_2)\}$ is overlapping.*[4]

4 Since $\mathscr{P} = \int \sigma(dy) \mathscr{P}_y$, that $\mathscr{P}_y(dV_2, dt_2)$ overlaps $\mathscr{P}(dV_2, dt_2)$ in fact follows from the fact that $\mathscr{P}_y(dV_2, dt_2)$ overlaps $\mathscr{P}_{y'}(dV_2, dt_2)$ for a.e. y'

Proof. Let $\bar{\Omega}=\{y|t_1<\infty, \mathscr{P}_y \text{ a.s.}\}$. Since the entrance of atoms into [0, 1] may be described by a pure jump Markov process, and hence by a strong Markov process [11], the $\mathscr{M}$ process begins anew after the stopping time t_1:

$$\mathscr{P}_0(dY(\cdot+t_1)|\,Y(t), t\leqq t_1)=\mathscr{P}_{Y(t_1)}(dY),$$

where $\mathscr{P}_0$ is any one of the measures $\mathscr{P}$, $\mathscr{P}_y$, $y\in\bar{\Omega}$. Consider the event $\mathscr{E}$ that the molecule collides with exactly one atom between the times t_1 and t_2, the collision occurring after time t_1+1/V_1 and producing a positive molecular velocity. Suppose V_1 and t_1 are given. Then V_2 and t_2 are determined by the time $\bar{t}$ of this collision and the (pre-collision) velocity $\bar{v}$ of the atom involved:

$$V_2=((M-m)/(M+m))V_1+(2m/(M+m))\bar{v}, \tag{1}$$

$$V_1(\bar{t}-t_1)+V_2(t_2-\bar{t})=2. \tag{2}$$

The occurrence of $\mathscr{E}$ corresponds to

$$-((M-m)/2m)V_1<\bar{v}<0, \tag{3}$$

$$\tilde{t}(V_1, V_2)+t_1+1/V_1<\bar{t}<t_1+2/V_1, \tag{4}$$

where

$$V_1\tilde{t}=\max\{((V_1-\bar{v})/(V_1+\bar{v})), 0\}.$$

(By (1), (3) corresponds to a positive post-collision molecular velocity.) Eqs. (1) and (2) define a $1-1$ map from the region (3), (4) onto the (V_2, t_2)-set $S(V_1, t_1)$:

$$0<V_2<((M-m)/(M+m))V_1,\ t_1+2/V_1<t_2<t_1+1/V_1$$
$$+(1-(V_1-V_2)\tilde{t})/V_2.$$

Let $\mathscr{P}'_0$ be the restriction of $\mathscr{P}_0$ to $\mathscr{E}$.

Since

$$dV_2dt_2=\partial(V_2, t_2)/\partial(\bar{v}, \bar{t})d\bar{v}d\bar{t}$$

where

$$\partial(V_2, t_2)/\partial(\bar{v}, \bar{t})=(2m/(M+m))((V_1-V_2)/V_2)>0$$

for the relevant values of the variables, and since the distribution of $(\bar{v}, \bar{t})$ is equivalent to Lebesgue measure (i.e., is given by a positive density), it follows that $\mathscr{P}'_0(dV_2, dt_2|V_1, t_1)$, the $\mathscr{P}'_0$ distribution of (V_2, t_2) given V_1 and t_1, is equivalent to Lebesgue measure on $S(V_1, t_1)$. Since the sets $S(V_1, t_1)$, $V_1>0$, $t_1>0$, are non-disjoint, Lemma 1 follows.

Lemma 2. (i) *For any* $y, y'\in\bar{\Omega}$, $P^t(y, \cdot)$ *and* $P^t(y', \cdot)$ *are overlapping for* t *sufficiently large.*

(ii) *For any* $y\in\bar{\Omega}$, $P^t(y, \cdot)$ *and* $\sigma(\cdot)$ *are overlapping for* t *sufficiently large.*

Proof. Applying the strong Markov property to the stopping time t_2, we obtain from Lemma 1 that the family of measures $\{\mathscr{P}_y\restriction\mathscr{F}_{\geqq t_2},\ y\in\bar{\Omega};\ \mathscr{P}\restriction\mathscr{F}_{\geqq t_2}\}$, where

$\mathscr{F}_{\geqq t_2} = \mathscr{F}(\{t_2;\ Y(t+t_2),\ t \geqq 0\})$ is the σ-algebra describing the process after t_2, is overlapping.[5] Lemma 2 follows easily.

Theorem 1. *For any $\tau > 0$, $\mathscr{M}_\tau$ is an ergodic aperiodic Harris chain.*

Proof. It follows immediately from Lemma 2 (ii) that $\mathscr{M}_\tau$ is a Harris chain. Suppose it has a nontrivial invariant set A. Then for a.e. $y \in A$ and $y' \in A^c$ (the complement of A) $P^{n\tau}(y, \cdot)$ does not overlap $P^{n\tau}(y', \cdot)$ for any $n = 1, 2, 3, \ldots$, contradicting Lemma 2 (i). Thus $\mathscr{M}_\tau$ is ergodic. Finally, since $\mathscr{M}_{n\tau}$ is ergodic for all $n = 1, 2, 3, \ldots, \mathscr{M}_\tau$ is aperiodic.

Remark. In case $M < m$, we may obtain a version of Lemma 1 – involving, in addition to t_1, t_2, V_1, and V_2, the times t_3, $t_4, \ldots$, of subsequent arrival of the molecule at the wall at 1, its speeds $V_3, V_4, \ldots$, at these times, and the distributions of $(V_k, t_k + n\tau)$ for appropriate k and n – whose proof employs many incoming atoms. We may then obtain, in place of Lemma 2 (i), that for any $y, y' \in \bar{\Omega}$, there exists an integer n such that $P^t(y, \cdot)$ and $P^{t+n\tau}(y', \cdot)$ are overlapping for t sufficiently large. Theorem 1 follows as before, with the contradiction for the proof of ergodicity being that $P^{n\tau}(y, \cdot)$ cannot overlap $P^{n\tau}(y', \cdot)$ if A is invariant. The details for the case $M < m$ are left to the reader. (The case $M = m$, which must also be treated separately, is fairly easy.)

Corollary 1. (*Convergence to equilibrium for $\mathscr{M}$*)

(i) $\|P^t(y, \cdot) - \sigma(\cdot)\| \to 0$ *as* $t \to \infty$ *for* σ a.e. $y \in \Omega\ [0, 1]$. (5)

(ii) *More generally, there exists a set* $\tilde{\Omega} \subset \Omega[0, 1]$, *with* $\sigma(\tilde{\Omega}) = 1$, *such that* $\|\gamma P^t - \sigma\| \to 0$ *as* $t \to \infty$, *for any probability measure* γ *on* $\Omega[0, 1]$ *with* $\gamma(\tilde{\Omega}) = 1$. *Here* $\gamma P^t(\cdot) = \int \gamma(dy) P^t(y, \cdot)$ *is the distribution at time* t *starting from* γ.

Proof. It follows from Theorem 1 that

$$\|P^n(y, \cdot) - \sigma(\cdot)\| \to 0 \quad \text{as} \quad n \to \infty \tag{6}$$

for σ a.e. $y \in \Omega[0, 1]$. Let $\tilde{\Omega}$ be the set on which (6) holds. Then $\sigma(\tilde{\Omega}) = 1$ and for $y \in \tilde{\Omega}$ (5) follows from the stationarity of σ and the fact that $\lambda \to \lambda P^t$ is a contraction for all $t \geqq 0$: Let $t = n + \tilde{t}$, $0 \leqq \tilde{t} < 1$. Then

$$\begin{aligned} \|P^t(y, \cdot) - \sigma\| &= \|(\delta_y - \sigma) P^t\| = \|(\delta_y - \sigma) P^n P^{\tilde{t}}\| \\ &\leqq \|(\delta_y - \sigma) P^n\| = \|P^n(y, \cdot) - \sigma(\cdot)\|, \end{aligned}$$

where δ_y is the unit (point) measure at y. Using dominated convergence, (ii) follows from (i):

$$\|\gamma P^t - \sigma\| = \|\int \gamma(dy)(P^t(y, \cdot) - \sigma(\cdot)\| \leqq \int \gamma(dy) \|P^t(y, \cdot) - \sigma(\cdot)\|.$$

From Corollary 1 we obtain (see Appendix B)

Corollary 2. *$\mathscr{M}$, and hence $\mathscr{A}$ and $\mathscr{B}$, are Bernoulli.*

Theorem 2. (*Convergence to equilibrium for $\mathscr{B}$*)

$\|\nu^t_{(Q_0, V_0)} - \nu\| \to 0$ *as* $t \to \infty$ *for* ν a.e. $(Q_0, V_0) \in [0, 1] \times \mathbb{R}$.

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5 "$\restriction$" indicates restriction

Proof. Let $\gamma(\cdot)=\sigma(\cdot\,|\,Q=Q_0, V=V_0)$. Then

$$\sigma(\cdot)=\int \nu(dQ_0, dV_0)\sigma(\cdot\,|\,Q=Q_0, V=V_0)$$

so that $\gamma(\tilde{\Omega})=1$ for ν a.e. $(Q_0, V_0)\in[0, 1]\times\mathbb{R}$. Since $\nu^t_{(Q_0,V_0)}-\nu$ is the restriction of $\gamma P^t-\sigma$ to the (Q, V) sub-σ-algebra, and since restriction to a sub-σ-algebra is variation norm decreasing, Theorem 2 follows from Corollary 1 (ii).

Appendix A

Let $H\subset\Omega$ be the set of configurations for which the molecule hits the wall at 1 infinitely often.

Theorem A. $\mu(H)=1$.

Proof. We assume first that $M\neq m$. Let $\Omega_{N,U}\subset\Omega$ be the set of configurations in which there are no more than N atoms in $[0, 1]$ and all the particles in $[0, 1]$ have speed smaller than $U>0$. Since $\Omega_{N,U}\uparrow\Omega$, it will suffice to show that $\mu(H\cap\Omega_{N,U})=\mu(\Omega_{N,U})$, i.e., that $\mu_{N,U}(H)=1$, where $\mu_{N,U}$ is the conditional measure given $\Omega_{N,U}$.

It follows from Poincare recurrence that for $\mu_{N,U}$ a.e. $x\in\Omega$, $\phi_n(x)\in\Omega_{N,U}$ infinitely often (i.e., for infinitely many values of $n=1, 2, 3, \ldots$). Let $\tau_1<\tau_2<\tau_3<\ldots$, be the successive integer return times to $\Omega_{N,U}$. $\tau_k<\infty$, $\mu_{N,V}$ a.e..

The key observation is that there exists a set $A_{N,U}\subset\Omega$, depending only upon atoms first entering $[0, 1]$ at a time $0\leqq t<1$, such that on $A_{N,U}\cap\Omega_{N,U}$ the molecule hits the wall at 1 at a time between 0 and 1. If $M>m$, we could take A to be the event that a sufficiently fast atom enters $[0, 1]$ between $t=0$ and $t=1/2$ and no other atom enters between $t=0$ and $t=1$. (This fast atom will impart a great deal of energy to the molecule, which will then quickly push out the atoms in $[0, 1]$.) If $M<m$, it may be necessary to send in many atoms in order to push out all atoms in $[0, 1]$.

Let f_k be the indicator function of the event $\phi_k(x)\in A_{N,U}$ and let $g_k(x)=f_{\tau_k(x)}(x)$. g_k is the indicator function of the event "A occurs at time τ_k." The g_k's form a sequence of independent identically distributed random variables, with

$$\int d\mu_{N,U} g_k=\mu_{N,U}(A)>0.$$

Therefore $g_k=1$ infinitely often, $\mu_{N,U}$ a.e.. Since $g_k=1$ implies that the molecule hits the wall at 1 between times τ_k and τ_k+1, we obtain that $\mu_{N,U}(H)=1$. This completes the proof for the case $M\neq m$.

If $M=m$, the theorem follows easily from the fact that if the wall at 1 is removed we obtain an ergodic, in fact, Bernoulli system [3].

Appendix B

By a *stationary Markov process* (Γ, π, P^t) we mean a Markov process on the state space Γ, with transition probability $P^t(\xi, d\eta)$ and stationary probability measure π, $\pi P^t=\pi$. We show that a stationary Markov process with convergence to equilib-

rium defines a Bernoulli system (*B*-flow for continuous time, *B*-shift for discrete time).

We assume that Γ is a Polish space, i.e., a separable topological space whose topology is compatible with a metric with respect to which it is complete, and, in the case of continuous time, that the process has (a version with) sample paths which are right continuous and have left limits.

Theorem B. *A stationary Markov process* (Γ, π, P^t) *satisfying*

$$\|P^t(\xi, \cdot) - \pi\| \to 0 \quad \text{as} \quad t \to \infty, \ \pi \text{ a.e.}$$

defines a Bernoulli system.

Proof. We first prove the theorem for discrete time, i.e. for a stationary Markov chain (Γ, π, P). Let $(\Gamma_\infty, \pi_\infty, S)$ be the corresponding dynamical system:

$$\Gamma_\infty = \bigtimes_{i=-\infty}^{\infty} \Gamma^{(i)}, \ \Gamma^{(i)} = \Gamma, \ \pi_\infty = \bigtimes_{i=-\infty}^{\infty} \pi^{(i)}, \ \pi^{(i)} = \pi,$$

S = the shift, $(S\underline{\xi})_i = \xi_{i+1}$ where $\underline{\xi} = (\xi_i) \in \Gamma_\infty$. Denote by $\bar{\mathscr{F}}$ the σ-algebra on which π is defined. Then π_∞ is a measure on the product σ-algebra $\mathscr{F} = \times \bar{\mathscr{F}}$. Let $\mathscr{F}_n$ be the sub-σ-algebra of $\mathscr{F}$ generated by ξ_n and let $\mathscr{F}_n^m$ be the sub-σ-algebra generated by ξ_k, $n \leqq k \leqq m$. For any finite partition $Q = \{Q_j\}$ of Γ, let Q_n be the partition into atoms $Q_{n,j} = \{\xi_n \in Q_j\}$, and let $Q_n^m = \bigvee_{k=n}^{m} Q_k$. We identify Q_n^m with the obvious σ-algebra.

To establish that $(\Gamma_\infty, \pi_\infty, S)$ is isomorphic to a Bernoulli shift, it is sufficient to show Q_0 is a weak Bernoulli partition [6, 12] for every finite partition Q of Γ.

For $\mathscr{I}$, $\mathscr{H} \subset \mathscr{F}$, let

$$\varrho(\mathscr{I}, \mathscr{H}) = \int \pi_\infty(d\underline{\xi}) \|(\pi_\infty(\cdot \mid \mathscr{H}(\underline{\xi})) - \pi_\infty(\cdot)) \restriction \mathscr{I}\|,$$

where $\pi_\infty(\cdot \mid \mathscr{H}(\underline{\xi}))$ is the conditional probability given the "fiber of the σ-algebra $\mathscr{H}$ containing $\underline{\xi}$." The weak Bernoulli property for Q is equivalent to the condition that

$$\sup_m \varrho(Q_n^{n+m}, Q_{-m}^0) \to 0 \quad \text{as} \quad n \to \infty.$$

It is easy to see that ϱ is increasing in both arguments: $\mathscr{I} \subset \mathscr{I}'$ implies $\varrho(\mathscr{I}, \mathscr{H}) \leqq \varrho(\mathscr{I}', \mathscr{H})$ and $\varrho(\mathscr{H}, \mathscr{I}) \leqq \varrho(\mathscr{H}, \mathscr{I}')$. Therefore

$$\begin{aligned} \varrho(Q_n^{n+m}, Q_{-m}^0) \leqq \varrho(\mathscr{F}_n^\infty, \mathscr{F}_{-\infty}^0) &\overset{(*)}{=} \varrho(\mathscr{F}_n, \mathscr{F}_0) \\ &= \int \pi(d\xi) \|P^n(\xi, \cdot) - \pi(\cdot)\| \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

where the equality (*) follows from the Markov property. This completes the proof for discrete time.

Let $(\Gamma_\infty, \pi_\infty, S_t)$ be the dynamical system for the continuous time case: For $\underline{\xi} \in \Gamma_\infty$, $\underline{\xi} = (\xi_t)$ is a right continuous sample path with left limits, π_∞ is the product measure, and S_t the shift (translation) by time t. To say that $(\Gamma_\infty, \pi_\infty, S_t)$ is Bernoulli is to say that $(\Gamma_\infty, \pi_\infty, S_1)$ is Bernoulli. But $(\Gamma_\infty, \pi_\infty, S_1)$ is isomorphic to the system corresponding to the stationary Markov chain, with state space $D[0, 1]$ of paths $\underline{\tilde{\xi}} = (\xi_t)$, $0 \leqq t \leqq 1$, which are right continuous with left limits, induced by the map

$\Gamma_\infty \to D[0, 1]$, $\xi \to \tilde{\xi} = \xi \restriction [0, 1]$. (This is the Markov chain $\tilde{\xi}_0, \tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_k, \ldots$, where $\tilde{\xi}_k = (\widetilde{S_k \xi})$). Since this Markov chain satisfies the convergence condition of Theorem B (by the Markov property for the continuous time process) and $D[0, 1]$ is a Polish space[6] [13], Theorem B for continuous time follows from Theorem B for discrete time.

We remark that the condition on the state space Γ was necessary for $(\Gamma_\infty, \pi_\infty)$ to be a Lebesgue space – the usual starting point in abstract ergodic theory. If $(\Gamma_\infty, \pi_\infty)$ were not a Lebesgue space, $(\Gamma_\infty, \pi_\infty, S)$ clearly could not be isomorphic to a Bernoulli shift, which acts on a Lebesgue space.

Acknowledgement. We wish to thank Errico Presutti for very helpful suggestions in the course of this work.

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Communicated by A. Jaffe

Received January 29, 1982

6 More precisely, the appropriate induced σ-algebra on $D[0, 1]$ is the Borel σ-algebra for $D[0, 1]$ regarded as a Polish space